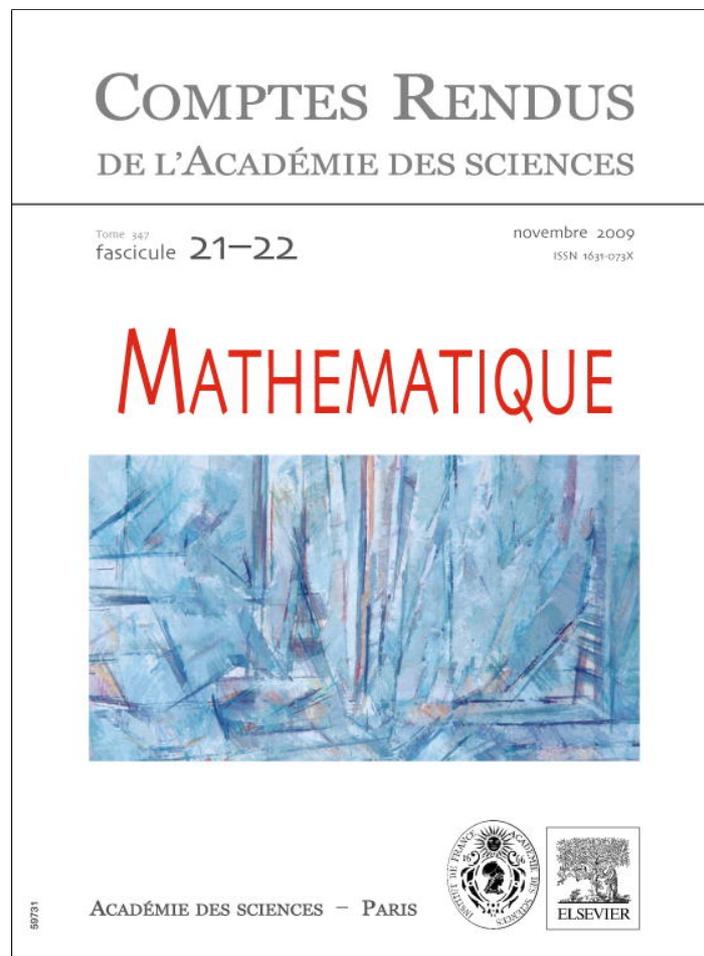


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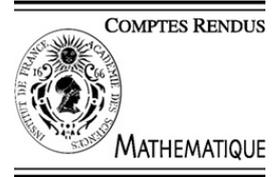
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Differential Geometry/Dynamical Systems

# Liouville and geodesic Ricci solitons

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## Abstract

On a tangent bundle endowed with a pseudo-Riemannian metric of complete lift type two classes of Ricci solitons are obtained: a 1-parameter family of shrinking Liouville Ricci solitons if the base manifold is Ricci flat and a steady geodesic Ricci soliton if the base manifold is flat. A nonexistence result of geodesic Ricci solitons for the tangent bundle of a non-flat space form is also provided. **To cite this article:** *M. Crasmareanu, C. R. Acad. Sci. Paris, Ser. I 347 (2009).*

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## Résumé

**Solitons de Ricci de type Liouville et de type géodésique.** Pour le fibré tangent d'une variété équipée d'une métrique pseudo-Riemannienne ayant un relèvement complet, deux classes de solitons de Ricci sont décrits : une famille à 1 paramètre de solitons de Ricci de type Liouville contractants si la variété de base est Ricci plate, et un soliton de Ricci de type géodésique nul si celle-ci est plate. Un résultat de non-existence de solitons de Ricci géodésiques est également obtenu dans le cas du fibré tangent d'une variété non plate. **Pour citer cet article :** *M. Crasmareanu, C. R. Acad. Sci. Paris, Ser. I 347 (2009).*

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## 1. Introduction

Fix  $M$  a smooth  $n (> 1)$ -dimensional manifold. A *Ricci soliton* on  $M$  is a triple  $(g, V, \lambda)$  consisting in a Riemannian metric, a vector field and a real scalar satisfying the Ricci equation:

$$L_V g + 2S_g + 2\lambda g = 0$$

where  $S_g$  is the associated Ricci tensor of  $g$  and  $L_V$  is the Lie derivative with respect to  $V$ . The Ricci soliton is said to be *shrinking*, *steady* or *expanding* according as  $\lambda$  is negative, zero or positive respectively. With  $V$  a Killing vector field it results that Ricci solitons are generalizations of Einstein metrics; also for Ricci flat metrics ( $S_g = 0$ ) it results that  $V$  is an homothetic Killing vector field. Compact Ricci solitons are fixed points of the *Ricci flow*, a very effective tool for studying the topology of manifolds [3].

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The aim of this Note is to obtain Ricci solitons in tangent bundles of Riemannian manifolds having as vector field exactly:

- (1) the Liouville vector field  $\Gamma$  which is a global vector field on any tangent bundle, independent of any metric on the base manifold,
- (2) the geodesic spray  $\Gamma_g$  which depends on the Riemannian metric of the base. Sometimes,  $\Gamma_g$  is called the *transversal Liouville vector field* [1, p. 231], or the *horizontal Liouville vector field* [5, p. 272], but we prefer this geometrical name.

Therefore we call *Liouville* and *geodesic* respectively, these types of Ricci solitons and let us remark that these classes are disjoint since  $\Gamma$  is a vertical vector field on  $TM$  while  $\Gamma_g$  belongs to a complementary distribution, called horizontal. We consider tangent bundles since already several studies were dedicated to the compact case.

On the tangent bundle of a Riemannian manifold there are lots of very interesting metrics [4,10], but we restrict to the pseudo-Riemannian metric of [6] and [7]; see also [8] and [9]. A strong motivation of this choice is the fact that the paper [6] contains a computation of the Ricci tensor of this metric, useful for our study. Another argument is that the present paper is dedicated to the memory of N. Papaghiuc, 1947–2008.

## 2. The tangent bundle with a complete lift

Fix a Riemannian metric  $g$  on the manifold  $M$ . A local system of coordinates  $(x) = (x^i) = (x^1, \dots, x^n)$  on  $M$  yields a system of coordinates  $(x, y) = (x^i, y^j)$  on the tangent bundle  $TM$ . The Levi-Civita connection  $(\Gamma_{jk}^i)$  of  $g$  defines a splitting  $T(TM) = VTM \oplus HTM$  into vertical and horizontal vectors respectively. The integrable distribution  $VTM$  has the basis  $(\frac{\partial}{\partial y^i})$  while the horizontal distribution  $HTM$  is spanned by  $(\frac{\delta}{\delta x^i})$  where  $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma_{i0}^j \frac{\partial}{\partial y^j}$ ,  $\Gamma_{i0}^j = \Gamma_{ih}^j y^h$ . Throughout the paper the transvecting with  $y^i$  will be denoted by a zero.

Consider now the kinetic energy  $t(x, y) = \frac{1}{2} \|y\|^2 = \frac{1}{2} g_{ij}(x) y^i y^j$  and also the smooth functions  $u, v : [0, \infty) \rightarrow \mathbb{R}$  such that [6, p. 229]  $u(t) > 0$  and  $u(t) + 2tv(t) > 0$  for every  $t$ . The above conditions assure that the symmetric  $(0, 2)$ -type tensor field of  $TM$ ,  $G_{ij} = u(t)g_{ij} + v(t)g_{0i}g_{0j}$  is positive definite. Then, the pseudo-Riemannian metric defined in [6, p. 229] is:

$$G\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 0, \quad G\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = G\left(\frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^i}\right) = G_{ij}. \tag{1}$$

The tangent bundle carries a remarkable global vector field  $\Gamma$ , called Liouville, which is independent of any Riemannian metric on the base manifold, namely,  $\Gamma = y^i \frac{\partial}{\partial y^i}$ . Also, the Riemannian metric  $g$  yields the geodesic spray  $\Gamma_g = y^a \frac{\delta}{\delta x^a}$ . Both  $\Gamma$  and  $\Gamma_g$  are null vector fields for  $G$ . For the Lie derivative from (Ricci) we need the Lie brackets of these vector fields with the local frame fields  $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$ :

$$\begin{aligned} \left[\Gamma, \frac{\delta}{\delta x^i}\right] &= 0, & \left[\Gamma, \frac{\partial}{\partial y^i}\right] &= -\frac{\partial}{\partial y^i}, \\ \left[\Gamma_g, \frac{\delta}{\delta x^i}\right] &= \Gamma_{i0}^j \frac{\delta}{\delta x^j} + R_{0i0}^b \frac{\partial}{\partial y^b}, & \left[\Gamma_g, \frac{\partial}{\partial y^i}\right] &= -\frac{\delta}{\delta x^i} + \Gamma_{i0}^b \frac{\partial}{\partial y^b}. \end{aligned} \tag{2}$$

Also  $\Gamma(G_{ij}) = 2t(u'g_{ij} + v'g_{i0}g_{j0}) + 2vg_{i0}g_{j0}$  and  $\Gamma_g(G_{ij}) = \Gamma_{i0}^b G_{bj} + \Gamma_{j0}^b G_{ib}$ .

Beginning with the Liouville vector field we have three cases:

- (I) (Ricci) on  $(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})$  yields  $S_G(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = 0$ ,
- (II) (Ricci) on  $(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})$  yields  $S_G(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = 0$ .

The Ricci tensor  $S_G$  of this metric is computed at the page 231 of [6] and exactly the above components are the only non-vanishing. Therefore the pair  $(G, \Gamma)$  belongs to a Ricci soliton if and only if  $G$  is Ricci flat which, according to Theorem 2 of the cited paper, is equivalent with the fact that  $(M, g)$  is Ricci flat and  $v = u'$ .

The last case, (III), (Ricci) on  $(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j})$  yields  $\Gamma(G_{ij}) + (2\lambda + 1)G_{ij} = 0$  which gives  $2t(u'g_{ij} + u''g_{i0}g_{j0}) + 2u'g_{i0}g_{j0} = -(2\lambda + 1)(ug_{ij} + u'g_{i0}g_{j0})$ . Applying Lemma 1 of [5] we derive the same equation  $2tu' = -(2\lambda + 1)u$

with the solution  $u(t) = t^{-(\lambda+\frac{1}{2})}$  which satisfies the condition  $u(t) > 0$ . The condition  $u + 2tu' > 0$  is equivalent with  $\lambda < 0$ .

**Theorem 2.1.** *If the Riemannian manifold  $(M, g)$  is Ricci flat then the tangent bundle carries a 1-parametric family of shrinking Ricci solitons  $(G_\lambda, \Gamma, \lambda)$  for  $\lambda < 0$  where  $G_\lambda(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = G_\lambda(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = 0$  and:*

$$G_\lambda\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = G_\lambda\left(\frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^i}\right) = t^{-(\lambda+\frac{3}{2})}\left[tg_{ij} - \left(\lambda + \frac{1}{2}\right)g_{i0}g_{j0}\right]. \tag{3}$$

$(TM, G_\lambda)$  is also Ricci flat.

Several and very interesting examples of Ricci flat metrics are given in [2]. From Proposition 1 of the cited paper we obtain that for the pseudo-Riemannian metric  $G_\lambda$  the vector fields  $(\frac{\delta}{\delta x^i})$  are covariant constant (parallel) with respect to vector fields  $(\frac{\partial}{\partial y^i})$  and  $\nabla_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial y^j} = \Gamma_{ij}^h \frac{\partial}{\partial y^h}$ .

For the geodesic vector field the same three cases occur:

- (I) (Ricci) on  $(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})$  yields:  $G_{ij} + S_G(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = 0$ .
- (II) (Ricci) on  $(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})$  gives:  $2S_G(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = R_{0i0}^b G_{bj} + R_{0j0}^b G_{bi}$ .
- (III) (Ricci) on  $(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j})$  yields:  $\Gamma_g(G_{ij}) + 2\lambda G_{ij} - \Gamma_{i0}^b G_{bj} - \Gamma_{j0}^b G_{ib} = 0$ .

This equation reduces to  $\lambda G_{ij} = 0$  which gives  $\lambda = 0$ .

Using the expression of the Ricci tensor from [6, p. 231] the first equation means:

$$u - \frac{(n-1)(u'-v)}{u+2tv} = 0, \quad v + \frac{(1-n)\alpha}{2u^2(u+2tv)} = 0 \tag{4}$$

with  $\alpha$  from the cited paper while the second equation reduces to  $2R_{ij} = uR_{0i0j} + v(R_{0j00}g_{i0} + R_{0i00}g_{j0})$ .

In the following we assume that the base metric  $g$  has constant sectional curvature  $c$ . With  $R_{ij} = c(n-1)g_{ij}$ ,  $R_{0i0j} = c(2tg_{ij} - g_{i0}g_{j0})$  we get  $c[2(tu - n + 1)g_{ij} - ug_{i0}g_{j0}] = 0$ .

**Theorem 2.2.** *Let  $(M, g)$  be a flat manifold and  $u, v$  smooth functions satisfying the differential system (4). Then, on that tangent bundle  $(TM, G)$  we have a steady geodesic Ricci soliton.*

*If  $M_n(c)$  is a space form with  $n > 1$  and  $c \neq 0$  then on  $TM$  there are no geodesic Ricci solitons having  $G$  as element.*

The system (4) is equivalent with:

$$2u(u+2tv)u'' - 3u(u')^2 - 2u^2v' + 5uv^2 - 2uu'v - 4tuu'v' - 2t(u')^2v + 2tv^3 = 0$$

which is divisible by  $u' - v$ ; But  $v = u'$  is not a solution of our system since from (4<sub>1</sub>) it results  $u = 0$ . From Corollary 8 of [9, p. 284] it results that, although  $(M, g)$  is flat, the tangent bundle  $(TM, G)$  is not.

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