Seminar 3

1. Daca V si W sunt spatii liniare peste acelasi camp $\mathbb{K}$, demonstrati ca $V \times W$ poate fi inzestrat cu o structura canonica de spatiu liniar peste $\mathbb{K}$. Generalizati pentru produsul cartezian a m spatii vectoriale peste acelasi camp. Consecinta: $\mathbb{K}^m$ este spatiu liniar peste $\mathbb{K}$.
   In particular, $\mathbb{R}^n$ impreuna cu operatiile $+ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ si $\cdot : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, definite prin
   $$(x^1, x^2, \cdots, x^n) + (y^1, y^2, \cdots, y^n) = \big(x^1 + y^1, x^2 + y^2, \cdots, x^n + y^n\big)$$
   si
   $$\alpha(x^1, x^2, \cdots, x^n) = (\alpha x^1, \alpha x^2, \cdots, \alpha x^n)$$
   este un spatiu liniar real.

2. Studiati daca $\mathbb{R}^n$, impreuna cu operatiile $\oplus : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ si $\cdot : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, definite prin
   $$(x^1, x^2, \cdots, x^n) \oplus (y^1, y^2, \cdots, y^n) = \big(x^1 + y^1, 2(x^2 + y^2), \cdots, n(x^n + y^n)\big)$$
   si
   $$\alpha(x^1, x^2, \cdots, x^n) = (\alpha x^1, \alpha x^2, \cdots, \alpha x^n)$$
   formeaza un spatiu liniar real.

3. Studiati daca urmatoarele sisteme de vectori sunt liniar independente, respectiv sisteme de generatori.
   (a) $S_1 = \{u_1 = (1, -1, 2),\ u_2 = (2, 1, -1),\ u_3 = (-1, -2, 3)\}$ in $\mathbb{R}^3$;
   (b) $S_2 = \{v_1 = (1, 2, 3),\ v_2 = (3, -1, 4),\ v_3 = (2, 5, -1)\}$ in $\mathbb{R}^3$;
   (c) $S_3 = \{w_1 = (1, -1, 2),\ w_2 = (3, 1, 0)\}$ in $\mathbb{R}^3$;
   (d) $S_4 = \{a_1 = (-2, 3, 1),\ a_2 = (3, -1, 5),\ a_3 = (1, -4, 3)\}$ in $\mathbb{R}^3$;
   (e) $S_5 = \{b_1 = (1, 1, 1, 1),\ b_2 = (1, -1, -1, 1),\ b_3 = (1, -1, 1, -1)\}$ in $\mathbb{R}^4$;
   (f) $S_6 = \{c_1 = (3, -5, 2, -3),\ c_2 = (2, -3, 1, 0),\ c_3 = (4, -1, 5, 6),\ c_4 = (9, -9, 8, 3)\}$ in $\mathbb{R}^4$.

4. Verificati ca urmatoarele sisteme de vectori din $\mathbb{R}^3$ sunt baze si determinati coordonatele vectorului $u = (1, -3, 2)$ in raport cu aceste baze:
   (a) $B_1 = \{a_1 = (1, 0, 0),\ a_2 = (1, 1, 0),\ a_3 = (1, 1, 1)\}$;
   (b) $B_2 = \{a_1 = (1, -1, 0),\ a_2 = (-4, 6, -10),\ a_3 = (-1, 3, -9)\}$;
   (c) $B_3 = \{a_1 = (1, 1, 0),\ a_2 = (0, 1, 0),\ a_3 = (0, -5, 5)\}$.
   (d) Scrieti matricele de trecere de la baza canonica B_c a lui $\mathbb{R}^3$ la B_1, de la B_1 la B_2 si respectiv de la B_3 la B_c.

5. Fie $(V, +, \cdot)$ un spatiu liniar peste campul $\mathbb{K}$. Daca $x_1, x_2, \cdots, x_n \in V$ sunt liniar independenti, sa se studieze liniara independenta a sistemelor de vectori:
   (a) $S_1 = \{v_1 = x_1 + x_2,\ v_2 = x_2 + x_3, \cdots,\ v_n = x_n + x_1\}$;
   (b) $S_2 = \{a_1 = x_1,\ a_2 = x_1 + x_2,\ \cdots,\ a_n = x_1 + x_2 + \cdots + x_n\}$.

6. Determinati conditiile ce trebuie sa fie indeplinite de $a, b \in \mathbb{R}$ astfel incat vectorii urmatori sa fie liniar independenti in $\mathbb{R}^3$:
   (a) $v_1 = (a, 1, 1),\ v_2 = (1, a, 1),\ v_3 = (1, 1, a)$;
   (b) $u_1 = (1, a, 2),\ u_2 = (1, 1, b)$;
   (c) $w_1 = (a, 2, 3),\ w_2 = (0, 4, 6),\ w_3 = (0, 0, b)$.