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*Biharmonic Maps and Submanifolds in Certain Geometric Contexts*
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# Scientific report
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In October 2011, the team members started the research on the above mentioned project, aiming to achieve project objectives.

The first proposed objective is *Study of biharmonic submanifolds in 7-dimensional Sasakian space forms and in the complex projective space* $CP^3$. The people working on this objective are: Cezar Oniciuc and Dorel Fetcu. The objective was achieved successfully and the research results were published in the paper *Biharmonic integral C-parallel submanifolds in 7-dimensional Sasakian space forms*, recently accepted in the Tohoku Mathematical Journal.

In this paper, they continued the study of integral submanifolds integrated into Sasakian spatial forms. This study was initially started by T. Sasahara, who determined all the biharmonic integral surfaces for the 5-dimensional unit sphere endowed with the usual structure of the Sasakian space form. Firstly, the authors derived the equations (one for the tangent part and one for the normal part) characterizing the biharmonic integral submanifolds and the equations for the biharmonic C-parallel integral submanifolds of maximum dimension in a certain Sasakian space form. By using these equations, the authors classified all the 3-dimensional biharmonic C-parallel integral submanifolds in a Sasakian space form of dimension 7. Furthermore, by considering the unit sphere $S^7$, endowed with the usual structure of Sasakian space form, having holomorphic curvature 1, and then the one with a modified structure (defined by Tanno), with a Sasakian space form, the authors found the parametric equations of the biharmonic C-parallel submanifolds. In the last section of the paper, by employing the Hopf fibration from the $S^7$ to the complex projective space $S^7$, the authors determined all the biharmonic parallel Lagrangian submanifolds in $CP^3$.

Another work in progress is "*Biharmonic submanifolds with parallel mean curvature vector in* $S^n \times R$", authors: Cezar Oniciuc, Harold Rosenberg and Dorel Fetcu. In a recent paper by Y.-L. Ou and Z.-P. Wang, they have studied the biharmonic surfaces of constant mean curvature (CMC surfaces) in 3-dimensional spaces endowed with Thurston geometries. In particular, they have studied the biharmonic CMC surfaces in $S^2 \times R$. In general, the CMC surfaces in product spaces of type $M^2(c) \times R$, where $M^2(c) \times R$ is a simple connected surface of constant constant sectional curvature $c$ (or, more generally, submanifolds with parallel mean curvature vector in $M^2(c) \times R$, called PMC submanifolds) have been intensively studied (for example, by Abresch, Alencar, do Carmo, Rosenberg etc.). In the above mentioned work in progress, the authors obtained a

Simons type formula for PMC submanifolds in $M^n \times R$. Then, using this formula and some known result obtained by Yau, the proved a "gap" type result for the mean curvature of a PMC biharmonic submanifold in $S^n \times R$. Here, the mean curvature is either 1 or takes values in an interval $(0, a]$, where $a$ is a constant strictly less than 1. In the second part of the paper, all the PMC biharmonic surfaces $S^n \times R$ were classified, with the aid of some know reduction results.

This paper was presented the Cezar Oniciuc at the seminar *Geometry and Analysis* in the Department of Mathematics, University of Cagliari, Italy.

Another work close to completion is *On complete submanifolds with parallel mean curvature in product spaces*, by Dorel Fetcu and Harold Rosenberg. In this article, they obtained an equation of Simons type for submanifolds with parallel mean curvature vector field in normal bundle (PMC submanifolds) in product spaces of type $M^n(c) \times R$. Then, the authors have used this formula to show some new results on the co-dimension reduction for some of these submanifols. More specifically, they calculated the Laplacean of the second fundamental form of a PMC submanifold and then, using the obtained formula, they found necessary conditions for a $m$-dimensional PMC submanifold in a product space to become a constant mean curvature hypersurface in $M^{m+1}(c)$. They also demonstrated some "gap" type theorems for the minimal submanifolds in product spaces, and, more generally, for submanifolds having the parallel mean curvature vector field in normal bundle

This paper was presented by Dorel Fetcu at two seminars: the *Sextas matematicas* seminar in the Department of Mathematics, Federal University of Rio de Janeiro, and the *Differential Geometry* seminar in the Instituto Nacional de Matematica Pura e Aplicada (IMPA), Brazil.

The paper "*New results toward the classification of biharmonic submanifolds in $S^n$*", authors Adina Balmus, Stefano Montaldo and Cezar Oniciuc, was presented at *The 10th International Workshop on Differential Geometry and its Applications, August 26-30, 2011, Constanta*. The paper includes some old results obtained by the authors, as well as some new ones, such as the classification of the biharmonic parallel submanifolds in an Euclidean unit sphere, or some co-dimension reduction results. In addition, the paper contains the proofs of two results obtained by J.H. Chen on biharmonic hypersurfaces in $S^n$. The original demonstrations were published in an article written in Chinese. They have been translated recently by Juan Yang. The authors of the above mentioned presented paper have used here a rather global and invariant approach, eliminating some of the unnecessary steps that were no longer required in this new setting.

In an article in progress, authored by Eric Loubeau, Cezar Oniciuc and Iulian Stoleriu, they studied the existence and the properties of periodic solutions of a system of nonlinear differential equations of the second order that appears in the study biharmonic applications.
The investigation method employed here is based on the theory of (local) centre manifold. The original system of differential equations (which is equivalent to a system

of four differential equations of first order) is reduced to a system of two first order differential equations. According to the centre manifold theory, at least in the vicinity of an equilibrium, the dynamics of the system will be topologically equivalent to the dynamics of the reduced system dynamics equations, which is much easier to analyze and there is plenty of literature available. Using appropriate variable changes, the original system was firstly transformed into a standard form, which is appropriate for the application of the theory of (local) centre manifold. One obtained an approximation for the local centre manifold and approximations for the flow dynamics on this manifold. Because the solution trajectories of the reduced system (including periodic orbits) are transported homeomorphic into trajectories (periodic orbits) for initial system, we deduce that, at least in the vicinity of the equilibrium point, the initial system of equations admits periodic solutions. Furthermore, by using the method of Lindstedt, one can approximate the period of the oscillations generated by the reduced dynamical.

**Accepted papers:**

1) D. Fetcu, C. Oniciuc. *Biharmonic integral C-parallel submanifolds in 7-dimensional Sasakian space forms,* accepted in Tohoku Mathematical Journal.

**Work in progress:**

1) D. Fetcu, C. Oniciuc, H. Rosenberg. *Biharmonic submanifolds with parallel mean curvature vector in $S^n$ x R.*
2) D. Fetcu, H. Rosenberg. *On complete submanifolds with parallel mean curvature in product spaces.*
3) A. Balmus, S. Montaldo, C. Oniciuc. *New results toward the classification of biharmonic submanifolds in $S^n$.*
4) E. Loubeau, C. Oniciuc, I. Stoleriu, *On biharmonic maps from torus into spheres.*