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Cubics associated to a Euclidean plane curve via curvature

Mircea Crasmareanu

Abstract. To each point of a natural parametrized curve C in the Euclidean plane we associate a cubic curve. We analyze when this cubic is an elliptic curve. The main tool of our study is the curvature of C and the logarithmic spiral appears as a remarkable example.

1. Introduction

In a series of recent papers ([3]-[5]) the geometry of a given curve C is studied in various frameworks through differential equations satisfied by the functions defining C. The present paper deals with this line of research into the Euclidean plane $\mathbb{E}^2$ by connecting it with the elementary algebraic geometry.

More precisely, by fixing a naturally parametrized curve C in $\mathbb{E}^2$, its third-order differential equation suggests a cubic curve $\Gamma(s)$ associated to a fixed value of the arc-length parameter s. The problem studied now is when this cubic $\Gamma(s)$ is an elliptic curve and this implies the study of the sign for the discriminant $\tilde{\Delta}$ of a second order algebraic equation. All three cases for $\tilde{\Delta}$ are analyzed and/or exemplified and hence some special cubics are obtained.

It is worth remarking that the expression of $\tilde{\Delta}$ involves the value of the curvature function k of C and its derivative k'; a starting hypothesis in our study is that $k > 0$. Hence, our study can be considered as building a bridge between the plane differential geometry and algebraic geometry through differential equations. We note also that in two cases the studied first-order differential equation with k as unknown is of Bernoulli-type.

Moreover, the equation $\tilde{\Delta} = 0$ implies two remarkable facts: 1) the initial curve C is a logarithmic spiral with the tangent angle equal to $\cot^{-1}(2)$; ii) its k is an isoparametric function on the positive real line $(0, +\infty)$ with the canonical Riemannian metric induced by the product of positive numbers.

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2. Cubic curves associated to points of a given Euclidean plane curve

The setting of this paper is the Euclidean plane $\mathbb{E}^2 := (\mathbb{R}^2, \langle , \rangle_{can})$ with the canonical inner product:

(2.1)
$$\langle u, v\rangle_{can} = x^1y^1+x^2y^2, \quad u = (x^1, x^2) \in \mathbb{R}^2, \quad v = (y^1, y^2) \in \mathbb{R}^2, \quad 0 \leq \|u\|^2 = \langle u, u\rangle.$$

Fix a real interval $I \subseteq \mathbb{R}^* := [0, +\infty)$ and consider $C \subset \mathbb{E}^2$ a naturally parametrized curve of vectorial equation:

(2.2)
$$C : r(s) = (x(s), y(s)) = x(s)\bar{i}+y(s)\bar{j}, \quad \|r'(s)\| = 1, \quad s \in I, \quad \bar{i} = (1, 0), \bar{j} = (0, 1).$$

The Frenet apparatus of the curve C consists in (the Frenet frame $\{T = T(s), N = N(s)\}$, the curvature function $k = k(s)$) and it is provided by the regularity of the vectorial map through the relations:

$$\left\{ \begin{array}{l} T(s) = r'(s), \quad N(s) = i \cdot T(s) = (-y'(s), x'(s)), \\ k(s) = \langle T'(s), N(s)\rangle_{can} = x'(s)y''(s) - y'(s)x''(s). \end{array} \right. \tag{2.3}$$

The fundamental theorem of plane curves ([1, p. 53]) gives the expression of the derivative of r as:

$$r'(s) = (-\sin K(s), \cos K(s)), \quad K(s) := \int_{s_0}^{s} k(t)dt. \tag{2.4}$$

Then a straightforward computation gives that the component functions $x(\cdot)$, $y(\cdot)$ of r satisfy the third order differential equation:

$$\mathcal{E}^3 : kU''' - k'U'' + k^3U' = 0, \quad k'(s) = x'(s)y'''(s) - y'(s)x'''(s). \tag{2.5}$$

Third-order explicit autonomous ordinary differential equations $\dddot{x} = F(\ddot{x}, \dot{x}, x)$ are called sometimes *jerk* or *jerky equations* since the quantity $\dddot{x}$ is called jerk in Mechanics. An important feature of jerk equations is provided by their *possible* chaotic behaviour.

From now on, we suppose that the curvature function is strictly positive, $k(s) > 0$ for all $s \in I = [0, L \leq +\infty]$; for example (after the Theorem 6.7. of [1, p. 198]) C is a simple, closed and *convex* curve. A geometrical interpretation of the inequality $k(s) > 0$ is that the trace of C lies on only one side of the tangent line of C in every point $r(s)$. The above differential equation becomes:

$$\mathcal{E}^3 : U''' - \frac{k'}{k}U'' + k^2U' = 0. \tag{2.6}$$

The idea of our work is, that inspired by this expression, to associate to a given point $r(s) \in C$ the cubic curve:

$$\Gamma(s) : y^2 = x^3 - \frac{k'(s)}{k(s)}x^2 + k^2(s)x = x\left(x^2 - \frac{k'(s)}{k(s)}x + k^2(s)\right). \tag{2.7}$$

The method to associate an object to every point of a given manifold M is the starting point of the theory of *bundles over* M. Recall, after [9, p. 3], that a general cubic curve $\Gamma : y^2 = x^3 + ax^2 + bx + c$ is an *elliptic curve* if the polynomial function of the right-hand-side is nonsingular i.e. it has *distinct* roots. For our $\Gamma(s)$ since $k > 0$ the root $x_1 = 0$ can not be a multiple root. Hence we need to discuss the sign of the discriminant:

$$\tilde{\Delta}(s) = \left(\frac{k'(s)}{k(s)}\right)^2 - 4k^2(s) \tag{2.8}$$

and if $\tilde{\Delta}(s) \neq 0$ for all s we have a bundle of groups over our curve C.

This discriminant $\tilde{\Delta}$ can be expressed in terms of K as:

$$\tilde{\Delta} = \left(\frac{K''}{K'}\right)^2 - 4(K')^2. \tag{2.9}$$

Recall also that for a function $F \in C^3(I, \mathbb{R})$ its *Schwarzian derivative* is ([2]):

$$S_F := \frac{F'''}{F'} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2 \tag{2.10}$$

and hence, the discriminant is:

$$\tilde{\Delta} = \frac{2}{3}\left(\frac{K'''}{K'} - S_K\right) - 4(K')^2. \tag{2.11}$$

3. The case $\tilde{\Delta}(s) < 0$

We discuss this case through two examples:

Example 3.1 We start with the circles $\mathcal{C}(O(0,0), R > 0)$ with:

$$\mathcal{C}(O, R) : r(s) = R\left(\cos\frac{s}{R}, \sin\frac{s}{R}\right), \quad k \equiv \frac{1}{R} > 0 \tag{3.1}$$

and therefore the elliptic curve Γ does not depend on s but on R:

$$\Gamma_R : y^2 = x\left(x^2 + \frac{1}{R^2}\right). \tag{3.2}$$

Hence, the unit circle S^1 has as associated elliptic curve:

$$\Gamma_1 : y^2 = x^3 + x \tag{3.3}$$

which is a remarkable one; for example, is a CM-elliptic curve ([7]). Several of its properties and invariants are available at: https://www.lmfdb.org/EllipticCurve/Q/64/a/4. Also:

$$\int_0^\infty \frac{dx}{y} = \int_0^\infty \frac{dx}{\sqrt{x^3 + x}} = \frac{8}{\sqrt{\pi}}\left(\Gamma\left(\frac{5}{4}\right)\right)^2 \simeq 3.70815. \tag{3.4}$$

□

Example 3.2 Adapting to our setting the *Cornu spiral* from [1, p. 54] we consider:

(3.5)
$$C : r(s) = \left(-\int_0^s \sin\left(\tau + \frac{\tau^2}{2}\right) d\tau, \int_0^s \cos\left(\tau + \frac{\tau^2}{2}\right) d\tau \right), \quad k(s) = 1 + s \geq 1$$

for which:

$$\tilde{\Delta}(s) = \left(\frac{1}{1+s}\right)^2 - 4(1+s)^2 < 0 \tag{3.6}$$

since $0 < 1/(1+s) \leq 1 + s < 2(1+s)$ for any $s \geq 0$. The associated elliptic curve is:

$$\Gamma_{Cornu}(s) : y^2 = x\left(x^2 - \frac{x}{1+s} + (1+s)^2\right) \tag{3.7}$$

with the interesting particular case $\Gamma_{Cornu}(0) : y^2 = x(x^2 - x + 1)$ which is the curve 5.7.2 of [8, p. 315]. The change $x = X + \frac{1}{3}$ yields its Weierstrass form: $\Gamma_{Cornu}(0) : y^2 = X^3 + \frac{2}{3}X + \frac{7}{27}$ and again, it is studied in: https://www.lmfdb.org/EllipticCurve/Q/24/a/5. □

4. The case $\tilde{\Delta}(s) > 0$

The elliptic curve of this situation can not be expressed in a *normal form*:

Proposition 4.1 *The elliptic curve* $\Gamma(s)$ *can not be expressed as*:

$$\Gamma(s) : y^2 = x(x-1)(x-\lambda(s)), \quad \lambda(s) \notin \{0, 1\}. \tag{4.1}$$

Proof We must discuss the differential equation:

$$1 - \frac{k'}{k} + k^2 = 0 \tag{4.2}$$

which is of Bernoulli-type. Its general solution depends of a strictly positive constant $\mathcal{C} > 0$:

$$k_{\mathcal{C}}(s) = \frac{e^{\mathcal{C}+s}}{\sqrt{1 - e^{2\mathcal{C}+2s}}} \tag{4.3}$$

but the existence of the square root implies $s \in (-\infty, -\mathcal{C} < 0)$, a contradiction with the initial hypothesis $s \geq 0$. □

A class of examples for this case is discussed in the next section. Now, we perform the reduction to the Weierstrass form with the change $x = X + \frac{k'}{3k}$:

(4.4)
$$\Gamma(s) : y^2 = X^3 + \left(k^2 - \frac{(k')^2}{3k^2}\right) X + \frac{kk'}{3} - \frac{2(k')^3}{27k^3} = X^3 + pX + q, p < k^2 - \frac{4k^2}{3} = -\frac{k^2}{3} < 0.$$

Its discriminant as elliptic curve is ([6]):

$$\Delta(s) := 4p^3 + 27q^2 = 4k^6 - (k'k)^2 < 0. \tag{4.5}$$

5. The case $\tilde{\Delta}(s) = 0$

There are two cases $k' = \pm 2k^2$ but a positive solution is provided by only one situation:

Proposition 5.1 *The only curve satisfying $\tilde{\Delta}(s) = 0$ is a logarithmic spiral depending on a real constant $\mathcal{C} > 0$. The corresponding cubic $\Gamma_{\mathcal{C}}(s)$ admits a polynomial parametrization.*

Proof The classical definition of the logarithmic spiral $LogS$ is that $LogS$ is the plane trajectory of a point P moving along a straight line $d \ni O$ with the velocity proportional to the Euclidean distance $d(O, P)$ while simultaneous, the line d uniformly rotates about the point O. Hence, the parametrization of the logarithmic spiral involves two constants:

$$LogS(R > 0, \omega > 0) : \tilde{r}(t) = Re^{\omega t}e^{it} = Re^{\omega t}(\cos t, \sin t) = e^{\ln R + (\omega + i)t} \tag{5.1}$$

with the curvature:

$$k(t) = \frac{1}{\|\tilde{r}'(t)\|} = \frac{1}{R\sqrt{1+\omega^2}e^{\omega t}}. \tag{5.2}$$

Therefore, the arc-length parameter is given by:

$$s(t) = R\frac{\sqrt{1+\omega^2}}{\omega}(e^{\omega t} - 1), \quad r(s) = \left(R + \frac{\omega s}{\sqrt{1+\omega^2}}\right)e^{it(s)} \tag{5.3}$$

and hence, the curvature is:

$$k(s) = \frac{1}{R\sqrt{1+\omega^2} + \omega s}, \quad K(s) = \frac{1}{\omega}\ln\left(1 + \frac{\omega s}{R\sqrt{1+\omega^2}}\right) = t(s). \tag{5.4}$$

The first relation (5.4) shows that, from the curvature point of view, the logarithmic spiral is an inverse of the Cornu spiral.

The Bernoulli-type equation $k' = -2k^2$ admits the solution:

$$k(s) = \frac{1}{2s + \mathcal{C}}, \quad \mathcal{C} > 0 \tag{5.5}$$

and hence, our curve belongs to the 1-parametric family of logarithmic spirals $LogS\left(\frac{\mathcal{C}}{\sqrt{5}}, 2\right)$. We recall that the parameter $\omega = 2$ controls the direction in which the spiral wraps itself while the parameter $R = \frac{\mathcal{C}}{\sqrt{5}}$ controls the size of $LogS$.

It is worth to point out that now it is easy to express the *evolute* of the logarithmic spiral:

$$Ev(LogS) : r_{ev}(s) := r(s) + \frac{1}{k(s)}N(s) = R\omega e^{\omega t(s)}(-\sin t(s), \cos t(s)) =$$

$$= R\omega e^{\omega t}\left(\cos\left(\frac{\pi}{2}+t\right),\sin\left(\frac{\pi}{2}+t\right)\right) \tag{5.6}$$

which gives the well-known result that this new curve is also a logarithmic spiral:

$$LogS(R\omega e^{-\frac{\omega\pi}{2}},\omega) : r_{ev}(T) = R\omega e^{\omega(T-\frac{\pi}{2})}(\cos T,\sin T). \tag{5.7}$$

The associated singular cubic curve is :

$$\Gamma_{\mathcal{C}}(s) : y^2 = x\left(x+\frac{1}{2s+\mathcal{C}}\right)^2. \tag{5.8}$$

and is polynomially parametrized by:

$$\Gamma_{\mathcal{C}}(s) : \hat{r}_{(\mathcal{C},s)}(t) = \left(t^2, t\left(t^2+\frac{1}{2s+\mathcal{C}}\right)\right), t\in\mathbb{R}. \tag{5.9}$$

□

Example 5.2 Suppose that $s=0$ and $\mathcal{C}>0$ is the rational number $\mathcal{C}=\frac{p}{q}\in\mathbb{Q}^*$ with $p,q\in\mathbb{N}^*:=\{1,2,...\}$. Then the cubic $\Gamma_{\mathcal{C}}(s)$ contains two sequences of integral points:

$$\Gamma_{\frac{p}{q}}(s)\ni(x_n=n^2p^2, y_n^{\pm}=\pm n(n^2p^3+q))\in\mathbb{N}^*\times\mathbb{N}^*. \tag{5.10}$$

The value $s=0$ corresponds to the starting point $\tilde{r}(0)=(R,0)\in LogS$. □

Remarks 5.3 i) Recall the geometrical meaning of ω for the logarithmic spiral: the angle between the position vector field $r(t)$ and the velocity vector field $r'(t)$ is a constant one, say φ. Then $\omega=\cot\varphi$. For our spiral this *tangent angle* is then $\varphi=\cot^{-1}(2)=26.57°$. The complex number $z=2+i=\sqrt{5}e^{i\arctan(1/2)}\in\mathbb{C}$ appearing in the exponential expression (5.1) is a Gaussian prime number having $P(X)=X^2-4X+5$ as minimal polynomial.
ii) Also for the spiral above we have:

$$\langle T,N\rangle_{can}=0,\quad \langle T',N\rangle_{can}=k>0,\quad \langle T'',N\rangle_{can}=k'=-2k^2<0. \tag{5.11}$$

iii) The curvature of the cubic (5.9) is:

$$\hat{k}_{(\mathcal{C},s)}(t)=\frac{6t^2-A}{[4t^2+(3t^2+A)^2]^{\frac{3}{2}}},\quad A=\frac{1}{2s+\mathcal{C}}>0. \tag{5.12}$$

iv) We recall after [1, p. 55-56] that if the vectorial function r is of class C^3 then *the local canonical form* of the curve C in the point $r(s_0)$ is:

$$r(s)=r(s_0)+(s-s_0)T(s_0)+\frac{(s-s_0)^2}{2!}k(s_0)N(s_0)+\frac{(s-s_0)^3}{3!}[k'(s_0)N(s_0)-k^2(s_0)T(s_0)]+R(s),$$

$$\lim_{s\to s_0}\frac{R(s)}{(s-s_0)^3}=(0,0). \tag{5.13}$$

It follows for our spiral that:
(5.14)

$$r(s) = r(s_0)+(s-s_0)T(s_0)+\frac{(s-s_0)^2}{2!}k(s_0)N(s_0)-\frac{(s-s_0)^3k^2(s_0)}{3!}[2N(s_0)+T(s_0)]+R(s).$$

v) The vanishing of $\tilde{\Delta}$ means the equation $(k')^2 = 4k^4$ which can be generalized on an arbitrary Riemannian manifold (M^n, g). Namely, we ask for smooth functions $f \in C^\infty(M)$ such that their gradient vector field ∇f satisfies:

$$\|\nabla f\|_g^2 = 4f^4. \tag{5.15}$$

Recall that $f \in C^\infty(M)$ is called *isoparametric function* on (M, g) if it satisfies the system of partial differential equations:

$$\Delta_g f = a(f), \quad \|\nabla f\|_g^2 = b(f) \tag{5.16}$$

for a continuous scalar function a respectively a C^2 scalar function b. Hence, our condition $\tilde{\Delta} = 0$ defines an isoparametric function on $(M^1 = (0, +\infty), g_{can})$ since from $k' = \pm 2k^2$ it results also $\Delta k = k'' = 8k^3$.
vi) In the section 2 we go from the linear differential equation (2.6) to the algebraic equation $\tilde{\Delta} = 0$. By reversing this trip we associate to the algebraic equation $\tilde{\Delta} = 0$ the linear differential equation:

$$U'' - \frac{k'}{k}U' + k^2 = 0 \tag{5.17}$$

with the unknown U; equivalently:

$$\left(\frac{U'}{k}\right)' = -k = -K'. \tag{5.18}$$

with the final solution: $U_{\mathcal{C}} = \mathcal{C} - K^2/2 \leq \mathcal{C}$, $\mathcal{C} \in \mathbb{R}$. $\square$

Now, we return to the second case with an example:

Example 5.4 Any logarithmic spiral with $\omega > 2$ satisfies $\tilde{\Delta}(s) > 0$. The associated elliptic curve is:

$$\Gamma(s) : y^2 = x(x - x_+(s))(x - x_-(s)), \quad 2x_\pm(s) = \frac{\sqrt{\omega^2 - 4} \pm \omega}{R\sqrt{1+\omega^2} + \omega s}. \tag{5.19}$$

Recall also that given the triple $e_1 < e_2 < e_3$ and its associated elliptic curve $E(e_1, e_2, e_3) : y^2 = (x - e_1)(x - e_2)(x - e_3)$ its *real period* is the positive number:

$$\Omega(E(e_1, e_2, e_3)) := 2\int_{e_3}^{+\infty} \frac{dx}{y} = \frac{2\pi}{AGM(\sqrt{e_3 - e_1}, \sqrt{e_3 - e_2})} \tag{5.20}$$

with $AGM(u > 0, v > 0)$ the arithmetic-geometric mean of the positive numbers u an v. For our $\Gamma(s)$:

$$(e_3 - e_2)(s) = x_+(s), \quad (e_3 - e_1)(s) = (x_+ - x_-)(s) = \frac{\omega}{R\sqrt{1+\omega^2} + \omega s}. \tag{5.21}$$

□

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6. References

REFERENCES

1. H. Alencar; W. Santos and S. N. Gregório: *Differential geometry of plane curves*. American Mathematical Society, 2022. Zbl 1511.53001
2. M. Crasmareanu: *Higher order curvatures of plane and space parametrized curves*. Algorithms, Special Issue: Machine Learning in Computational Geometry **15** (2022), no. 11, paper no. 436.
3. M. Crasmareanu: *The adjoint map of Euclidean plane curves and curvature problems*. Tamkang J. Math. **55** (2024), no. 4, 331–337. Zbl 1554.53003
4. M. Crasmareanu: *Flow-selfdual curves in a geometric surface*. Ital. J. Pure Appl. Math. **51** (2024), 99–105.
5. M. Crasmareanu: *Differential equations of spacelike parametrized curves in the Lorentz plane*. Turk. J. Math. Comput. Sci. **16** (2024), no. 1, 1–5.
6. M. Crasmareanu; C.-L. Pripoae and G.-T. Pripoae: *CR-selfdual cubic curves*. Mathematics **13** (2025), no. 2, paper no 317.
7. R. Ross: *K_2 of Fermat curves and values of L-functions*. C. R. Acad. Sci. Paris **312** (1991), no. 1, 1–5. Zbl 0744.14006
8. J.-P. Serre: *Propriétés galoisiennes des points d'ordre fini des courbes elliptiques*. Invent. Math. **15** (1972), 259–331. Zbl 0235.14012
9. T. R. Shemanske: *Modern cryptography and elliptic curves. A beginner's guide*. American Mathematical Society, 2017. Zbl 1392.11001

Mircea Crasmareanu
Faculty of Mathematics
Department of Mathematics
University "Al. I. Cuza"
Iaşi, 700506, Romania
mcrasm@uaic.ro
http://www.math.uaic.ro/∼mcrasm
ORCID: 0000-0002-5230-2751