Konuralp Journal of Mathematics, 13 (x) (2025?) xxx-xxx

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath
e-ISSN: 2147-625X

KJM_Cover.jpg

The flow-equiaffine curvature of curves in a geometric surface Submitted: February 19, 2025, Revised: November 25, 2025

Mircea Crasmareanu[1]

[1] *Faculty of Mathematics, University "Al. I. Cuza", 700506, Iaşi, Romania*
http://www.math.uaic.ro/∼mcrasm , ORCID ID: 0000-0002-5230-2751

Abstract

Given a curve γ in an orientable two-dimensional Riemannian manifold, we introduce and investigate a deformation of the equiaffine curvature function. Due to the similarity of this construction with that performed for the geodesic curvature of γ in [6] we call flow-equiaffine curvature this new tool. The Euclidean and hyperbolic plane geometries are considered as examples.

Keywords: *two-dimensional Riemannian manifold; equiaffine curvature; flow-equiaffine curvature.*
2010 Mathematics Subject Classification: *53A04; 53A45; 53A55.*

1. Introduction

The theory of geometric flows is a recent and fascinating topic of study in geometric analysis. The simplest of these is the *curve shortening flow* and the survey [1] has been around for more than 20 years. Remember that the main geometric element in this last flow is the well-known curvature of planar curves. It appears important to search for variations of the curvature, or in the terminology of [10], for *deformations of the typical curvature*, in order to restart this challenge.

This brief note's proposal is to introduce such a deformation into generic orientable two-dimensional Riemannian geometry rather than plane Euclidean geometry. To be more specific, our starting framework is a triple (M^2, g, γ) made up of a smooth curve γ on the orientable Riemannian surface (M^2, g), which we refer to as a geometric surface. By rotating the Frenet frame of the standard Frenet theory for γ at an angle exactly determined by the parameter t of γ, we distort it and this new frame was studied in [4] under the name of *flow-frame*; it produces a new curvature function k^f called *flow-curvature*, which is always lower than the usual (geodesic) curvature of γ. We point out that the Euclidean variant of the flow-curvature is the contents of the paper [3], the hyperbolic geometry is treated in [5] in the hyperboloid model while the spherical variant is covered in the paper [6]. We have developed also the Lorentzian counterpart for the spacelike curves in [2].

Here we add to our setting the symplectic 2-form Ω generated by the metric g and the complex endomorphism J naturally associated to the given orientation of M. The symplectic geometry of the pair (M, Ω) produces a new curvature for γ, called *equiaffine curvature* and denoted k_e, which is already a classical topic, see [9]; see also the very interesting paper [12]. Hence, we mimic the deformation "(geodesic) curvature k to flow-curvature k^f" for the equiaffine curvature k_e obtaining *the flow-equiaffine curvature* k_e^f which is our new object to study. It is worth noting that the present computations are more harder than the ones of [4] since if in the Riemannian case we use the arc length parameter, the symplectic geometry imposes another parameter, called *equiaffine* and involving the integral of $k^{\frac{1}{3}}$.

The contents of this work is as follows. The following section presents this new geometrical tool and an objective is to find/study curves with a vanishing k_e^f; we call these as *flow-equiaffine geodesics* of the triple (M, g, Ω). In fact, the differential equation (with k as unkonwn) providing these curves is a very complicated one, and apart of the constant solution $k \equiv 1$ we obtain only a first integral for it. The second section contains concrete examples for the Euclidean and hyperbolic geometries.

2. The flow-curvature and the flow-equiaffine curvature

The setting of this paper is a *geometric surface* i.e. ([11, p. 322]) a smooth, orientable two-dimensional Riemannian manifold (M^2, g). Due to its orientability M carries an almost complex structure J; in fact J is integrable and for an arbitrary point $p \in M$ we consider $J_p : T_pM \to T_pM$ as being the multiplication with the complex unit $i \in \mathbb{C}$. Let ∇ be the Levi-Civita connection of g.

***Email addresses*:** mcrasm@uiac.ro

Fix also a smooth curve $\gamma: I \subseteq \mathbb{R} \to M$ which we suppose to be *regular*: $\gamma'(t) \in T_{\gamma(t)}M \setminus \{0\}$; let $\bar{s}$ be the arc-length parameter i.e. $\|\gamma'(\bar{s})\| = 1$ for $\bar{s} \in [0, L]$. Let $\mathfrak{X}(\gamma)$ be the $C^\infty(I)$-module of vector fields along γ i.e. smooth maps $X : I \to TM$ with $X(t) \in T_{\gamma(t)}M$ for all $t \in I$. It follows the *unit tangent vector field* $T \in \mathfrak{X}(\gamma)$ with:

$$T(t) := \frac{\gamma'(t)}{\|\gamma'(t)\|}. \tag{2.1}$$

Therefore, the *Frenet frame* of γ is $\mathscr{F} := \begin{pmatrix} T \\ N := J(T) \end{pmatrix} \in \mathfrak{X}(\gamma) \times \mathfrak{X}(\gamma)$.

The Riemannian geometry of γ is described by its *geodesic curvature* $k : I \to \mathbb{R}$ provided by the *Frenet equations*:

$$\nabla_{T(t)}\mathscr{F}(t) = \begin{pmatrix} 0 & k(t) \\ -k(t) & 0 \end{pmatrix} \mathscr{F}(t) = k(t) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathscr{F}(t) \tag{2.2}$$

which means:

$$k(t) := \frac{g(\nabla_{\gamma(t)}T(t), N(t))}{\|\gamma'(t)\|} = \frac{g(\nabla_{\gamma(t)}\gamma'(t), J(\gamma'(t)))}{\|\gamma'(t)\|^3}. \tag{2.3}$$

In the paper [4] we introduce a new frame, called *flow-frame*, as the rotation of the Frenet frame with exactly the angle t. The moving equations of this new frame yield a new tool, called *the flow-curvature* of γ:

$$k^f(t) := k(t) - \frac{1}{\|\gamma'(t)\|} = \frac{g(\nabla_{\gamma'(t)}T(t) - N(t), N(t))}{\|\gamma'(t)\|} < k(t). \tag{2.4}$$

For the example of the Euclidean plane, the circles and the logarithmic spirals are the flow-geodesics i.e. $k^f \equiv 0$.

Recall also the pair (g, J) yields *the symplectic form* $\Omega(\cdot,\cdot) := -g(\cdot, J\cdot) = g(J\cdot,\cdot)$ and whence:

$$k(t) := \frac{\Omega(\gamma'(t), \nabla_{\gamma'(t)}\gamma'(t))}{\|\gamma'(t)\|^3}. \tag{2.5}$$

For example, if γ is a *J-planar curve* i.e. there exist the smooth functions $\alpha, \beta \in C^\infty(M)$ such that $\nabla_\gamma \gamma' = \alpha\gamma' + \beta J(\gamma')$ then $k(t) = \frac{\beta(t)}{\|\gamma'(t)\|}$.

This symplectic form gives another geometry on the underlying manifold M, namely *the equiaffine geometry*, having the 5-dimensional Lie group $SA(2) := SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ as the structural group. So, for our fixed curve γ there exist:
i) a specific condition of being *nondegenerate*: the tangent vectors $\gamma'(t), \nabla_{\gamma'(t)}\gamma'(t)$ are linear independent for each $t \in I$; a consequence is that γ is not a geodesic of g i.e. k is not zero,
ii) an *equiaffine parameter* s such that:

$$\Omega(\bar{e}_1(s) := \gamma'(s), \bar{e}_2(s) := \nabla_{\gamma'(s)}\gamma'(s)) = 1. \tag{2.6}$$

The formula providing this new parameter is:

$$s(t) = \int_{t_0}^{t} [\Omega(\gamma'(u), \nabla_{\gamma'(u)}\gamma'(u))]^{\frac{1}{3}} du. \tag{2.7}$$

The motivation of this choice is the invariance of the 1-form $u \to [\Omega(\gamma'(u), \nabla_{\gamma'(u)}\gamma'(u))]^{\frac{1}{3}} du \in \mathfrak{X}(\gamma)^*$ with respect to C^2-reparametrizations of curves and $SA(2)$-changes of local coordinates on the manifold M.

The adapted frame $\mathscr{F}_e$ is *unimodular*:

$$\mathscr{F}_e := \begin{pmatrix} \bar{e}_1 \\ \bar{e}_2 \end{pmatrix} \in \mathfrak{X}(\gamma) \times \mathfrak{X}(\gamma), \quad \mathscr{F}_e(s) \in SL(2,\mathbb{R}), \quad \forall s \in \tilde{I}. \tag{2.8}$$

The tangent vectors $\bar{e}_1(s), \bar{e}_2(s) \in T_{\gamma(s)}M$ are called *the equiaffine tangent* and *equiaffine normal* to γ at s, respectively. The moving equations of the equiaffine frame are:

$$\nabla_{\gamma'(s)}\mathscr{F}_e(s) = \begin{pmatrix} 0 & 1 \\ -k_e(s) & 0 \end{pmatrix} \mathscr{F}_e(s) \tag{2.9}$$

and hence we have *the equiaffine curvature* of (parametrized by $\bar{s}$!) γ, [9]:

$$k_e(s) = \frac{1}{9k(\bar{s}(s))^{\frac{8}{3}}} \left[3k(\bar{s}(s))k''(\bar{s}(s)) - 5(k'(\bar{s}(s)))^2 + 9(k(\bar{s}(s)))^4\right] \tag{2.10}$$

where $\bar{s}(s)$ is the inverse function of the equiaffine arc function (2.7) equivalently:

$$s(\bar{s}) = \int_0^{\bar{s}} [k(u)]^{\frac{1}{3}} du. \tag{2.11}$$

which gives the elements of the equiaffine frame:

$$\bar{e}_1(s) = [k(\bar{s}(s))]^{-\frac{1}{3}} T(\bar{s}(s)), \quad \bar{e}_2(s) = -\frac{1}{3}[k(\bar{s}(s))]^{-\frac{5}{3}} k'(\bar{s}(s))T(\bar{s}(s)) + [k(\bar{s}(s))]^{\frac{1}{3}} N(\bar{s}(s)). \tag{2.12}$$

For the Euclidean plane geometry the conics are the curves with constant equiaffine curvature: i) if γ is an ellipse then $k_e > 0$; ii) if γ is a hyperbola then $k_e < 0$; iii) the parabolas are equiaffine-geodesics i.e. $k_e \equiv 0$.

This short note defines a new curvature function for γ having as model the formula (1.4) of the flow-curvature in which t is replace by s:

Definition 2.1 The *flow-equiaffine curvature* of γ is the smooth function $k_e^f : \bar{I} \to \mathbb{R}$ given by the equiaffine-variant of (1.4):

$$k_e^f(s) := k_e(s) - \frac{1}{\frac{d\bar{s}}{ds}(s)} = k_e(s) - \frac{ds}{d\bar{s}}(s) = k_e(s) - [k(\bar{s}(s))]^{\frac{1}{3}}. \tag{2.13}$$

A curve with $k_e^f \equiv 0$ will be called *flow-equiaffine geodesic*. If the flow-equiaffine curvature has a constant sign then the *flow-equiaffine evolute of* γ is the curve $Ev_\gamma^f : \bar{I} \to \mathbb{R}^2$ given by:

$$Ev_\gamma^f(s) := \gamma(\bar{s}(s)) + \frac{1}{k_e^f(s)}\bar{e}_2(s). \tag{2.14}$$

Remark 1.2 i) Now, the data (M, g, Ω, γ) has four curvatures: k, k^f, k_e, k_e^f and hence an interesting future subject can be the possible cases of relationships between these functions, apart of (2.4) and (2.13).
ii) We can connect our new curvature with a notion introduced by M. Gromov in [7, p. 213] and concerning with hypersurfaces M^n in a weighted Riemannian manifold $(\tilde{M}^{n+1}, g, f \in C_+^\infty(\tilde{M}))$. More precisely, the *weighted mean curvature* of M is the difference:

$$H^f := H - g(\tilde{N}, \tilde{\nabla} f) \tag{2.14}$$

where H is the usual mean curvature of M and $\tilde{N}$ is the unit normal to M. We get directly:

$$k_e^f(s) = k_e(s) - g(N(\bar{s}(s)), \bar{e}_2(s)). \tag{2.16}$$

□

In the following we restrict our study to the flow-equiaffine geodesics. It follows directly that the equiaffine curvature of a flow-equiaffine geodesic satisfies the identity $k_e = (k \circ \bar{s})^{\frac{1}{3}}$ which means the nonlinear second-order differential equation (SODE):

$$k''(\bar{s}) - \frac{5}{3k(\bar{s})}\left(k'(\bar{s})\right)^2 + 3k^2(\bar{s})(k(\bar{s}) - 1) = 0, \quad k(\bar{s}) > 0 \tag{2.17}$$

admitting the constant solution $k \equiv 1 \equiv k_e$; from (1.11) it results that then $\bar{s}$ is also the equiaffine parameter of γ which in turn, means that the frames $\mathscr{F}$, $\mathscr{F}_e$ coincide.

For searching non-constant solutions, performing the substitution $k' = v = v(k)$ with $\dot{v} := \frac{dv}{dk}$ we derive the differential equation of Bernoulli type:

$$2v\dot{v} - \frac{10}{3k}v^2 + 6k^2(k-1) = 0. \tag{2.18}$$

A new substitution $w = v^2$ yields the linear differential equation:

$$\dot{w} - \frac{10}{3k}w + 6k^2(k-1) = 0 \tag{2.19}$$

admitting the integrating factor $\rho = k^{-\frac{10}{3}}$ and leading finally to:

$$v(k) = \pm[Ck^{\frac{10}{3}} - 9k^3(k+2)]^{\frac{1}{2}}, \quad C \in \mathbb{R}. \tag{2.20}$$

It results a first integral of the initial SODE (2.17):

$$\int \frac{dk}{\sqrt{Ck^{\frac{10}{3}} - 9k^3(k+2)}} = D \pm \bar{s}, \quad D \in \mathbb{R}. \tag{2.21}$$

In order to deal with the fractional power $\frac{10}{3}$ we consider a new smooth function K provided by the substitution $k = K^3$; from (2.11) it results that the equiaffine parameter function s is the anti-derivative of K. Returning to (1.20) it follows a relation involving an elliptic integral:

$$\int \frac{dK}{K^2\sqrt{CK^2 - 9K(K^3+2)}} = \frac{1}{3}(D \pm \bar{s}). \tag{2.22}$$

We finish this section with the remark that the SODE (2.17) falls in the class of SODEs providing a *geometry of paths* a la Cartan, for which we refer to the chapter 8 of the book [13]. More precisely, let $M^e(g, \Omega)$ be the set of all pairs $(\bar{s}, k)$ which we suppose to be a smooth manifold. Then, we can apply the Theorem 3.8(iii) of the cited book and it results the existence and uniqueness of a normal projective connection on $M^e(g, \Omega)$ such that the set of its geodesics is exactly the set of all solutions of (1.16). We can call it *the flow-equiaffine normal projective connection* of (M, g, Ω).

3. Examples

Example 3.1 The Euclidean geometry $\mathbb{E}^2 := (\mathbb{R}^2, g_{can} = dx^2 + dy^2)$ has the symplectic geometry given by the area form:

$$\Omega(X,Y) := \det[X,Y], \quad X,Y \in \mathbb{R}^2 \tag{3.1}$$

where X, Y are thought as columns matrices i.e. $X,Y \in M_{2,1}(\mathbb{R})$. It follows that our special parameters $\bar{s}$, s are chosen so that the closest rotation from T to N, respectively from $\bar{e}_1$ to $\bar{e}_2$, to be counter-clockwise.

As curve let the circle $\mathscr{C}(O,R>0)$ with:

$$\gamma(\bar{s}) = R\left(\cos\frac{\bar{s}}{R}, \sin\frac{\bar{s}}{R}\right), \quad \bar{s} \in [0, L = 2\pi R]. \tag{3.2}$$

From $\bar{s}(s) = R^{\frac{1}{3}} s$ it results that $s \in [0, 2\pi R^{\frac{2}{3}}]$ and:

$$\gamma(s) = R\left(\cos\frac{s}{R^{\frac{2}{3}}}, \sin\frac{s}{R^{\frac{2}{3}}}\right) \tag{3.3}$$

implying:

$$\begin{cases} \bar{e}_1(s) = \gamma'(s) = R^{\frac{1}{3}}\left(-\sin\frac{s}{R^{\frac{2}{3}}}, \cos\frac{s}{R^{\frac{2}{3}}}\right), \bar{e}_2(s) = \gamma''(s) = R^{-\frac{1}{3}}\left(-\cos\frac{s}{R^{\frac{2}{3}}}, -\sin\frac{s}{R^{\frac{2}{3}}}\right), \\ k_e \equiv \frac{1}{R^{\frac{4}{3}}}, \quad k_e^f \equiv \frac{1}{R^{\frac{4}{3}}} - \frac{1}{R^{\frac{1}{3}}} \end{cases} \tag{3.4}$$

and hence the unit circle S^1 is the unique flow-equiaffine geodesic of circle type of $\mathbb{E}^2$; if $R \in (0,1)$ then $k_e^f > 0$ while $R > 1$ gives $k_e^f < 0$. Also, we can compute the flow-equiaffine evolute of the any circle with $R \neq 1$:

$$Ev^f_{\mathscr{C}(O,R)}(s) = \frac{R^2}{R-1}\left(\cos\frac{s}{R^{\frac{2}{3}}}, \sin\frac{s}{R^{\frac{2}{3}}}\right) \tag{3.5}$$

and then, this curve is different to the initial circle.

Recall that in $SL(2,\mathbb{R})$ we have the unique diagonal-unipotent-orthogonal decomposition:

$$SL(2,\mathbb{R}) \ni \Gamma := \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}\begin{pmatrix} 1 & \rho \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{3.6}$$

For the equiaffine frame of the circle it follows: $a(s) := R^{\frac{1}{3}}$=constant> 0, $\rho(s) = 0$=constant and $\theta(s) := \frac{3\pi}{2} - \frac{s}{R^{\frac{2}{3}}} \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$.

The circle example is useful to illustrate the *affine isoparametric inequality* for a curve γ which is closed, strictly convex and positively oriented: let $\mathscr{A}(\gamma)$ be the enclosed aria and *the equiaffine length* $L_e > 0$ to be defined by $s \in [0, L_e(\gamma)]$. Then:

$$(L_e(\gamma))^3 \leq 8\pi^2 \mathscr{A}(\gamma) \tag{3.7}$$

with equality if and only if γ is an ellipse. For our circle we have:

$$\mathscr{A}(R) = \pi R^2, \quad L_e(R) = \frac{2\pi R}{R^{\frac{1}{3}}} = 2\pi R^{\frac{2}{3}} \tag{3.8}$$

and the equality holds in (3.7).

Let now the constant $\omega > 0$ and the curve be the ω-*clothoid*:

$$\gamma(\bar{s}) = \left(\int_0^{\bar{s}} \cos\frac{\omega u^2}{2} du, \int_0^{\bar{s}} \sin\frac{\omega u^2}{2} du\right), \quad \bar{s} \in (0, +\infty) \tag{3.9}$$

having the linear curvature $k(\bar{s}) = \omega\bar{s}$ and then:

$$\bar{s}(s) = \left(\frac{4}{3\omega^{\frac{1}{3}}} s\right)^{\frac{3}{4}}, \quad k_e(s) = \frac{9\omega^2\bar{s}(s)^4 - 5}{9\omega^{\frac{2}{3}}\bar{s}(s)^{\frac{8}{3}}}, \quad k_e^f(s) = \frac{9\omega^2\bar{s}(s)^4 - 5}{9\omega^{\frac{2}{3}}\bar{s}(s)^{\frac{8}{3}}} - [\omega\bar{s}(s)]^{\frac{1}{3}}. \tag{3.10}$$

□

Example 3.2 Fix the Poincaré upper half plane model of the hyperbolic plane geometry $\mathbb{H}^2 := \left(\mathbb{C}_+ = \{z \in \mathbb{C}; Im\, z > 0\}, ds = \frac{|dz|}{Im\, z}\right)$ and the Euclidean circle $\mathscr{C}(M_0(x_0,y_0), R>0)$: $(x-x_0)^2 + (y-y_0)^2 = R^2$. Hence $y_0 > R > 0$ and x_0 can be zero since the horizontal translations are hyperbolic isometries. We use the trigonometric parametrization: $\gamma(t) = (0,y_0) + Re^{it}$ and then the curvature is constant:

$$k \equiv \frac{y_0}{R} > 1. \tag{3.11}$$

With $R \to y_0$ we obtain that the circles tangent to the real axis have $k \equiv 1$ and then there are horocycles of $\mathbb{H}^2$. In conclusion the unit speed horocycles are flow-equiaffine geodesics of $\mathbb{H}^2$; as example we have the horizontal line: $\gamma(\bar{s}) = \bar{s} + i = (\bar{s}, 1) \in \mathbb{H}^2$, which is a horocycle centered at ∞.

For $k \equiv \frac{y_0}{R} > 1$ the canonical parameter is given by the function:

$$\bar{s}(t) = \frac{2}{\sqrt{k^2-1}} \arctan\left(\frac{1 + k\tan\frac{t}{2}}{\sqrt{k^2-1}}\right). \tag{3.12}$$

The equidistant curves from the vertical semi-axis of $\mathbb{H}^2$ are the Euclidean lines $\gamma_\theta : y = cx$ with $c = \cot\theta > 0$ and having the constant curvature: $k \equiv \sin\theta$. Starting with the parametrization $\gamma_\theta(t) = (t, ct)$ we find the new parameters and curvatures:

$$\bar{s} = \frac{\sqrt{c^2+1}}{c}\ln t, \quad s = (\sin\theta)^{\frac{1}{3}}\bar{s} \quad k_e \equiv (\sin\theta)^{\frac{4}{3}}, \quad k_e^f \equiv (\sin\theta)^{\frac{4}{3}} - (\sin\theta)^{\frac{1}{3}} < 0. \tag{3.13}$$

□

Example 3.3 It is easy to generalize the first part of the previous example to the class of diagonal Riemannian metrics:

$$g = A^2(x,y)dx^2 + B^2(x,y)dy^2 \tag{3.14}$$

when the curve is "the meridian" $\gamma : y = constant = y_0$. Then:

$$\|\gamma'(t)\| = A(t, y_0), \quad k(t) = -\frac{A_y}{AB}(t, y_0) \tag{3.15}$$

which yields that we have $k \equiv 1$ for:

$$A(x,y) = e^{-\int_{y_i}^{y} B(x,u)du} \tag{3.16}$$

where y_i means an initial value of y; we must add the condition $A(x, y_0) = 1$ and then we consider $y_i = y_0$. The (Gaussian) curvature of this Riemannian metric is:

$$\mathscr{K}(x,y) = -1 - \frac{1}{AB}\left(\frac{B_x}{A}\right)_x \tag{3.17}$$

and then, if $\frac{B_x}{A}$ does not depend on x then g is a pseudospherical metric.

We connect this class of metrics with our study in the following way: in the very old paper [8] it is proved that if $A = B = \sqrt{y} > 0$ (in the domain $y > 0$; the curvature is $\mathscr{K}(x,y) = \frac{1}{2y^3} > 0$) then the geodesics of the metric are parabolas. Our flow-equiaffine geodesics are the *flow*-generalization of parabolas and hence it is a natural problem if a generalization of the previous equality yields remarkable curves from the flow-equiaffine point of view. □

Article Information

Acknowledgements: The author would like to express his sincere thanks to the Editor-in-Chief and to two anonymous reviewers for their helpful comments and suggestions.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Copyright Statement: Author own the copyright of his work published in the journal and their work is published under the CC BY-NC 4.0 license.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Funding This research received no external funding.

References

[1] Chou Kai-Seng; Zhu Xi-Ping, *The curve shortening problem*, Boca Raton, FL: Chapman & Hall/CRC, 2001. Zbl 1061.53045
[2] Crasmareanu Mircea, *The flow-curvature of spacelike parametrized curves in the Lorentz plane*, Proceedings of the International Geometry Center, 15(2022), no. 2, 100-108. Zbl 1523.53024
[3] Crasmareanu Mircea, *The flow-curvature of plane parametrized curves*, Commun. Fac. Sci. Univ. Ankara Ser. A1 Math. Stat., 72(2023), no. 2, 417-428. MR 4615669
[4] Crasmareanu Mircea, *The flow-curvature of curves in a geometric surface*, Commun. Korean Math. Soc., 38(2023), no. 4, 1261-1269. Zbl 1528.53003
[5] Crasmareanu Mircea, *The flow-geodesic curvature and the flow-evolute of hyperbolic plane curves*, Int. Electron. J. Geom. (IEJG), 16(2023), no. 1, 225-231. Zbl 1516.30056
[6] Crasmareanu Mircea, *The flow-geodesic curvature and the flow-evolute of spherical curves*, Turkish J. Math., 48(2024), no. 4, 749-756. Zbl 1541.53006
[7] Gromov Mikhael, *Isoperimetry of waists and concentration of maps*, Geom. Funct. Anal., 13(2003), no. 1, 178-215; erratum ibid. 18(2008), no. 5, 1786-1786. Zbl 1044.46057
[8] Kasner E., *Surfaces whose geodesics may be represented in the plane by parabolas*, Trans. Am. Math. Soc., 6(1905), 141-158.
[9] Olszak Karina; Olszak Zbigniew, *The equi-affine and Frenet curvatures of curves in pseudo-Riemannian 2-manifolds*, Colloq. Math., 150(2017), no. 1, 103-112 (2017). Zbl 1410.53062
[10] Mazur Barry, *Perturbations, deformations, and variations (and "near-misses") in geometry, physics, and number theory*, Bull. Am. Math. Soc., 41(2004), no. 3, 307-336. Zbl 1057.11033
[11] O'Neill Barrett, *Elementary differential geometry. Revision of the 2nd ed. 1997*, Amsterdam: Elsevier/Academic Press, 2006. Zbl 1208.53003
[12] Rovenski V., *The curve shortening flow in the metric-affine plane*, Mathematics, 8(2020), no. 5, 701.
[13] Sharpe R. W., *Differential geometry: Cartan's generalization of Klein's Erlangen program*, Graduate Texts in Mathematics, 166, Berlin: Springer, 1997. Zbl 0876.53001