Seminar 5: produs vectorial si mixt

1. Fie vectorii $\bar{a} = 2\bar{i} + 3\bar{j} - \bar{k}$, $\bar{b} = \bar{i} - \bar{j} + 3\bar{k}$ si $\bar{c} = \bar{i} + \bar{j} + \bar{k}$. Baza $\{\bar{i}, \bar{j}, \bar{k}\}$ este ortonormata si pozitiva.
   (a) Calculati $\bar{a} \times \bar{b}$, $\bar{a} \times (\bar{b} \times \bar{c})$, $(\bar{a} \times \bar{b}) \times \bar{c}$, $(\bar{a}, \bar{b}, \bar{c})$. Sunt vectorii $\bar{a}, \bar{b}$ coliniari? Dar vectorii $\bar{a}, \bar{b}, \bar{c}$ coplanari? Este produsul vectorial asociativ?
   (b) Determinati vectorul liber $\bar{u}$ care satisface conditiile: $\bar{u} \perp \bar{a}$, $\bar{u} \perp \bar{b}$, $\| \bar{u} \| = \sqrt{138}$, $\bar{u}$ formeaza un unghi obtuz cu $\bar{i}$.
   (c) Care e baza reciproca bazei $\mathscr{B} = \{\bar{a}, \bar{b}, \bar{c}\}$? Determinati coordonatele vectorului $\bar{w} = 2\bar{i} - \bar{j} - \bar{k}$ in raport cu $\mathscr{B}^*$.
   (d) Calculati aria paralelogramului construit pe vectorii $\bar{a}, \bar{b}$ aplicati in acelasi punct, cat si volumul paralelipipedului construit pe vectorii $\bar{a}, \bar{b}, \bar{c}$ aplicati in acelasi punct.

2. Verificati urmatoarele doua identitati:
   (a) identitatea *Lagrange*: $\| \bar{u} \times \bar{v} \|^2 + < \bar{u}, \bar{v} >^2 = \| \bar{u} \|^2 \| \bar{v} \|^2$, $\forall \bar{u}, \bar{v} \in \mathscr{V}^3$.
   (b) identitatea *Jacobi*: $\bar{a} \times (\bar{b} \times \bar{c}) + \bar{b} \times (\bar{c} \times \bar{a}) + \bar{c} \times (\bar{a} \times \bar{b}) = \bar{0}$, $\forall \bar{a}, \bar{b}, \bar{c} \in \mathscr{V}^3$.

3. Fie vectorii $\bar{a}, \bar{b}, \bar{c}$ necoliniari doi cate doi. Demonstrati ca $\bar{a} + \bar{b} + \bar{c} = \bar{0}$ daca si numai daca $\bar{a} \times \bar{b} = \bar{b} \times \bar{c} = \bar{c} \times \bar{a}$. Aplicati acest rezultat pentru a demonstra teorema sinusurilor intr-un priunghi:
   Dat $\triangle ABC$, notam in mod uzual lungimile laturilor cu a, b, c si masurile unghiurilor cu A, B, C. Atunci
   $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

4. Fie punctele A, B, C puncte fixate in $\mathscr{S}$. Sa se arate ca vectorul $\bar{w} = \overrightarrow{MA} \times \overrightarrow{BC} + \overrightarrow{MB} \times \overrightarrow{CA} + \overrightarrow{MC} \times \overrightarrow{AB}$ nu depinde de alegerea lui M. Determinati marimea acestui vector, directia si sensul sau.

5. Fie punctele A, B, C, D. Determinati punctul V care satisface
   $$\overrightarrow{VA} \times \overrightarrow{VB} = \overrightarrow{VB} \times \overrightarrow{VC} = \overrightarrow{VC} \times \overrightarrow{VD}.$$
   Rezolvare Fie $\bar{w} = \overrightarrow{VA} \times \overrightarrow{VB} = \overrightarrow{VB} \times \overrightarrow{VC} = \overrightarrow{VC} \times \overrightarrow{VD}$. Observam ca $\bar{w} \perp (VAB)$, $\bar{w} \perp (VBC)$, $\bar{w} \perp (VCD)$. Acest lucru e posibil doar daca cele trei plane coincid, deci punctele A, B, C, D sunt coplanare si formeaza un patrulater.
   $\| \overrightarrow{VA} \times \overrightarrow{VB} \| = \| \overrightarrow{VB} \times \overrightarrow{VC} \| \Leftrightarrow \sigma(VAB) = \sigma(VBC) \Leftrightarrow V$ se afla pe dreapta suport a medianei din B a triunghiului BAC. Analog din $\| \overrightarrow{VB} \times \overrightarrow{VC} \| = \| \overrightarrow{VC} \times \overrightarrow{VD} \|$ rezulta ca V se afla pe dreapta suport a medianei din C a triunghiului BCD. Deci punctul cautat este punctul de intersectie al celor doua drepte specificate anterior.

6. Dati vectorii $\bar{a} \neq \bar{0}, \bar{b} \perp \bar{a}$, rezolvati ecuatiile vectoriale

$$\bar{a} \times \bar{x} = \bar{b},\ \ \bar{a} \times (\bar{a} \times \bar{x}) = \bar{b},\ \ \bar{a} \times (\bar{a} \times (\cdots \times (\bar{a} \times \bar{x}) \cdots) = \bar{b}.$$

7. Rezolvati sistemele de ecuatii vectoriale:

   (a)
$$\begin{cases} \bar{a} \times \bar{x} & = \bar{b}, \\ < \bar{a}, \bar{x} > & = m, \end{cases}$$
   pentru $\bar{a} \neq \bar{0}$, $\bar{b} \perp \bar{a}$ si $m \in \mathbb{R}$ dati.

   (b)
$$\begin{cases} \bar{a} \times \bar{x} & = \bar{b}, \\ < \bar{c}, \bar{x} > & = m, \end{cases}$$
   unde $\bar{a} \neq \bar{0}$, $\bar{c} \neq \bar{0}, \bar{b} \perp \bar{a}$ si $m \in \mathbb{R}$ sunt dati.

   (c)
$$\begin{cases} < \bar{a}, \bar{x} > & = m, \\ < \bar{b}, \bar{x} > & = n, \\ < \bar{c}, \bar{x} > & = p, \end{cases}$$
   unde $\bar{a}, \bar{b}, \bar{c}$ sunt trei vectori necoplanari dati si m, n, p sunt trei numere reale date.

8. Se considera o baza $\{\bar{a}, \bar{b}, \bar{c}\}$ si $\{\bar{a}^*, \bar{b}^*, \bar{c}^*\}$ baza ei reciproca.

   (a) Demonstrati ca pentru orice vector $\bar{u} \in \mathscr{V}_3$, avem:
$$\begin{aligned} \bar{u} &= < \bar{u}, \bar{a} > \bar{a}^* + < \bar{u}, \bar{b} > \bar{b}^* + < \bar{u}, \bar{c} > \bar{c}^*, \\ \bar{u} &= < \bar{u}, \bar{a}^* > \bar{a} + < \bar{u}, \bar{b}^* > \bar{b} + < \bar{u}, \bar{c}^* > \bar{c}. \end{aligned}$$

   (b) Folosind relatiile de mai sus, demonstrati ca
$$\bar{u} \times \bar{v} = \frac{1}{(\bar{a}, \bar{b}, \bar{c})} \begin{vmatrix} \bar{a} & < \bar{u}, \bar{a} > & < \bar{v}, \bar{a} > \\ \bar{b} & < \bar{u}, \bar{b} > & < \bar{v}, \bar{b} > \\ \bar{c} & < \bar{u}, \bar{c} > & < \bar{v}, \bar{c} > \end{vmatrix},$$
$$(\bar{u}, \bar{v}, \bar{w}) = \frac{1}{(\bar{a}, \bar{b}, \bar{c})} \begin{vmatrix} < \bar{u}, \bar{a} > & < \bar{v}, \bar{a} > & < \bar{w}, \bar{a} > \\ < \bar{u}, \bar{b} > & < \bar{v}, \bar{b} > & < \bar{w}, \bar{b} > \\ < \bar{u}, \bar{c} > & < \bar{v}, \bar{c} > & < \bar{w}, \bar{c} > \end{vmatrix}.$$

9. Fie O, A, B, C necoplanare. Fie H proiectia lui O pe (ABC) si Ω centrul sferei circumscrise tetraedrului $OABC$. Exprimati vectorii $\overrightarrow{OH}$ si $\overrightarrow{O\Omega}$ in functie de vectorii $\bar{a} = \overrightarrow{OA}$, $\bar{b} = \overrightarrow{OB}$, $\bar{c} = \overrightarrow{OC}$.