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NEW CR-STRUCTURES ON THE UNIT TANGENT BUNDLE

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Abstract. One obtains a $CR-$ structure on the unit tangent bundle over a real space form in the case where the tangent bundle is endowed with a special Riemannian metric and an adapted integrable almost complex structure. The Bochner type tensor of this CR-structure vanishes in the case where the constant curvature is negative.

1. Introduction

Let (M, g) be a Riemannian manifold and (TM, G) its tangent bundle endowed with the metric G. Using the Levi Civita connection of g it can be defined on TM the usual almost complex structure J. If G is the Sasaki metric then (TM, J, G) is almost Kählerian. The restriction of J to T_1M, thought of as real hypersurface of TM, induces an almost $CR-$ structure. There are known some results concerning the complex involutivity of J, which is equivalent to have a $CR-$ structure on T_1M ([1] , [2] , [4] , [7]).

In this paper we have changed the metric G on the tangent bundle by using an $M-$ tensor field on TM depending on the energy density. It is more convenient to use an almost complex structure J on TM, related to the considered metric. In [6] it is obtained a Kähler Einstein structure on TM in the case where (M, g) has constant (negative) sectional curvature. The main statement of this article is to prove that we have a $CR-$ structure on the unit tangent bundle (with the new metric and new almost complex structure) if and only if the base manifold has constant sectional curvature c. Moreover, T_1M is a Bochner flat manifold if and only if c is negative.

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2. The Pseudoconvex CR-structures

Definition. *A $CR-$structure on the differentiable manifold M, is defined by a complex vector subbundle $H(M)$ in the complexification T^cM of the tangent bundle of M so that:*

(i) $A(M) \cap H(M) = \{0\}$ where $A(M) = \overline{H(M)}$.

(ii) $H(M)$ is involutive, i.e. for two $H(M)$-valued complex vector fields Z, W, the bracket $[Z, W]$ is $H(M)$-valued too.

Denoting by J the operator on the decomplexification $R(M)$ of $H(M)$, corresponding to the multiplication by i, the condition of complex involutivity can be expressed by:

(i) $[X, Y] - [JX, JY] \in \Gamma(R(M))$; $\forall X, Y \in \Gamma(R(M))$

(ii) $N_J(X, Y) = [JX, JY] - [X, Y] - J\{[JX, Y] + [X, JY]\} = 0$; $\forall X, Y \in \Gamma(R(M))$

where $\Gamma(R(M))$ denotes the set of cross-sections of $R(M)$.

Remark. Since (i) it follows, putting JY in the place of Y and using the property $J^2 = -I_{R(M)}$, that:

$$[JX, Y] + [X, JY] \in \Gamma(R(M)) \ ; \ \forall X, Y \in \Gamma(R(M))$$

thus the operations in the expression of N_J in (ii) are well defined.

The fundamental example of $CR-$structures is supplied by the real submanifolds of the complex manifolds.

Let $\widetilde{M}$ be a complex manifold of complex dimension m and let $M \subset \widetilde{M}$ be a real submanifold. The tangent space T_pM in a fixed point $p \in M$, is a real subspace in the tangent space $T_p\widetilde{M}$, which is a complex vector space. Then we may define:

$$H(T_pM) = \{X_p \in T_pM \mid iX_p \in T_pM\}$$

Remark. Generally, the dimension of $H(T_pM)$ varies simultaneously with $p \in M$.

Assuming that dim $(H(T_pM)) = k$, a constant number independent of $p \in M$, then $H(M) = \bigcup\limits_{p \in M} H(T_pM)$ is a complex vector bundle on M and it defines, naturally, a $CR-$structure on M. When M is a real hypersurface in a complex manifold, it is not necessary the assumption for the fibre dimension of $H(M)$ to be constant in order to have an induced $CR-$structure.

Let $(M, H(M))$ be a $CR-$structure of hypersurface type, on the $(2n+1)-$ dimensional, oriented manifold M and let η be a $1-$form having the decomplexification $R(M)$ of $H(M)$ as its null bundle. Due to the orientation of M, we can choose the 1-form η defined on the whole M.

Definition. *A hypersurface type CR-structure is pseudoconvex* if η is a contact form, i.e. $\eta \wedge (d\eta)^n \neq 0$.

Remark. It follows easily that, in the case of a pseudoconvex $CR-$structure of hypersurface type, the restriction of dη to $R(M)$ is nondegenerate.

It is known that we may introduce an almost contact structure subordinated to the pseudoconvex $CR-$ structure. $\xi \in \chi(M)$ and $\phi \in \mathcal{T}_1^1(M)$ are defined by:

$$\eta(\xi) = 1 \ , \ i_\xi d\eta = 0 \ , \ \eta \circ \phi = 0 \ , \ \phi\xi = 0 \ , \ \phi^2 = -I + \eta \otimes \xi. \qquad (1)$$

Considering $\phi X = J(X - \eta(X)\xi)$; $\forall X \in \chi(M)$ we have

$$TM \ = \ \text{span}\ [\xi] \oplus R(M).$$

In [3] the authors found a torsion free connection ∇ on M related in a natural way to the almost contact structure (ϕ, ξ, η), subordinated to the pseudoconvex $CR-$structure $(M, H(M))$. They proved the following

Theorem. *Given (ϕ, ξ, η) as above there exists a unique torsion free connection ∇ so that*

$$\begin{cases} (\nabla_X \eta)(Y) = \dfrac{1}{2} d\eta(X,Y) \ , \ \nabla_X d\eta = 0 \ , \ \nabla_X \xi = 0 \\ (\nabla_X \phi)Y = 2\eta(X)\psi Y - \dfrac{1}{2} d\eta(X, \phi Y)\xi \ \ , \ \forall X, Y \in \chi(M), \end{cases} \qquad (2)$$

where $\psi = \frac{1}{2}\mathcal{L}_\xi \phi$.

This connection is called the torsion free canonical connection associated to the almost contact structure (ϕ, ξ, η).

Remark. The almost contact structure (ϕ, ξ, η) subordinated to the pseudoconvex CR-structure $(M, H(M))$ is not unique, so the associated torsion free canonical connection is not unique.

Definition. *A change* $\eta \mapsto \eta' = \varepsilon e^f \eta$, $f \in C^\infty(M)$ *and* $\varepsilon = \pm 1$ *of the* $1-$*form* η *is called a gauge transformation.*

Remark. The complex involutivity is invariant with respect to gauge transformations.

From now on we shall take $\varepsilon = +1$. The case $\varepsilon = -1$ can be discussed in a similar way.

Given an almost contact structure (ϕ, ξ, η) subordinated to the pseudoconvex $CR-$structure $(M, H(M))$ and the associated torsion free connection ∇ on M, let R be the curvature tensor field of ∇, defined by

$$R_{XY}Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \ ; \ X, Y, Z \in \chi(M).$$

Introduce the auxiliary (pseudo-) Riemannian metric a on $R(M)$

$$a(X, Y) = d\eta(\phi X, Y) \ ; \ X, Y \in \Gamma(R(M)).$$

The Ricci tensor field $\rho(R)$ of ∇ is defined, in the usual way by:

$$\rho(R)(Y, Z) = \text{trace } (X \mapsto R_{XY} Z) \ ; \ X, Y, Z \in \chi(M). \tag{3}$$

Denote $\tau(R) = \text{trace } (\rho(R))$ a scalar curvature of the restriction of R to $R(M)$ obtained by using the partial metric a on $R(M)$.

In [3] is proved the following result:

Theorem. *Let* (ϕ, ξ, η) *be an almost contact structure on* M *subordinated to the pseudoconvex* $CR-$*structure* $(M, H(M))$. *Then the tensor field*

$$\begin{aligned} B(R)_{XY}Z = R_{XY}Z &+ L(X,Z)Y - L(Y,Z)X + L(Y,\phi Z)\phi X - \\ &- L(X,\phi Z)\phi Y - \{L(X,\phi Y) - L(Y,\phi X)\}\phi Z - \\ &- d\eta(X,Y)KZ - \frac{1}{2}d\eta(X,Z)KY + \frac{1}{2}d\eta(Y,Z)KX + \\ &+ \frac{1}{2}d\eta(X,\phi Z)\phi KY - \frac{1}{2}d\eta(Y,\phi Z)\phi KX, \\ & X, Y, Z \in \Gamma(R(M)) \end{aligned} \tag{4}$$

is invariant under the action of gauge transformation, where

$$L(X,Y)=\frac{1}{2(n+2)}\{\rho(R)(X,Y)+2d\eta(\phi X,\psi Y)\}+ \\ +\frac{1}{8(n+1)(n+2)}\tau(R)d\eta(X,\phi Y)\ , \tag{5}$$

and

$$\frac{1}{2}d\eta(KX,Y)=L(X,Y). \tag{6}$$

Remark. Doing the computations in local coordinates, we obtain that trace $(B(R)_{XY})=0$ and trace $(X\mapsto B(R)_{XY}Z)=0$.

Definition. *$B(R)$ is called the Bochner type tensor associated to the pseudoconvex $CR-$ structure $(M,H(M))$.*

3. The Unit Tangent Bundle

Let (M,g) be an (n+1)-Riemannian manifold and $\tau:TM\longrightarrow M$ be its tangent bundle. A local chart $(U,\phi)=(U,x^1,\ldots,x^{n+1})$ on M induces a local chart $(\tau^{-1}(U),\Phi)=(\tau^{-1}(U),x^1,\ldots,x^{n+1},\ y^1,\ldots,y^{n+1})$ on TM where the local coordinates $x^i,y^i;i=1,\ldots,n+1$ are defined as follows. The first n+1 local coordinates $x^i=x^i\circ\tau;i=1,\ldots,n+1$ on TM are the local coordinates in the local chart (U,ϕ) of the base point of a tangent vector from $\tau^{-1}(U)$. The last n+1 coordinates $y^i;i=1,\ldots,n+1$ are the vector space coordinates of the same tangent vector with respect to the natural local basis in the corresponding tangent space, defined by the local chart (U,ϕ). The tangent bundle TM of a Riemannian manifold (M,g) can be organized as a Riemannian manifold in many ways. The most known is given by the Sasaki metric on TM defined by g. It is well known we have a direct sum decomposition

$$TTM=VTM\oplus HTM \tag{7}$$

of the tangent bundle TTM into the vertical distribution $VTM=\ker\tau_*$ and the horizontal distribution HTM defined by the Levi Civita connection D

of g. The vector fields $(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^{n+1}})$ define a local frame for VTM and the vector fields $(\frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^{n+1}})$ where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{i0} \frac{\partial}{\partial y^h} \ ; \ \Gamma^h_{i0} = \Gamma^h_{ik} y^k$$

and Γ^h_{ik} are the Christoffel symbols defined by the Riemannian metric g, define a local frame for HTM.

The distributions VTM and HTM are isomorphic each other and it is possible to derive an almost complex structure on TM which, together with the Sasaki metric, determines a structure of almost Kählerian manifold on TM.

Consider the energy density (kinetic energy or "forza viva", according to the terminology used by Levi Civita):

$$t = \frac{1}{2} g_{ik}(x) y^i y^k \tag{8}$$

defined on TM by the Riemannian metric g of M, where g_{ik} are the components of g in the local chart (U, ϕ). Let $u, v : [0, \infty) \longrightarrow I\!R$ be two real smooth functions having positive values. In [6] it is considered the following M-tensor field of type (0,2) on TM, defined by components:

$$G_{ij} = u(t) g_{ij} + v(t) g_{i0} g_{j0} \tag{9}$$

where $g_{i0} = g_{ik}(x) y^k$. The matrix (G_{ij}) is symmetric and positive defined and has the inverse

$$H^{kl} = \frac{1}{u} g^{kl} + w y^k y^l \tag{10}$$

where g^{kl} are the components of the inverse of the matrix (g_{ij}) and:

$$w = w(t) = -\frac{v}{u(u + 2tv)} \ . \tag{11}$$

Throughout this paper we shall use also the following M-tensor fields on TM obtained by "lowering" or "uppering" the indices by the "musical" isomorphism between TM and T^*M defined by g :

$$\begin{cases} H_{ij} = g_{ik} H^{kl} g_{lj} = \frac{1}{u} g_{ij} + w g_{i0} g_{j0} \\ G^{kl} = g^{ki} G_{ij} g^{jl} = u g^{kl} + v y^k y^l \\ H^i_k = H^{ih} g_{hk} = H_{kh} g^{hi} = \frac{1}{u} \delta^i_k + w y^i g_{k0} \\ G^i_k = G^{ih} g_{hk} = G_{kh} g^{hi} = u \delta^i_k + v y^i g_{k0} \end{cases} \ . \tag{12}$$

The following Riemannian metric it is also considered on $TM(see[6])$:

$$G(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = G_{ij},\ G(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = H_{ij},\ G(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}) = 0,\ G(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}) = 0\,. \tag{13}$$

Remark that VTM and HTM are orthogonal each other with respect to G but the Riemannian metrics induced from G on VTM and HTM are not the same, so the considered metric G on TM is no longer a metric of Sasaki type.

An almost complex structure J is defined on TM by:

$$J\frac{\delta}{\delta x^i} = G_i^k \frac{\partial}{\partial y^k}\ ;\ J\frac{\partial}{\partial y^i} = -H_i^k \frac{\delta}{\delta x^k} \tag{14}$$

Consider the unit tangent bundle T_1M as the bundle of the unit tangent vectors at M, i.e. if $v \in TM$, then $v \in T_1M \iff g(v,v) = 1$. Locally, if $v = y^i \frac{\partial}{\partial x^i}$, we conclude that the unit tangent bundle $\pi : T_1M \longrightarrow M$ is a hypersurface in TM, given in the local coordinates by the equation:

$$g_{ij}(x) y^i y^j = 1. \tag{15}$$

Thus, on the unit tangent bundle the density energy is constant $(t = \frac{1}{2})$ and hence we conclude that u and v are constants and $w = -\frac{v}{u(u+v)}$.

Let $N \in \chi(TM)$ be the unitary and normal vector field with respect to the metric G. N is parallel with the gradient of the function $f : TM \longrightarrow \mathbb{R}$, $f(x,y) = g_{ij} y^i y^j - 1$ which is null on T_1M. One gets

$$N = \sqrt{u+v}\ y^i \frac{\partial}{\partial y^i}\ . \tag{16}$$

Let us remark that the vector fields $\frac{\delta}{\delta x^i}$, $Y_i = \frac{\partial}{\partial y^i} - \frac{1}{\sqrt{u+v}} g_{i0} N$; $i = 1, \dots, n{+}1$ are orthogonal to N and they form a generator system for $\chi(T_1M)$. Since

$$y^i Y_i = 0 \tag{17}$$

we conclude that the vector fields $Y_i; i = 1, \dots, n+1$ are not independent.

By using the formulas

$$\xi = -JN\ ,\ JX = \phi(X) + \eta(X)N\ ;\ X \in \chi(T_1M). \tag{18}$$

we compute the expression of the almost contact structure subordinated to the $CR-$ structure:

$$\begin{cases} \xi = \dfrac{1}{\sqrt{u+v}} y^i \dfrac{\delta}{\delta x^i}, \quad \eta(\dfrac{\delta}{\delta x^i}) = \sqrt{u+v}\ g_{i0}\ , \ \eta(Y_i) = 0 \\ \phi(\dfrac{\delta}{\delta x^i}) = G_i^k Y_k, \ \phi(Y_i) = -H_i^k \dfrac{\delta}{\delta x^k} + \dfrac{1}{\sqrt{u+v}} g_{i0}\xi \ ; \ i,j = 1,...,n+1. \end{cases} \tag{19}$$

In the following we project the vector fields $\frac{\delta}{\delta x^i}$; $i = 1, \ldots, n+1$ on $R(T_1M) = ker\eta$. Thus, denote by $X_i = \frac{\delta}{\delta x^i} - \eta(\frac{\delta}{\delta x^i})\xi = \frac{\delta}{\delta x^i} - \sqrt{u+v}\ g_{i0}\xi$.

Denote also by

$$\varepsilon_i^j = \delta_i^j - g_{i0} y^j \ . \tag{20}$$

With these notations the sections $X_i,\ Y_i$; $i = 1, \ldots, n+1$ may be written in the same way as

$$X_i = \varepsilon_i^k \frac{\delta}{\delta x^k}\ ,\ Y_i = \varepsilon_i^k \frac{\partial}{\partial y^k}\ . \tag{21}$$

We claim now that the Nijenjuis tensor of J vanishes on the set of sections in $R(T_1M)$. The expression of N_J is given by the formulas:

$$\begin{cases} N_J(Y_i, Y_j) = -\dfrac{1}{u^2}(R^k_{0ij} - g_{j0}R^k_{0i0} + g_{i0}R^k_{0j0})\dfrac{\partial}{\partial y^k} \\ N_J(X_i, X_j) = \quad (R^k_{0ij} - g_{j0}R^k_{0i0} + g_{i0}R^k_{0j0})\dfrac{\partial}{\partial y^k} \\ N_J(Y_i, Y_j) = \quad \dfrac{1}{u^2}(R^k_{0ij} - g_{j0}R^k_{0i0} + g_{i0}R^k_{0j0})\dfrac{\delta}{\delta x^k} \end{cases} . \tag{22}$$

Theorem. *The unit tangent bundle T_1M of a Riemannian manifold (M, g) of dimension $n+1 \geq 3$ has a Cauchy-Riemann structure obtained from the almost Kählerian structure of TM, defined above, if and only if the base manifold M has constant sectional curvature.*

Proof. The vanishing of the Nijenhuis tensor field N_J defined by the relations (22) yields

$$R^k_{0ij} - g_{j0}R^k_{0i0} + g_{i0}R^k_{0j0} = 0$$

which implies that M has constant sectional curvature c. (see [2], [4], [7]) ■

4. The Bochner type tensor field

In this section we shall find the condition under which the unit tangent bundle of a Riamannian manifold with constant sectional curvature is Bochner flat. To do this, we need the following formulas for the brackets of the vector fields $\frac{\delta}{\delta x^i}$, $Y_i; \imath = 1, \ldots, n+1$:

$$[Y_i, Y_j] = g_{i0}Y_j - g_{j0}Y_i,\ [Y_i, \frac{\delta}{\delta x^j}] = -\Gamma^k_{ij}Y_k,\ [\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}] = c(g_{i0}Y_j - g_{j0}Y_i) \tag{23}$$

We also need the expression of the differentiation of the 1-form η :

$$d\eta(Y_i, Y_j) = 0,\ d\eta(X_i, X_j) = 0,\ d\eta(Y_i, X_j) = \sqrt{u+v}\ u_{ij}\ , \tag{24}$$

where $u_{ij} = g_{ij} - g_{i0}g_{j0}$.

Theorem. *The canonical connection adapted to the almost contact structure (ϕ, ξ, η) presented in the Section 2. has the following expression:*

$$\left\{ \begin{array}{ll} \nabla_{Y_i}Y_j = -g_{j0}Y_i & \nabla_{Y_i}X_j = -g_{j0}X_i - \frac{u_{ij}}{2}\sqrt{u+v}\ \xi \\ \nabla_{X_i}X_j = \varepsilon^s_i\Gamma^k_{sj}X_k & \nabla_{X_i}Y_j = \varepsilon^s_i\Gamma^k_{sj}Y_k + \frac{u_{ij}}{2}\sqrt{u+v}\ \xi \end{array} \right. . \tag{25}$$

Proof. We may use the formulas (2) for the adapted connection but is more convenient to get the connection ∇ computing $\eta(\nabla_X Y)$ and $d\eta(\nabla_X Y, Z)$.

First of all, we have the simple formulas:

$$G^k_i X_k = u\ X_i,\ G^k_i Y_k = u\ Y_i,\ H^k_i X_k = \frac{1}{u}X_i,\ H^k_i Y_k = \frac{1}{u}Y_i; \tag{26}$$

futhermore, we have other useful relation for ϕ and ψ

$$\left\{ \begin{array}{ll} \phi X_i = u\ Y_i & \phi Y_i = -\frac{1}{u}X_i \\ \psi X_i = -\frac{1}{2\sqrt{u+v}}(u - \frac{c}{u})X_i & \psi Y_i = \frac{1}{2\sqrt{u+v}}(u - \frac{c}{u})Y_i. \end{array} \right. \tag{27}$$

After a straightforward computation we get the expressions of $\eta(\nabla_X Y)$ and $d\eta(\nabla_X Y, Z)$; $X, Y, Z \in \Gamma(R(T_1M))$. Taking into account that $d\eta$ is non degenerated on $H(T_1M)$ one gets quite easily the formulas (25). ∎

We give now the most important result of this section.

Theorem. *Let (M,g) be a Riemannian manifold of dimension $n+1 \geq 3$ and T_1M be the unit tangent bundle with the Riemannian structure defined by (9).*

Then the gauge invariant B of $(1,3)$-type (the Bochner type tensor associated to the CR-structure induced on T_1M) vanishes, if and only if (M,g) has constant negative sectional curvature.

Proof. First of all we have to compute the expression of the curvature tensor R by using the sections from $\Gamma(R(T_1M))$. Using the formula for the curvature tensor field the relations (25) yiels to:

$$\begin{cases} R_{Y_iY_j}Y_k = u_{jk}Y_i - u_{ik}Y_j \ , \ R_{Y_iY_j}X_k = u_{jk}X_i - u_{ik}X_j \\ R_{X_iX_j}Y_k = c(u_{jk}Y_i - u_{ik}Y_j) \ , \ R_{X_iX_j}X_k = c(u_{jk}X_i - u_{ik}X_j) \\ R_{Y_iX_j}Y_k = -u_{ij}X_k \ , \ R_{Y_iX_j}X_k = c\ u_{ij}Y_k \ , \ i,j,k = 1,\dots,n+1. \end{cases} \tag{28}$$

The auxiliary metric a on $H(T_1M)$ is given by:

$$a(Y_i,Y_j) = \frac{1}{u}\sqrt{u+v} \ \ u_{ij}, \ a(X_i,X_j) = u\sqrt{u+v} \ \ u_{ij}, \ a(Y_i,X_j) = 0. \tag{29}$$

Making the usual contraction we get the Ricci tensor:

$$\begin{cases} \rho(R)(Y_i,Y_j) = n\ h_{ij} \ \ , \ \ \rho(R)(Y_i,X_j) = 0 \\ \rho(R)(X_i,Y_j) = 0 \ \ , \ \ \rho(R)(X_i,X_j) = c\ n\ h_{ij}. \end{cases} \tag{30}$$

Hence, $\tau(R) = \frac{n^2}{\sqrt{u+v}}(u + \frac{c}{u})$.

In order to find the Bochner tensor we have to compute the expressions of the tensor fields L and K, respectively.

For L one gets:

$$\begin{cases} L(Y_i,Y_j) = \dfrac{k_1}{2}u_{ij} \ \ , \ \ L(Y_i,X_j) = 0 \\ L(X_i,Y_j) = 0 \ \ , \ \ L(X_i,X_j) = \dfrac{k_2}{2}u_{ij} \end{cases} \tag{31}$$

where:

$$\begin{cases} k_1 = \dfrac{1}{n+2}\left[n+1 - \dfrac{c}{u^2} - \dfrac{n^2}{4(n+1)}(1+\dfrac{c}{u^2})\right] \\ k_2 = \dfrac{1}{n+2}\left[nc - u^2 + c - \dfrac{n^2}{4(n+1)}(u^2+c)\right]. \end{cases} \tag{32}$$

From the definition of K it follows:

$$KY_i = -k_1\frac{1}{\sqrt{u+v}}X_i \ , \ KX_i = k_2\frac{1}{\sqrt{u+v}}Y_i \ , \ i = 1,\dots,n+1. \tag{33}$$

By replacing now in (4) we obtain:

$$\begin{cases} B_{Y_iY_j}Y_k = (1+\dfrac{c}{u^2})\dfrac{(n+2)}{4(n+1)}(u_{jk}Y_i - u_{ik}Y_j) \\ B_{Y_iY_j}X_k = (1+\dfrac{c}{u^2})\dfrac{(n+2)}{4(n+1)}(u_{jk}X_i - u_{ik}X_j) \\ B_{Y_iX_j}Y_k = -(1+\dfrac{c}{u^2})\dfrac{(n+2)}{4(n+1)}u_{ij}X_k + \\ \qquad +(1+\dfrac{c}{u^2})\dfrac{n}{4(n+1)}(u_{jk}X_i + u_{ik}X_j + u_{ij}X_k) \\ B_{Y_iX_j}X_k = \ (u^2+c)\dfrac{(n+2)}{4(n+1)}u_{ij}Y_k - \\ \qquad -(u^2+c)\dfrac{n}{4(n+1)}(u_{jk}Y_i + u_{ik}Y_j + u_{ij}Y_k) \\ B_{X_iX_j}Y_k = (u^2+c)\dfrac{(n+2)}{4(n+1)}(u_{jk}Y_i - u_{ik}Y_j) \\ B_{X_iX_j}X_k = (u^2+c)\dfrac{(n+2)}{4(n+1)}(h_{jk}X_i - h_{ik}X_j). \end{cases} \tag{34}$$

If we claim that the unit tangent bundle to be Bochner flat, i.e. if we put $B(R) = 0$ we get $c = -u^2$.

Otherwise we have $u_{jk}Y_i = u_{ik}Y_j$; $\forall i,j,k = 1,\dots,n+1$ and $n(u_{ij}Y_k + u_{jk}Y_i + u_{ik}Y_j) - (n+2)u_{ij}Y_k = 0$, which imply $n = 1$, namely the dimension of the manifold M is 2. This proves the theorem. ∎

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