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Biharmonic maps and submanifolds in various geometrical contexts
Principal Investigator: Prof. Dr. Cezar Oniciuc

Scientific Report
(2011-2014)

I. Approaching the project's objectives. Published and accepted for publication articles.

O1. The first objective was the *Study of biharmonic submanifolds in 7-dimensional Sasakian space forms and in complex projective space form* CP^3. We reached the objective and our results can be found in the paper

1. D. Fetcu, C. Oniciuc, ***Biharmonic integral C-parallel submanifolds in 7-dimensional Sasakian space forms***, *Tohoku Mathematical Journal* 64(2), 195-222, 2012.

This paper continues the study of biharmonic integral submanifolds in Sasakian space forms. T. Sasahara was the first to study these submanifolds in the 5-dimensional unit sphere endowed with its canonical Sasakian structure and, therefore, viewed as a Sasakian space form with constant φ-sectional curvature equal to 1.
In our paper, we determined the equations (for the tangent and normal parts) that characterize biharmonic integral submanifolds and biharmonic integral C-parallel submanifolds of maximum dimension in a Sasakian space form. Then, using these equations, were classified all 3-dimensional biharmonic integral C-parallel submanifolds in a 7-dimensional Sasakian space form. Next, working in the unit sphere endowed with its canonical Sasakian structure, as well as with its deformed Sasakian structure defined by Tanno, we found the parametric equations of biharmonic integral C-parallel submanifolds. In the last section of the paper, using the Hopf fibration, we determined all biharmonic Lagrangian parallel submanifolds of CP^3.

O2. The second objective of the project, *Study of biharmonic submanifolds with additional properties in spheres*, was reached through the papers:

1. A. Balmus, S. Montaldo, and C. Oniciuc, ***Biharmonic PNMC submanifolds in spheres***, *Arkiv für Matematik* 51(2), 197-221, 2013.

In this paper were obtained some rigidity results for biharmonic PNMC submanifolds in S^n. PNMC submanifolds (i.e., submanifolds with normalized mean curvature vector field parallel in the normal bundle) are a generalization of PMC submanifolds (i.e., submanifolds with mean curvature vector field parallel in the normal bundle) and a rigidity result provides sufficient conditions for a PNMC submanifold to be PMC.
We characterized the biharmonicity of PNMC submanifolds in S^n in terms of their mean curvature function and Weingarten operator associated to normalized mean curvature vector field.

In the compact case, we proved rigidity results imposing certain restrictions to the norm of the Weingarten operator and the mean curvature.
In the complete non-compact case, imposing restrictions to the Ricci curvature and using the Omori-Yau Maximum Principle, similar results were obtained.
Then, we classified biharmonic PNMC submanifolds with at most two principal curvatures in the direction of the mean curvature vector field in spheres. In particular, we completely determined the biharmonic PNMC submanifolds in S^n.
Next, we investigated the type, in B.-Y. Chen's sense, of proper-biharmonic submanifolds in S^n. We proved that:

- a proper-biharmonic submanifold in S^n is of type 1 or 2 if and only if its mean curvature is either constant $f=1$ or $f \in (0,1)$, respectively;
- there are no proper-biharmonic PNMC submanifolds of type 3 in S^n.

2. R. Caddeo, S. Montaldo, C. Oniciuc, and P. Piu, ***Surfaces in three-dimensional space forms with divergence-free stress-bienergy tensor***, *Annali di Matematica Pura ed Applicata* 193(2), 529-550, 2014.

In this paper, the authors introduced the notion of a biconservative submanifold. A submanifold is called biconservative if the tangent part of the bitension field vanishes. Equivalently, a submanifold is biconservative if and only if the divergence of the stress-bienergy tensor vanishes. The stress-bienergy tensor field comes from a variational principle. The notion of a biconservative submanifold is less restrictive than that of a biharmonic submanifold, a fact that allowed us to obtain many examples. In this paper we explicitly found all biconservative surfaces in 3-dimensional Euclidean sphere, as well as in 3-dimensional hyperbolic space. An interesting fact is that that biconservative surfaces in R^3 and S^3 are rotational surfaces of a certain type.

3. E. Loubeau, C. Oniciuc, ***Biharmonic surfaces of constant mean curvature***, *Pacific Journal of Mathematics* 271(1), 213-230, 2014.

In this paper, we found a Simons type formula for the bitension field of a biharmonic map from a surface to a Riemannian manifold. When such a map is a Riemannian immersion, we obtained rigidity results for CMC surfaces, pointing out the influence of the Gaussian curvature on the pseudo-umbilicity. Next, we showed that the biharmonic hypothesis allows us to extend the classical theorem of Hopf on CMC surfaces to CMC surfaces in an arbitrary Riemannian manifold.

4. S. Montaldo, C. Oniciuc, and A. Ratto, ***Biconservative surfaces***, *Annali di Matematica Pura ed Aplicata*, to appear.

In this paper, we obtained geometric properties of biconservative surfaces in a Riemannian manifold. In particular, we studied the relationship between biconservative surfaces and the holomorphy of generalized Hopf function. We also obtained a complete classification of CMC biconservative in a 4-dimensional space form.

5. S. Montaldo, C. Oniciuc, and A. Ratto, ***Proper biconservative immersions into the Euclidean space***, *Journal of Geometric Analysis*, to appear.

In this paper, using techniques from the geometric equivariant theory, we studied biconservative $SO(p+1)\times SO(q+1)$ - invariant hypersurfaces in the Euclidean space R^{p+q+2} and biconservative SO^{p+1} - invariant in R^{p+2}. Moreover, we showed that none of the immersions in these two invariant families is proper-biharmonic.

6. E. Loubeau, C. Oniciuc, ***CMC proper-biharmonic surfaces of constant Gaussian curvature in spheres***, preprint 2014.

In this paper, we studied biharmonic CMC surfaces in spheres. Inspired by a paper of Miyata, we completely described these immersions and, in the flat case, we proved that, for any $h\in(0,1)$, there are proper-biharmonic CMC planes and cylinders in S^5, with $|H|=h$, and we found a necessary and sufficient condition, in terms of h, for the existence of CMC proper-biharmonic tori in S^5, with mean curvature h.

O3. The third objective of the project, *Study of submanifolds with parallel mean curvature vector field,* was reached through:

1. D. Fetcu, C. Oniciuc, and H. Rosenberg, ***Biharmonic Submanifolds with parallel mean curvature in*** $S^n\times R$, *Journal of Geometric Analysis* 23(4), 2158-2176, 2013.

In a recent paper, Y.-L. Ou and Z.-P. Wang studied biharmonic CMC surfaces in 3-dimensional Thurston geometries. In particular, they studied these surfaces in $S^2\times R$. In general, CMC surfaces in product spaces of type $M^n(c)\times R$, where $M^n(c)$ is an n-dimensional space form with constant sectional curvature c, and, even more generally, PMC surfaces in these spaces were intensively studied (for example, by U. Abresch, H. Rosenberg, M. do Carmo, R. Tribuzy, F. Urbano, etc.).
In our paper, computing the Laplacian of the squared norm of the shape operator in the direction of the mean curvature vector field, we obtained a Simons type formula for PMC submanifolds in such a space and we used it to prove a gap theorem for the mean curvature of a proper-biharmonic PMC submanifold in $S^n\times R$, that shows that either the mean curvature is equal to 1 or its range is $(0,a]$, where $a<1$ is a constant. Next, we classified all proper-biharmonic PMC surfaces in $S^n\times R$.
Our results were presented at the *Geometry and Analysis* seminar at the Department of Mathematics, University of Cagliari, Italy, by the PI of this project.

2. D. Fetcu, H. Rosenberg, ***On complete submanifolds with parallel mean curvature in product spaces***, *Revista Matematica Iberoamericana*, 29(4), 1283-1306, 2013.

This paper is concerned with PMC submanifolds in . We first found a Simons type formula for these submanifolds and then we used it to prove reduction of codimension results. More exactly, we computed the Laplacian of the squared norm of the second fundamental form and we used this formula to find sufficient conditions for an m-dimensional PMC submanifold in $M^n(c)\times R$ to be a hypersurface in $M^{m+1}(c)$. These conditions are given in terms of the norm of the second fundamental form and the angle between the mean curvature vector field and the unit vector field tangent to R. We have also proved gap theorems for minimal and, in general, for PMC submanifolds in product spaces. These results were presented by D. Fetcu at the *Sextas Matematicas* seminar at the Department of Mathematics of the Federal University of Rio de Janeiro, Brazil, and at the *Differential Geometry* seminar at the National Institute of Pure and Applied Mathematics (IMPA), Rio de Janeiro, Brazil.

3. A. Balmus, S. Montaldo, and C. Oniciuc, ***New results toward the classification of biharmonic submanifolds in*** S^n, *Analele Stiintifice ale Universitatii Ovidius Constanta – seria Matematica* 20(2), 89-113, 2012.

This papers brings together some of the authors' older results and also some new ones, like the classification of biharmonic parallel submanifolds in the unit Euclidean sphere or reduction of codimension results. Moreover, the paper also contains the proofs of two results of J. H. Chen concerning biharmonic hypersurfaces in S^n. The original proofs can be found in a paper written (and published) in Chinese. These proofs were translated by Juan Yang. In our paper, we presented in an invariant global formalism.
We presented this paper at *The 10th International Workshop on Differential Geometry and its Applications, 26-30 August 2011, Constanta*.

4. D. Fetcu, A. L. Pinheiro, ***Biharmonic surfaces with parallel mean curvature in complex space forms***, *Kyoto Journal of Mathematics*, to appear.

This paper is concerned with the biharmonicity of PMC surfaces in complex space forms. First, using a holomorphic differential defined on such a surface, introduced by D. Fetcu in a paper from 2012, we find some properties on the pseudo-umbilicity and the flatness of a biharmonic PMC surface. Then, using these results, we obtain the complete classification of proper-biharmonic complete pmc surfaces with non-negative Gaussian curvature. Thus, we prove that such surfaces only exist in the complex projective space form, are totally real, and of three distinct types: pseudo-umbilical surfaces, or Lagrangian surfaces in CP^2, or the direct product of two complete helices in CP^3. In the first two cases, the surfaces are described by using their horizontal lifts to odd dimensional spheres, via Hopf fibration, and, in the third one, for the two curves are determined the curvatures and the complex torsions and it is shown that these results also implies their existence.

O4 & O5. For these two objectives, *Study of biharmonic maps in Lie groups* and *Analytical study of certain equations of biharmonic type*, we mention:

1. E. Loubeau, C. Oniciuc, and I. Stoleriu, ***On biharmonic maps from torus into spheres***, in preparation.

In this paper, we study the existence and qualitative properties of periodic solutions of a two non-linear differential equations of second order system that appears in the study of biharmonic maps. This system is autonomous, with two degrees of freedom, time reversible, and it admits a first integral. Since the eigenvalues of the linearized system are imaginary and non-semi-simple, the classical Liapunov method of the central manifold can not be used in this case. Using the Liapunov-Schmidt reduction technique, it was proved that in small neighborhood of the interest equilibrium point, this system admits periodic solutions. For convenient initial values, these solutions can be approximated (Lindstedt-Poincare method) until the desired order. The next step is to see if the observed periodic solutions are different from the harmonic ones. This can be observed by studying the bifurcation diagram, obtained by artificially introducing two real parameters in the initial system. The periodic solutions that are different from the harmonic ones will correspond to some biharmonic maps from R^2 to S^2.

2. M. Crasmareanu, I. Stoleriu, ***Nonholonomic dynamics of second order and the Heisenberg spinning particle***, *International Journal of Geometric Methods in Modern Physics* 9(7), 1-9, 2012.

In this paper, it is studied a system of differential equations that models the nonholonomic dynamics of second order of Heisenberg particle.

O6. The sixth objective of the project, *Study of direct images of certain coholomogical objects through classes of morphisms on manifolds*, was reached through the paper

1. J. I. Burgos Gil, G. Freixas, and R. Litcanu, ***Generalized holomorphic analytic torsion***, *Journal of the European Mathematical Society* 16(3), 463-535, 2014.

To formulate and prove an arithmetic version of Grothendieck-Riemann-Roch Theorem, a generalization of the notion of analytic torsion as defined by Ray and Singer was necessary, thus being defined the superior analytical torsion classes. Gillet and Soule observed that, once a coherent theory of these classes is defined, the arithmetic Grothendieck-Riemann-Roch Theorem is a natural consequence.
The Grothendieck-Riemann-Roch Theorem provides a formula that establishes the relationship between direct images and characteristic classes of vector bundles on complex manifolds or, more generally, on algebraic manifolds. If the vector bundles are endowed with Hermitian metrics, Chern-Weil theory associates to characteristic classes a system of representatives given by differential forms.
In general, the formula given by the Grothendieck-Riemann-Roch Theorem does not remain valid at the level of of these characteristic forms. This problem was solved by introducing the Bott-Chern singular classes, in the case of closed immersions, and the superior analytic torsion classes, in the case of smooth morphisms.
In this paper, we extend the holomorphic analytic torsion classes defined by J.-M. Bismut and K. Köhler to the case of arbitrary projective morphisms between smooth complex manifolds. In order to do this, we propose an axiomatic definition and we classify the possible generalized analytic torsion, that correspond to this definition. Thus, we prove that any such theory biunivocally corresponds to a characteristic aditive class associated to vector bundles.
The extension of the analytic torsion defined by Bismut and Köhler is obtained in the context of the generalized theory corresponding to the characteristic class $-R/2$, R being the classical R-genus that appears in the definition of analytic torsion.
As applications of the axiomatic characterization, we obtain simpler proofs of some properties of analytic torsion classes, we give a characterization of the R-genus, and construct direct images for generalized Hermitian structures through projective morphisms.
The axiomatic approach and the obtained applications are essential ingredients in proving a Grothendieck-Riemann-Roch type arithmetic theorem, for arbitrary projective morphisms. This result will be published in a subsequent article. In particular, we expect to obtain relevant results on direct images of the various structures endowed with Hermitian metrics through closed immersions with peculiar properties.

O7. As a step towards reaching the objective *Editing a monograph on biharmonic maps on manifolds*, we mention that C. Oniciuc elaborated and defended his Habilitation Thesis ***Biharmonic submanifolds in space forms***. This thesis was presented at the Institute of Mathematics Simion Stoilow of the Romanian Academy, Bucharest, on 25 July 2012. The thesis contains all known results obtained by the author and his collaborators on biharmonic submanifolds. This work will be submitted for publication as a monograph.

List of published and accepted for publication papers

1. D. Fetcu, C. Oniciuc, ***Biharmonic integral C-parallel submanifolds in 7-dimensional Sasakian space forms***, *Tohoku Mathematical Journal, 64 (2), 195-222, 2012.*
2. Balmus, S. Montaldo, and C. Oniciuc, ***Biharmonic PNMC submanifolds in spheres***, *Arkiv für Matematik, 51 (2), 197-221, 2013.*
3. R. Cadeo, S. Montaldo, C. Oniciuc, and P. Piu, ***Surfaces in three-dimensional space forms with divergence-free stress-bienergy tensor,*** *Annali di Matematica Pura ed Applicata (4), 193 (2), 529-550, 2014.*
4. E. Loubeau, C. Oniciuc, ***Biharmonic surfaces of constant mean curvature,*** *Pacific J. Math., 271 (1), 213-230, 2014.*
5. S. Montaldo, C. Oniciuc, and A. Ratto, ***Biconservative surfaces***, *Annali di Matematica Pura ed Aplicata*, to appear.
6. S. Montaldo, C. Oniciuc, and A. Ratto, ***Proper biconservative immersions into the Euclidean space***, *Journal of Geometric Analysis*, to appear.
7. D. Fetcu, C. Oniciuc, and H. Rosenberg, ***Biharmonic Submanifolds with Parallel Mean Curvature in*** $S^n \times R$, *Journal of Geometric Analysis, 23 (4), 2158–2176, 2013.*
8. D. Fetcu, H. Rosenberg, ***On complete submanifolds with parallel mean curvature in product spaces***, *Revista Matematica Iberoamericana*, 29(4), 1283-1306, 2013.
9. Balmus, S. Montaldo, and C. Oniciuc, ***New results toward the classification of biharmonic submanifolds in*** S^n, *Analele stiintifice ale Universitatii Ovidius Constanta - seria Matematica, 20 (2), 89-113, 2012.*
10. D. Fetcu, A. L. Pinheiro, ***Biharmonic surfaces with parallel mean curvature in complex space forms,*** *Kyoto Journal of Mathematics, to appear.*
11. M. Crasmareanu, I. Stoleriu, ***Nonholonomic dynamics of second order and the Heisenberg spinning particle,*** *International Journal of Geometric Methods in Modern Physics, 9 (7), 1-9, 2012.*
12. J. I. Burgos Gil, G. Freixas si R. Litcanu, ***Generalized holomorphic analytic torsion***, *Journal of the European Mathematical Society, 16 (3), 463-535, 2014.*

List of submitted papers

1. E. Loubeau, C. Oniciuc, ***CMC proper-biharmonic surfaces of constant Gaussian curvature in spheres***.

II. Mobilities financed through the project

- May 2012: Professor Eric Loubeau, Université de Bretagne Occidentale, Brest, France, visited the Alexandru Ioan Cuza University of Iasi as an Invited Professor. He presented several conferences at the Workshop *Riemannian Geometry and Harmonic Maps*.

- 2-8 December 2012: Professor Stefano Montaldo, Università degli Studi di Cagliari, Italy, visited the Alexandru Ioan Cuza University of Iasi as an Invited Professor. He presented the conference *An Introduction to Wilmore Surfaces*.
- 4-15 November 2012, Cezar Oniciuc visited Université de Bretagne Occidentale, Brest, France, as an Invited Professor. He continued the collaboration with Professor Eric Loubeau on the study of biharmonic maps from R^2 to S^2.
- January-February 2012, Cezar Oniciuc visited Università degli Studi di Cagliari, Italy, where he studied biconservative hypersurfaces. He obtained a local classification for biconservative surfaces in 3-dimensional space forms. He also taught the course *Riemannian Geometry* at the Doctoral School of the Department of Mathematics and presented the conference *Biharmonic submanifolds.*
- 8-13 June 2012, Dorel Fetcu attended the conference *XIVth International Conference in Geometry, Integrability, and Quantization,* Varna, Bulgaria, with the presentation *Simons type formulas for submanifolds with parallel mean curvature in product spaces and applications,* based on the results obtained in three papers, joint works with Harold Rosenberg (IMPA, Rio de Janeiro) and Cezar Oniciuc.

  Dorel Fetcu presented a conference with the same title at the seminar of the Department of Mathematics of the Federal University of Bahia, Salvador, Brazil.
- 17-23 September 2012, Razvan Litcanu visited the Kyoto University, Japan, invited by Professor A. Moriwaki and Professor S. Kawaguchi. During this visit he attended the conference *Paris-Barcelona-Kyoto Seminar on Arakelov Geometry* (Kyoto, 18-21 September) where he presented results from the paper *Generalized holomorphic analytic torsion*.
- Some preliminary results related to the fifth objective of this project were presented by Iulian Stoleriu at two international conferences, *Conference on Applied Mathematics* – CAIM, Chisinau, Moldova, August 2012, and International Conference on *Controlled Deterministic and Stochastic Systems*, Iasi, July 2012, and at two seminars at the Alexandru Ioan Cuza University of Iasi and the Stranthclyde University, Great Britain.
- May 2013: Professor Eric Loubeau, Université de Bretagne Occidentale, Brest, France, visited the Alexandru Ioan Cuza University of Iasi as an Invited Professor.
- 11-18 September 2013: Professor Shun Maeta, Shumei University, Japan, and Professor Hajime Urakawa, Tohoku University, Japan, visited the Alexandru Ioan Cuza University of Iasi as Invited Professors. They presented the conferences *Biharmonic submanifolds and Chen's conjecture* (S. Maeta) and *Geometry of harmonic maps, biharmonic maps, and integrable systems* (H. Urakawa).
- 19-22 September 2013: Cezar Oniciuc attended *The 11th International Workshop on Differential Geometry and its Applications*, Ploiesti, with the presentation *Recent advances in the theory of biharmonic submanifolds.*
- 23 September – 9 October 2013: Cezar Oniciuc visited Università degli Studi di Cagliari, Italy, as an Invited Professor.
- November 2013: Professor Stefano Montaldo, Università degli Studi di Cagliari, Italy, visited the Alexandru Ioan Cuza University of Iasi as an Invited Professor.
- 20 March – 3 April 2014: Cezar Oniciuc visited the Federal University of Bahia, Salvador, Brazil, as an Invited Professor. He presented the conference *Biharmonic submanifolds in spheres.*
- 3-9 May 2014: Cezar Oniciuc visited Università degli Studi di Cagliari, Italy, for a research stage.

- 7-11 July 2014: Iulian Stoleriu attended *The 10th AIMS Conference on Dynamical Systems, Differential Equations and Aplications*, organized by the Autonomous University of Madrid, with the presentation *Periodic solutions for a pair of coupled oscillators at resonance.*
- 3-5 September 2014: Cezar Oniciuc attended the *International Workshop on Finite type submanifolds,* organized by the Istanbul Technical University, with the presentation *Biharmonic and biconservative immersions.*
- 8-12 September 2014: Cezar Oniciuc attended the conference *Real and Complex Differential Geometry,* organized by the Bucharest University, with the presentation *Surfaces in spheres: biharmonic and CMC.*
- 8-12 September 2014: Simona Nistor attended the conference *Real and Complex Differential Geometry,* organized by the Bucharest University.
- 16-29 September 2014: Cezar Oniciuc visited Università degli Studi di Cagliari, Italy, for a research stage.

III. Other activities

- Cezar Oniciuc defended his Habilitation Thesis *Biharmonic submanifolds in space forms* at the Institute of Mathematics Simion Stoilow of the Romanian Academy, Bucharest, on 25 July 2012.
- A mini-course on biharmonic submanifolds.

Dorel Fetcu taught the mini-course (6 hours) *Subvariedades biharmonicas em variedades Riemannianas* (in Portuguese), between 21 and 23 January 2013, to master and doctoral students of the Institute of Mathematics of the Federal University of Bahia in Salvador, as a part of the *Postgraduate Summer Program 2013* of the institute. He presented introductive notions from the theory of biharmonic maps, general results as, for example, the Euler-Lagrange equation associated to the bienergy integral and the second variational formula of the bienergy, existence results for proper-biharmonic submanifolds in real space forms (mainly obtained by A. Balmus, S. Mointaldo, and C. Oniciuc), and also explicit examples and classification results for such submanifolds in complex and Sasakian space forms, obtained by D. Fetcu, S. Montaldo, E. Loubeau, and C. Oniciuc.

IV. New team members

- Between June 2012 and September 2013, Ph.D. student **Andrei Cuzub** was hired as a member of the project's team. His research activity was focused on:

- Green forms and currents on complex projective manifolds. Forms with logarithmic type singularities on complex projective manifolds. Existence results and properties;
- Intersection theory on an algebraic manifold and, more generally, on a Noetherian regular scheme;
- Moduli problems. Algebraic stacks;
- Arakelov intersection theory on arithmetic manifolds;
- Belyi Theorem of characterization of algebraic curves defined over a numeric field.

The main objective was to obtain associated invariants to algebraic curves over numeric fields using Belyi morphisms and Arakelov intersection theory. More exactly, for a given curve over a numeric field, a Belyi morphism, and a model of the curve over the corresponding ring of integers, we study some Arakelov intersections, that can be obtained this way, looking for invariants with similar

properties to classical heights on algebraic manifolds.
The second objective was to obtain some invariants on algebraic curves using Belyi morphisms and intersection theory on algebraic stacks.
The study of these subjects led to the elaboration of a research project that was presented at the Doctoral School and will be developed into a future Ph.D. Thesis.
Also, between June 2012 and November 2012, Andrei Cuzub attended the symposium organized by the Faculty of Mathematics as a part of Alexadru Ioan Cuza University's Days and the *Informal Seminar Geometric News*, also organized at the Faculty of Mathematics.

- Since 1 October 2013, the Master student **Simona Elena Nistor** was hired as a team member of the project, replacing Andrei Cuzub in this team. At that time, she was a second year Master student of the *Research in Mathematics* section of the Faculty of Mathematics and she was working in the field of Riemannian manifolds. On July 2014 she defended her Dissertation with a subject from this field.
Between 20 September 2013 and 28 February 2014, Simona Nistor went to Università degli Studi di Cagliari, Italy, for a research stage at the Department of Mathematics.
In October 2014, Simona Nistor was enrolled in the Doctoral School, her advisor being Cezar Oniciuc.

Principal Investigator,
Professor Cezar Oniciuc