# Seminarii 5-6

1. In spatiul liniar $\mathbb{K}^n$ se considera multimea solutiilor unui sistem cu $m$ ecuatii liniare si $n$ necunoscute

$$V = \left\{ u = \left(x^1, x^2, \cdots, x^n\right) \in \mathbb{K}^n \mid \sum_{j=1}^{n} a^i_j x^j = b^i,\ i \in \overline{1,m} \right\}.$$

   (a) Demonstrati ca $V$ este un subspatiu liniar al lui $\mathbb{K}^n$ daca si numai daca $b^i = 0$, $\forall i \in \overline{1,m}$, deci daca si numai daca $V$ este multimea solutiilor unui sistem **omogen** de ecuatii liniare. Determinati o baza a acestuia si observati ca $dim(V) = n - rang(A)$, unde $A = (a^i_j)_{i\in\overline{1,m}, j\in\overline{1,n}} \in M_{m,n}(\mathbb{K})$ este matricea sistemului.

   (b) Determinati baze in subspatiile liniare ale solutiilor urmatoarelor sisteme omogene de ecuatii liniare, apoi completati-le la baze in spatiul liniar ambiant.

      i. $\begin{cases} x^1 - x^2 + x^3 & = 0 \\ x^1 + 2x^2 - 3x^4 & = 0 \end{cases}$

      ii. $\begin{cases} x^1 + 4x^2 + 2x^3 - 3x^5 & = 0 \\ 2x^1 + 9x^2 + 5x^3 + 2x^4 + x^5 & = 0 \\ x^1 + 3x^2 + x^3 - 2x^4 - 9x^5 & = 0 \end{cases}$

      iii. $\begin{cases} 2x^1 + 7x^2 + 3x^3 + x^4 & = 0 \\ 3x^1 + 5x^2 + 2x^3 + 2x^4 & = 0 \\ 9x^1 + 4x^2 + x^3 + 7x^4 & = 0 \end{cases}$

      iv. $\begin{cases} 3x^1 + 3x^2 - 4x^3 - 3x^4 & = 0 \\ 6x^2 + x^3 + x^4 & = 0 \\ 5x^1 + 4x^2 + 2x^3 + x^4 & = 0 \\ 2x^1 + 3x^2 + 3x^3 + 2x^4 & = 0 \end{cases}$

      v. $\begin{cases} 2x^1 - x^2 + 3x^3 - 2x^4 + 4x^5 & = 0 \\ 4x^1 - 2x^2 + 5x^3 + x^4 + 7x^5 & = 0 \\ 2x^1 - x^2 + x^3 + 8x^4 + 2x^5 & = 0 \end{cases}$

2. Care din urmatoarele submultimi sunt subspatii liniare in $\mathbb{R}^n$?

   (a) $V_1 = \left\{(x^1, x^2, x^3) \in \mathbb{R}^3 \mid 2x^1 + x^2 - x^3 = 0\right\}$;

   (b) $V_2 = \left\{(x^1, x^2, x^3) \in \mathbb{R}^3 \mid 2x^1 + x^2 - x^3 = 1\right\}$;

   (c) $V_3 = \left\{(x^1, x^2, x^3) \in \mathbb{R}^3 \mid \begin{cases} x^1 - x^2 + x^3 & = 1 \\ x^1 + 2x^2 - 3x^4 & = 2 \end{cases}\right\}$;

   (d) $V_{a,b,c} = \left\{(x^1, x^2) \in \mathbb{R}^2 \mid ax^1 + bx^2 + c = 0\right\}$, $a, b, c \in \mathbb{R}$;

   (e) $W_{a,b,c,d} = \left\{(x^1, x^2, x^3) \in \mathbb{R}^3 \mid ax^1 + bx^2 + cx^3 + d = 0\right\}$, $a, b, c, d \in \mathbb{R}$;

   (f) Dati interpretari geometrice pentru $V_{a,b,c}$ si $W_{a,b,c,d}$ si precizati conditii necesare si suficiente pentru ca $V_{a,b,c} = V_{a',b',c'}$ si respectiv $W_{a,b,c,d} = W_{a',b',c',d'}$;

   (g) Determinati conditii necesare si suficiente ca $\mathbb{R}^2 = V_{a,b,0} \oplus V_{a',b',0}$.

   (h) Demonstrati ca $Z_{a,b,c} = \left\{(at, bt, ct) \mid t \in \mathbb{R}\right\}$ este subspatiu liniar al lui $\mathbb{R}^3$ si dati interpretarea geometrica a acestuia. Determinati conditii necesare si suficiente pentru ca $Z_{a,b,c} = Z_{a',b',c'}$? In ce conditii $\mathbb{R}^3 = Z_{a,b,c} \oplus W_{a,b,c,0}$? Dar $\mathbb{R}^3 = Z_{a,b,c} \oplus Z_{a',b',c'} \oplus Z_{a'',b'',c''}$?

3. Determinati o baza in subspatiile liniare generate de urmatoarele multimi de vectori din $\mathbb{R}^n$. Scrieti apoi fiecare din aceste subspatii ca subspatiul solutiilor unui sistem omogen de ecuatii liniare.

   (a) $x_1 = (1,1,1,1)$, $x_2 = (1,1,1,2)$, $x_3 = (3,-5,7,2)$, $x_4 = (1,-7,5,-2)$;
   (b) $x_1 = (1,-1)$, $x_2 = (-1,6)$, $x_3 = (-1,1)$;
   (c) $x_1 = (-3,2,0)$, $x_2 = (-3,6,-15)$, $x_3 = (0,-4,15)$;
   (d) $x_1 = (1,1,1,1)$, $x_2 = (2,2,2,2)$, $x_3 = (3,3,3,3)$, $x_4 = (1,1,2,2)$.

4. Precizati valoarea de adevar a propozitiilor:

   (a) $\{f : [0,1] \to \mathbb{R}, \mid f(x) = a\sin x + b\cos x,\ x \in [0,1],\ a,b \in \mathbb{R}\}$ este subspatiu liniar in $\mathbb{R}^{[0,1]}$;
   (b) $\mathbb{K}_n[X]$ este subspatiu liniar in $\mathbb{K}[X]$;
   (c) $\left\{(x,y) \in \mathbb{R}^2 \mid y = x^2\right\}$ este subspatiu liniar in $\mathbb{R}^2$;
   (d) $\left\{f : [0,1] \to \mathbb{R}, \mid f(x) = ax^2 \ sau\ f(x) = x^3,\ a \in \mathbb{R}\right\}$ este subspatiu liniar in $\mathbb{R}^{[0,1]}$;
   (e) $\left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \mathcal{M}_{2,2}(\mathbb{R}) \mid a,b \in \mathbb{R}\right\}$ este subspatiu liniar in $\mathcal{M}_{2,2}(\mathbb{R})$.

5. Aratati ca multimea
$$V = \left\{ A \in \mathcal{M}_{3,3}(\mathbb{R}) \mid A = \begin{pmatrix} a & c & b \\ c & a-b & c \\ b & c & a \end{pmatrix},\ a,b,c \in \mathbb{R} \right\}$$
este subspatiu liniar al lui $\mathcal{M}_{3,3}(\mathbb{R})$. Determinati dimensiunea sa (peste $\mathbb{R}$) si o baza in $V$.

6. Demonstrati ca $\mathcal{S}_n(\mathbb{R}) = \{A \in \mathcal{M}_{n,n}(\mathbb{R}) \mid {}^tA = A\}$ si $\mathcal{A}_n(\mathbb{R}) = \{A \in \mathcal{M}_{n,n}(\mathbb{R}) \mid {}^tA = -A\}$ sunt subspatii liniare ale spatiului liniar real $\mathcal{M}_{n,n}(\mathbb{R})$. Determinati dimensiunile acestor doua subspatii de matrice si precizati cate o baza in fiecare dintre ele. Aratati ca $\mathcal{M}_{n,n}(\mathbb{R}) = \mathcal{S}_n(\mathbb{R}) \oplus \mathcal{A}_n(\mathbb{R})$,

7. Fie $V_1 = \{f : \mathbb{R} \to \mathbb{R} \mid f(-x) = f(x),\ \forall x \in \mathbb{R}\}$ si $V_2 = \{f : \mathbb{R} \to \mathbb{R} \mid f(-x) = -f(x),\ \forall x \in \mathbb{R}\}$. Demonstrati ca $V_1$ si $V_2$ sunst subspatii liniare ale spatiului liniar real al functiilor definite pe $\mathbb{R}$ cu valori in $\mathbb{R}$. Aratati ca $\mathbb{R}^{\mathbb{R}} = V_1 \oplus V_2$.

8. In $\mathbb{K}[X]$ se considera urmatoarele submultimi: $V_1 = \mathbb{K}[X^2]$ si $V_2$ multimea polinoamelor ce au in scrierea algebrica coeficientii puterilor pare ale lui $X$ egali cu 0, la care se adauga polinomul nul. Demonstrati ca $V_1$ si $V_2$ sunt subspatii liniare ale lui $\mathbb{K}[X]$ si ca $\mathbb{K}[X] = V_1 \oplus V_2$.

9. Fie $V_1$, $V_2$ subspatii liniare ale unui spatiu liniar $V$ cu $dimV_1 = dimV_2 = 6$ si $dimV = 10$. Care este cea mai mica valoare posibila pentru $dim(V_1 \cap V_2)$?

10. Fie $V_1, V_2, V_3$ subspatii liniare ale spatiului liniar $n$-dimensional $V$. Demonstrati ca $dim(V_1 \cap V_2 \cap V_3) \geq dimV_1 + dimV_2 + dimV_3 - 2n$.

11. Aflati baze in suma si intersectia subspatiilor $V_1$ si $V_2$ din $\mathbb{R}^n$:

    (a) $n = 4$, $V_1 = L(a_1,a_2,a_3)$, $V_2 = L(b_1,b_2,b_3)$, $a_1 = (1,2,1,1)$, $a_2 = (2,3,1,0)$, $a_3 = (3,1,1,-2)$, $b_1 = (0,4,1,3)$, $b_2 = (1,0,-2,-6)$, $b_3 = (1,0,3,5)$;
    (b) $n = 3$, $V_1 = L(a_1,a_2,a_3)$, $V_2 = L(b_1,b_2,b_3)$, $a_1 = (1,2,3)$, $a_2 = (0,1,1)$, $a_3 = (1,1,2)$, $b_1 = (4,3,1)$, $b_2 = (1,1,0)$, $b_3 = (5,3,2)$;
    (c) $n = 3$, $V_1 = L(a_1,a_2,a_3)$, $V_2 = L(b_1,b_2,b_3)$, $a_1 = (1,2,1)$, $a_2 = (1,1,-1)$, $a_3 = (1,3,3)$, $b_1 = (2,3,-1)$, $b_2 = (1,2,2)$, $b_3 = (1,1,-3)$;
    (d) $n = 2$, $V_1 = L(a_1,a_2,a_3)$, $V_2 = L(b_1,b_2,b_3)$, $a_1 = (1,3)$, $a_2 = (2,6)$, $a_3 = (1,4)$, $b_1 = (-1,1)$, $b_2 = (3,-3)$, $b_3 = (2,-2)$;

(e) $n = 3$, $V_1 = L(a_1, a_2, a_3)$, $V_2 = L(4, 2, 1)$, $a_1 = (-3, 2, 0)$, $a_2 = (-1, 4, 0)$, $a_3 = (-1, 4, 0)$, $b_1 = (5, 3, 13)$, $b_2 = (7, 0, 12)$, $b_3 = (-2, 3, 1)$;

12. Demonstrati ca $\mathbb{R}^n = V_1 \oplus V_2$ si determinati unica descompunere a vectorului $x$ dupa aceste subspatii, unde:

(a) $n = 4$, $V_1 = L(a_1, a_2, a_3)$, $V_2 = L(b_1, b_2, b_3)$, $a_1 = (2, 3, 11, 5)$, $a_2 = (1, 1, 5, 2)$, $a_3 = (0, 1, 1, 1)$, $b_1 = (2, 1, 3, 2)$, $b_2 = (1, 1, 3, 4)$, $b_3 = (5, 2, 6, 2)$, $x = (2, 0, 0, 3)$;

(b) $n = 2$, $V_1 = L(a_1, a_2)$, $V_2 = L(b_1)$, $a_1 = (-1, 1)$, $a_2 = (2, -2)$, $b_1 = (1, 4)$, $x = (2, 0)$;

(c) $n = 3$, $V_1 = L(a_1, a_2)$, $V_2 = L(b_1, b_2)$, $a_1 = (2, 1, 1)$, $a_2 = (1, 4, 1)$, $b_1 = (1, 1, 1)$, $b_2 = (4, 4, 4)$, $x = (1, 2, 3)$.