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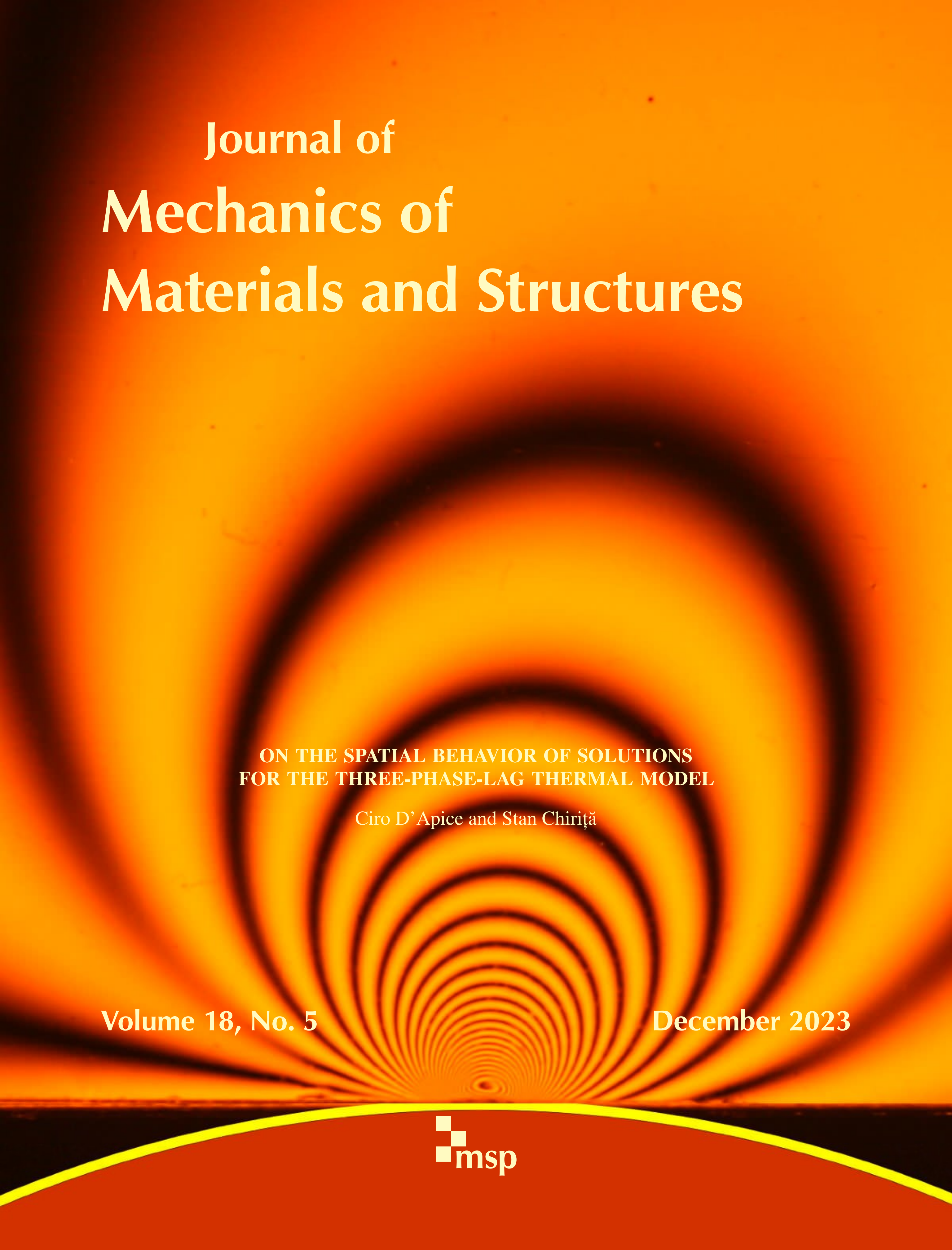

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# ON THE SPATIAL BEHAVIOR OF SOLUTIONS FOR THE THREE-PHASE-LAG THERMAL MODEL

CIRO D'APICE AND STAN CHIRIȚĂ

We present an extensive analysis on the spatial behavior of the solutions within the three-phase-lags model of a rigid heat conductor for a semi-infinite cylinder excited on its base. The relaxation time of the temperature gradient has a special significance here, namely (i) in the absence of this relaxation time, we manage to highlight a theorem of the domain of influence, that is, outside of a region adjacent to the charged base of the cylinder, all the thermal activity is vanishing, (ii) instead, when the relaxation time of the temperature gradient is present, then it is no longer possible to highlight the area of influence, but we can notice the Saint-Venant's effect. In this latter situation, we are able to describe the exponential decay with respect to the distance from the charged base for a measure of the solution, having a suitable time-dependent exponent to show the rapid decay of the effects when a small time leak has occurred. For the situations when the base of the cylinder is excited for a longer time, both the result expressing the domain of influence and the Saint-Venant type exponential decrease estimate provide insufficient information regarding the spatial behavior along the generator of cylinder. To deal with this shortcoming, we establish exponential decay estimates with a time-independent exponent that can be used to describe the spatial behavior even inside the domain of influence.

## 1. Introduction

Recently, Ostoja-Starzewski and Quintanilla [19] reported interesting results concerning the spatial behavior of solutions of a problem related to the Moore–Gibson–Thomson equation,

$$\tau \dddot{u} + \ddot{u} - k\Delta\dot{u} - k^{*}\Delta u = 0, \tag{1}$$

on a three-dimensional semi-infinite cylinder $R = D \times (0, \infty)$ subject to the homogeneous Dirichlet boundary condition $u(x, t) = 0$ over the lateral boundary surface $\partial D \times (0, \infty)$. (Here $\tau > 0$ is a relaxation time, $\Delta$ is the Laplace operator and $k$ and $k^{*}$ are positive parameters under restriction $k > \tau k^{*}$). More specifically, a domain of influence result is obtained and some exponential decays are reported for the solutions along certain space-time lines.

Moreover, the Moore–Gibson–Thompson equation or its versions have been intensively studied in recent years in relation to the problems of uniqueness, continuous data dependence or exponential decay of energy: Kaltenbacher et al. [16], Lasiecka and Wang [17; 18], Dell'Oro et al. [9], Dell'Oro and Pata [8], Pellicer and Said-Houari [21], Chen and Ikehata [2].

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It was suggested for the first time by Quintanilla [22] and then by Ostoja-Starzewski and Quintanilla [19] that the Moore–Gibson–Thomson equation (1) can be obtained as a particular case of the three-phase-lag model proposed by Roy Choudhuri [6] by a convenient choice of the three times of relaxation (something that would be equivalent to the fact that the relaxation times of the temperature gradient and the thermal displacement gradient are negligible). Besides, there is recent rich and intense research activity of the third-order in time differential equations, either separately or coupled with the elastic deformations of media, leading to valuable results concerning the well-posedness: Quintanilla [22], Pellicer and Quintanilla [20], Conti et al. [7], Bazarra et al. [1], Fernandez and Quintanilla [10; 11], Fernandez et al. [12].

Inspired by Quintanilla's observation in [22] and that in the paper by Ostoja-Starzewski and Quintanilla [19] concerning the spatial behavior of the solutions of the Moore–Gibson–Thomson equation, we were interested to see how the solutions of the corresponding equation behave in the model presented by Roy Choudhuri [6]. In this regard, we considered here a cylinder made of a three-phase-lag rigid conductor material that is excited on its base by a prescribed thermal flux, and its lateral boundary surface being free from thermal exchanges. Then we were interested in how the effects of the excited base are felt along the semi-infinite cylinder generators.

Our analysis in this paper on the spatial behavior in the semi-infinite cylinder shows that there is a consistent difference between what the particular three-phase-lag model indicated by Quintanilla predicts and what Roy Choudhuri's general model predicts. While in Ostoja-Starzewski and Quintanilla's paper a result expressing a so-called domain of influence is established, this is no longer possible in our present analysis. An important role in our analysis is played by phase-lag $\tau_T$ of the temperature gradient: (i) if $\tau_T$ is negligible then the considered model is of hyperbolic type and a theorem of domain of influence in the cylinder is established, (ii) if $\tau_T$ is not negligible, then the considered model no longer allows the description of a domain of influence, but instead it allows us to obtain appropriate exponential decrease estimates of Saint-Venant type.

The plan of the work is the following. Section 2 presents the basic system of differential equations describing the evolutionary behavior of the heat flux and of the thermal displacement in the line described by Roy Choudhuri [6]. The associated third-order in time differential equation in terms of thermal displacement is explicitly written. Section 3 formulates the initial boundary value problem associated with the model in concern and then describe its modified version that will be useful in the future analysis. Section 4 is dedicated to a possible measure of the solution under common assumptions on the thermal coefficients. Section 5 supposes that the phase-lag of the temperature gradient is negligible and, consequently, a domain of influence is established. Section 6 considers the general model as proposed by Roy Choudhuri [6] and establishes estimates describing the exponential decay of the excited base effects, both with a time-dependent exponent and with a uniform exponent in time.

## 2. Three-phase-lag heat conducting model for a rigid conductor

Tzou [23] has proposed a two-phase-lag model by generalizing Fourier's law of heat conduction $q_i(x,t) = -k_{ij}\theta_{,j}(x,t)$ in the following form,

$$q_i(x, t+\tau_q) = -k_{ij}(x) T_{,j}(x, t+\tau_T), \tag{2}$$

where the $q_i$ are the components of the heat flux vector, $T$ is the variation of the temperature from the constant reference temperature $T_0 > 0$, the $k_{ij}$ are the components of the thermal conductivity tensor and the delay time $\tau_q > 0$ is the phase-lag of the heat flux and the delay time $\tau_T > 0$ is the phase-lag of the temperature gradient.

On the other hand, Green and Naghdi [13; 14; 15] have introduced the thermal displacement by $\alpha(x,t) = \dot{T}(x,t)$ and they have proposed a heat conduction law as

$$q_i(x,t) = -\big[k_{ij}(x)T_{,j}(x,t) + k^*_{ij}(x)\alpha_{,j}(x,t)\big], \tag{3}$$

where the $k^*_{ij}$ are the components of the conductivity rate tensor.

Roy Choudhuri [6] combined the two above models and he proposed the three-phase-lags for the heat flux vector $q_i$, that is he considers the following generalized constitutive equation for heat conduction in order to describe the lagging behavior,

$$q_i(x, t+\tau_q) = -\big[k_{ij}(x)T_{,j}(x, t+\tau_T) + k^*_{ij}(x)\alpha_{,j}(x, t+\tau_\alpha)\big], \tag{4}$$

where $\tau_\alpha > 0$ is the phase-lag of the thermal displacement gradient.

Further, by taking Taylor's series expansion of (4) up to the first-order terms in $\tau_q$, $\tau_T$ and $\tau_\alpha$, Roy Choudhuri proposes the following generalized heat conduction law valid at a point $x$ at time $t$:

$$q_i(x,t) + \tau_q\dot{q}_i(x,t) = -k^*_{ij}(x)\alpha_{,j}(x,t) - \big(k_{ij}(x) + \tau_\alpha k^*_{ij}(x)\big)\dot{\alpha}_{,j}(x,t) - \tau_T k_{ij}(x)\ddot{\alpha}_{,j}(x,t). \tag{5}$$

Equation (5) serves as a generalized constitutive heat conduction law in which the elastic deformation term is ignored.

The three-phase-lags model for a rigid conductor as proposed by Roy Choudhuri [6] is based on the constitutive equation (5) and the well-known heat equation

$$-q_{i,i}(x,t) + r(x,t) = c(x)\dot{T}(x,t), \tag{6}$$

where $r(x,t)$ represents the heat source acting per unit volume and $c(x)$ is the specific heat per unit volume.

In terms of the thermal displacement $\alpha$, the three-phase-lags model for a rigid conductor as proposed by Roy Choudhuri [6] is based upon the following differential equation

$$c\tau_q\dddot{\alpha} + c\ddot{\alpha} - (k^*_{ij}\alpha_{,j})_{,i} - [(k_{ij} + \tau_\alpha k^*_{ij})\dot{\alpha}_{,j}]_{,i} - \tau_T(k_{ij}\ddot{\alpha}_{,j})_{,i} = 0, \tag{7}$$

where we assumed the vanishing of the heat source and, moreover, the dependence on the independent variables $x$ and $t$ was suppressed, but implicitly understood.

It can be easily seen that (7) generalizes the Moore–Gibson–Thompson equation in the sense that a neglect of the terms containing the relaxation times $\tau_T$ and $\tau_\alpha$ leads to equation (1). Consequently, all our analysis in this paper will remain valid in the case of the simpler model used in the studies by Kaltenbacher et al. [16], Lasiecka and Wang [17; 18], Pellicer and Said-Houari [21] and Chen and Ikehata [2].

The constitutive equation (5) is compatible with the second law of thermodynamic if [5]

R(i) the tensor $k_{ij}$ is positive semidefinite,

R(ii) the tensor $\kappa_{ij} = k_{ij} + (\tau_\alpha - \tau_q)k^*_{ij}$ is positive semidefinite.

## 3. Formulation of the initial boundary value problem and its modified form

Throughout this paper we shall assume that a semi-infinity cylindrical region $B = D \times (0, \infty)$ is filled by a homogeneous and anisotropic conductor material with three-phase-lag times. We choose a Cartesian coordinate system $Ox_1x_2x_3$ in such a way that the base of the cylinder is contained in the plane $x_3 = 0$, and the axis $Ox_3$ is parallel to the generators of the cylindrical surface. Throughout this paper we consider the initial boundary value problem $\mathcal{P}$ defined by the heat equation (6), with null heat source, the constitutive equation (5), the initial conditions

$$\alpha(x, 0) = 0, \quad \dot{\alpha}(x, 0) = 0, \quad q_i(x, 0) = 0 \quad \text{for all } x \in B \equiv D \times (0, \infty), \tag{8}$$

and the boundary conditions

$$\begin{aligned} &q_\rho(x_1, x_2, x_3, t)n_\rho = 0 && \text{for all } (x_1, x_2, x_3) \in [\partial D \times (0, \infty)] \text{ and } t \in (0, \infty), \\ &q_3(x_1, x_2, 0, t) = g(x_1, x_2, t) && \text{for all } (x_1, x_2) \in D_0 \text{ and } t \in (0, \infty). \end{aligned} \tag{9}$$

Here the $n_\rho$ are the components of the outward normal to the lateral surface of the cylinder and $g(x_1, x_2, t)$ is a prescribed smooth function and $D_0$ is the base section of the cylinder.

By a solution of the initial boundary value problem $\mathcal{P}$, corresponding to the given data $g(x_1, x_2, t)$ we mean the ordered array $\mathcal{S} = \{\alpha, q_i\}$ defined on $B \times (0, \infty)$ with the properties that

$$\alpha(x, t) \in C^{2,2}(B \times (0, \infty)), \quad q_i(x, t) \in C^{1,1}(B \times (0, \infty))$$

and which satisfy the field equations (5) and (6), the initial conditions (8) and the boundary conditions (9).

Throughout this paper we are interested in how the solution of the initial boundary value problem $\mathcal{P}$ behaves with respect to the distance $x_3$ at the loaded base $x_3 = 0$. In this sense, we want to identify appropriate measures associated with the solution $\mathcal{S} = \{\alpha, q_i\}$ of the problem in question $\mathcal{P}$ that describe its behavior in terms of the distance $x_3$ to the base acted by the specified load $g(x_1, x_2, t)$.

In order to be able to conveniently deal with topics regarding the spatial behavior of solutions, it is necessary to define a modified initial boundary value problem $\widetilde{\mathcal{P}}$ associated with the problem in question $\mathcal{P}$. In this sense we introduce the notation

$$\tilde{q}_i = q_i + \tau_q \dot{q}_i, \quad \tilde{\alpha} = \alpha + \tau_q \dot{\alpha}, \tag{10}$$

and note that the heat equation (6) implies

$$-\tilde{q}_{i,i} = c\ddot{\tilde{\alpha}} \quad \text{in } B \times (0, \infty), \tag{11}$$

while the constitutive equation (5) becomes

$$\tilde{q}_i = -k^*_{ij}\alpha_{,j} - (k_{ij} + \tau_\alpha k^*_{ij})\dot{\alpha}_{,j} - \tau_T k_{ij}\ddot{\alpha}_{,j} \quad \text{in } B \times (0, \infty). \tag{12}$$

Furthermore, in view of the initial conditions (8) and the heat equation (6), we have

$$\tilde{\alpha}(x, 0) = 0, \quad \dot{\tilde{\alpha}}(x, 0) = 0, \quad \ddot{\alpha}(x, 0) = q_{i,i}(x, 0) = 0 \quad \text{for all } x \in B \equiv D \times (0, \infty), \tag{13}$$

while the boundary conditions (9) furnishes

$$\begin{aligned}&\tilde{q}_\rho(x_1,x_2,x_3,t)n_\rho=0 && \text{for all } (x_1,x_2,x_3)\in[\partial D\times(0,\infty)] \text{ and } t\in(0,\infty),\\ &\tilde{q}_3(x_1,x_2,0,t)=\tilde{g}(x_1,x_2,t) && \text{for all } (x_1,x_2)\in D_0 \text{ and } t\in(0,\infty),\end{aligned} \tag{14}$$

with $\tilde{g}=g+\tau_q\dot{g}$.

It is useful for what follows to note that the constitutive equation (12) can be written in the form

$$\tilde{q}_i=-k^*_{ij}\tilde{\alpha}_{,j}-\kappa_{ij}\dot{\alpha}_{,j}-\tau_T k_{ij}\ddot{\alpha}_{,j} \quad \text{in } B\times(0,\infty), \tag{15}$$

where

$$\kappa_{ij}=k_{ij}+(\tau_\alpha-\tau_q)k^*_{ij}. \tag{16}$$

For future use, we write (15) as

$$\tilde{q}_i=\widetilde{Q}_i+\widetilde{R}_i, \tag{17}$$

where

$$\widetilde{Q}_i=-k^*_{ij}\tilde{\alpha}_{,j}-\kappa_{ij}\dot{\alpha}_{,j}, \tag{18}$$

and

$$\widetilde{R}_i=-\tau_T k_{ij}\ddot{\alpha}_{,j}. \tag{19}$$

Further, we note the following estimates

$$|\widetilde{Q}_3|\le(k^*_{3j}k^*_{3j})^{1/2}(\tilde{\alpha}_{,i}\tilde{\alpha}_{,i})^{1/2}+(\kappa_{3j}\kappa_{3j})^{1/2}(\dot{\alpha}_{,i}\dot{\alpha}_{,i})^{1/2}, \tag{20}$$

$$|\widetilde{R}_3|\le\tau_T(k_{3j}k_{3j})^{1/2}(\ddot{\alpha}_{,i}\ddot{\alpha}_{,i})^{1/2}. \tag{21}$$

In our further analysis we will need some of the following hypotheses upon the characteristic material coefficients:

(H1) The specific heat per unit volume is strictly positive, that is

$$c>0. \tag{22}$$

(H2) $k^*_{ij}$ is a positive definite tensor as

$$k^*_{rs}\xi_r\xi_s\ge k^*_m\xi_i\xi_i \quad \text{for all } (\xi_1,\xi_2,\xi_3)\ne 0. \tag{23}$$

(H3) $k_{ij}$ is a positive definite tensor as

$$k_{rs}\xi_r\xi_s\ge k_m\xi_i\xi_i \quad \text{for all } (\xi_1,\xi_2,\xi_3)\ne 0. \tag{24}$$

(H4) $\kappa_{ij}=k_{ij}+(\tau_\alpha-\tau_q)k^*_{ij}$ is a positive definite tensor, that is

$$\kappa_{rs}\xi_r\xi_s\ge \kappa_m\xi_i\xi_i \quad \text{for all } (\xi_1,\xi_2,\xi_3)\ne 0, \tag{25}$$

where $k^*_m$, $k_m$ and $\kappa_m$ are the smallest eigenvalues of the tensors $k^*_{ij}$, $k_{ij}$ and $\kappa_{ij}$, respectively.

The last two hypotheses represent a strengthening of the two consequences of compatibility with the second law of thermodynamics, while the second hypothesis (H2) is an extension of hypothesis (H3) for the conductivity rate tensor. Finally, the first hypothesis proves that a genuine dynamic thermal situation is considered.

## 4. Possible measure associated with a solution of the initial boundary value problem $\mathcal{P}$

We try to study our problem by using an associated "measure" of the solution $\mathcal{S} = \{\alpha, q_i\}$ like

$$F(x_3, t) = \int_0^t \int_0^s \int_{D_{x_3}} \tilde{q}_3(z)\dot{\tilde{\alpha}}(z)\, da\, dz\, ds, \quad x_3 > 0,\ t > 0, \tag{26}$$

where $D_{x_3}$ is the transverse section of the cylinder with the plane $x_3 = \text{constant}$. We note that

$$\frac{\partial F}{\partial t}(x_3, t) = \int_0^t \int_{D_{x_3}} \tilde{q}_3(s)\dot{\tilde{\alpha}}(s)\, da\, ds, \quad x_3 > 0,\ t > 0. \tag{27}$$

In what follows we try to prove that $F(x_3, t)$ can be considered like a measure of the solution $\mathcal{S} = \{\alpha, q_i\}$. In this connection we note that

$$\frac{\partial F}{\partial x_3}(x_3, t) = \int_0^t \int_0^s \int_{D_{x_3}} \big[\tilde{q}_{3,3}(z)\dot{\tilde{\alpha}}(z) + \tilde{q}_3(z)\dot{\tilde{\alpha}}_{,3}(z)\big]\, da\, dz\, ds. \tag{28}$$

Furthermore, on the basis of (11), the lateral boundary condition (14) and the divergence theorem we obtain

$$\frac{\partial F}{\partial x_3}(x_3, t) = -\int_0^t \int_0^s \int_{D_{x_3}} c\dot{\tilde{\alpha}}(z)\ddot{\tilde{\alpha}}(z)\, da\, dz\, ds + \int_0^t \int_0^s \int_{D_{x_3}} \tilde{q}_i(z)\dot{\tilde{\alpha}}_{,i}(z)\, da\, dz\, ds. \tag{29}$$

The relation (10) and the constitutive equation (12) furnish

$$\tilde{q}_i(z)\dot{\tilde{\alpha}}_{,i}(z) = -\frac{1}{2}\frac{\partial}{\partial z}\big[k^*_{ij}\tilde{\alpha}_{,i}(z)\tilde{\alpha}_{,j}(z) + \tau_T k_{ij}\dot{\alpha}_{,i}(z)\dot{\alpha}_{,j}(z) + \tau_q \kappa_{ij}\dot{\alpha}_{,i}(z)\dot{\alpha}_{,j}(z)\big] - \kappa_{ij}\dot{\alpha}_{,i}(z)\dot{\alpha}_{,j}(z) - \tau_q\tau_T k_{ij}\ddot{\alpha}_{,i}(z)\ddot{\alpha}_{,j}(z), \tag{30}$$

so that, from (29) and null initial data (13), we get

$$-\frac{\partial F}{\partial x_3}(x_3, t) = \frac{1}{2}\int_0^t \int_{D_{x_3}} \big[c\dot{\tilde{\alpha}}(s)^2 + k^*_{ij}\tilde{\alpha}_{,i}(s)\tilde{\alpha}_{,j}(s) + (\tau_q\kappa_{ij} + \tau_T k_{ij})\dot{\alpha}_{,i}(s)\dot{\alpha}_{,j}(s)\big]\, da\, ds + \int_0^t \int_0^s \int_{D_{x_3}} \big[\kappa_{ij}\dot{\alpha}_{,i}(z)\dot{\alpha}_{,j}(z) + \tau_q\tau_T k_{ij}\ddot{\alpha}_{,i}(z)\ddot{\alpha}_{,j}(z)\big]\, da\, dz\, ds. \tag{31}$$

Consequently, in view of our hypotheses (H1) to (H4), we deduce

$$-\frac{\partial F}{\partial x_3}(x_3, t) \geq \frac{1}{2}\int_0^t \int_{D_{x_3}} \big[c\dot{\tilde{\alpha}}(s)^2 + k^*_m\tilde{\alpha}_{,i}(s)\tilde{\alpha}_{,i}(s) + (\tau_q\kappa_m + \tau_T k_m)\dot{\alpha}_{,i}(s)\dot{\alpha}_{,i}(s)\big]\, da\, ds + \int_0^t \int_0^s \int_{D_{x_3}} \big[\kappa_m\dot{\alpha}_{,i}(z)\dot{\alpha}_{,i}(z) + \tau_q\tau_T k_m\ddot{\alpha}_{,i}(z)\ddot{\alpha}_{,i}(z)\big]\, da\, dz\, ds \geq 0, \quad \text{for all } x_3 > 0,\ t > 0, \tag{32}$$

and hence $F(x_3, t)$ is a nonincreasing function with respect to $x_3$ for all $t > 0$. As we will see later, this last inequality suggests that $F(x_3, t)$ can lead to a measure of the solution $\mathcal{S} = \{\alpha, q_i\}$ of our initial boundary value problem $\mathcal{P}$.

## 5. Model without phase-lag of the temperature gradient: domain of influence

Throughout this section we will suppose that there is no phase-lag of the temperature gradient, that is there will vanish the terms containing the relaxation time $\tau_T$. In what follows we will denote by $F_T(x_3, t)$ the corresponding function $F(x_3, t)$. Then the constitutive equation (15) becomes

$$\tilde{q}_i = -k^*_{ij}\tilde{\alpha}_{,j} - \kappa_{ij}\dot{\alpha}_{,j} \quad \text{in } B \times (0, \infty), \tag{33}$$

while the inequality (32) is

$$-\frac{\partial F_T}{\partial x_3}(x_3, t) \geq \frac{1}{2}\int_0^t \int_{D_{x_3}} \big[c\dot{\tilde{\alpha}}(s)^2 + k^*_m\tilde{\alpha}_{,i}(s)\tilde{\alpha}_{,i}(s) + \tau_q\kappa_m\dot{\alpha}_{,i}(s)\dot{\alpha}_{,i}(s)\big]\, da\, ds + \int_0^t\int_0^s\int_{D_{x_3}} \kappa_m\dot{\alpha}_{,i}(z)\dot{\alpha}_{,i}(z)\, da\, dz\, ds \geq 0, \quad \text{for all } x_3 > 0,\ t > 0. \tag{34}$$

Moreover, the relations (18) and (20) imply

$$|\tilde{q}_3| \leq (k^*_{3j}k^*_{3j})^{1/2}(\tilde{\alpha}_{,i}\tilde{\alpha}_{,i})^{1/2} + (\kappa_{3j}\kappa_{3j})^{1/2}(\dot{\alpha}_{,i}\dot{\alpha}_{,i})^{1/2}, \tag{35}$$

so that we get

$$\tilde{q}_3^2 \leq 2M^2\big[k^*_m\tilde{\alpha}_{,i}\tilde{\alpha}_{,i} + \tau_q\kappa_m\dot{\alpha}_{,i}\dot{\alpha}_{,i}\big], \tag{36}$$

where

$$M = \max\left(\left(\frac{k^*_{3j}k^*_{3j}}{k^*_m}\right)^{1/2}, \left(\frac{\kappa_{3j}\kappa_{3j}}{\tau_q\kappa_m}\right)^{1/2}\right). \tag{37}$$

Further, we use the Cauchy–Schwarz and the arithmetic-geometric mean inequalities into relation (27) in order to obtain

$$\left|\frac{\partial F_T}{\partial t}\right|(x_3, t) \leq \frac{1}{2}\int_0^t\int_{D_{x_3}} \Big[\varepsilon c\dot{\tilde{\alpha}}(s)^2 + \frac{1}{\varepsilon c}\tilde{q}_3(s)^2\Big]\, da\, ds \leq \frac{1}{2}\int_0^t\int_{D_{x_3}} \Big\{\varepsilon c\dot{\tilde{\alpha}}(s)^2 + \frac{2M^2}{\varepsilon c}\big[k^*_m\tilde{\alpha}_{,i}(s)\tilde{\alpha}_{,i}(s) + \tau_q\kappa_m\dot{\alpha}_{,i}(s)\dot{\alpha}_{,i}(s)\big]\Big\}\, da\, ds, \tag{38}$$

for any positive parameter $\varepsilon$. Now we choose $\varepsilon$ to be

$$\varepsilon = M\sqrt{\frac{2}{c}}, \tag{39}$$

so that, in view of the estimate (34), we obtain

$$\frac{1}{\varepsilon}\left|\frac{\partial F_T}{\partial t}\right|(x_3, t) \leq -\frac{\partial F_T}{\partial x_3}(x_3, t) \quad \text{for all } x_3 > 0,\ t > 0. \tag{40}$$

This last inequality is equivalent to the differential inequalities

$$\frac{1}{\varepsilon}\frac{\partial F_T}{\partial t}(x_3, t) + \frac{\partial F_T}{\partial x_3}(x_3, t) \leq 0 \quad \text{for all } x_3 > 0,\ t > 0, \tag{41}$$

and

$$-\frac{1}{\varepsilon}\frac{\partial F_T}{\partial t}(x_3, t) + \frac{\partial F_T}{\partial x_3}(x_3, t) \leq 0 \quad \text{for all } x_3 > 0,\ t > 0. \tag{42}$$

Let us first choose $t_0 > 0$ and $x_3^0 \geq \varepsilon t_0$. If we set $t = t_0 + (x_3 - x_3^0)/\varepsilon$ in (41) it results in

$$\frac{d}{dx_3}\left[F_T\left(x_3, t_0 + \frac{x_3 - x_3^0}{\varepsilon}\right)\right] \leq 0, \tag{43}$$

and hence $F_T(x_3, t_0 + (x_3 - x_3^0)/\varepsilon)$ is a nonincreasing function with respect to $x_3$. Thus, if we recall that $0 \leq x_3^0 - \varepsilon t_0 \leq x_3^0$, it results that

$$F_T(x_3^0, t_0) \leq F_T(x_3^0 - \varepsilon t_0, 0) = 0. \tag{44}$$

Further, we set $t = t_0 - (x_3 - x_3^0)/\varepsilon$ in (42) so that it follows that

$$\frac{d}{dx_3}\left[F_T\left(x_3, t_0 - \frac{x_3 - x_3^0}{\varepsilon}\right)\right] \leq 0, \tag{45}$$

and hence $F_T(x_3, t_0 - (x_3 - x_3^0)/\varepsilon)$ is a nonincreasing function with respect to $x_3$. Since $x_3^0 \leq x_3^0 + \varepsilon t_0$ it results that

$$F_T(x_3^0, t_0) \geq F_T(x_3^0 + \varepsilon t_0, 0) = 0. \tag{46}$$

Consequently, from the relations (44) and (46), we deduce that

$$F_T(\infty, t_0) = \lim_{x_3 \to \infty} F_T(x_3, t_0) = 0 \quad \text{for all } t_0 > 0, \tag{47}$$

and hence, by an integration of the relation (34) over $(x_3, \infty)$, we deduce that

$$F_T(x_3, t) \geq \frac{1}{2}\int_0^t \int_{B_{x_3}} \left[c\dot{\tilde{\alpha}}(s)^2 + k_m^* \tilde{\alpha}_{,i}(s)\tilde{\alpha}_{,i}(s) + \tau_q \kappa_m \dot{\alpha}_{,i}(s)\dot{\alpha}_{,i}(s)\right] dv\, ds \\ + \int_0^t \int_0^s \int_{B_{x_3}} \kappa_m \dot{\alpha}_{,i}(z)\dot{\alpha}_{,i}(z)\, dv\, dz\, ds \geq 0, \quad \text{for all } x_3 > 0,\ t > 0, \tag{48}$$

where $B_{x_3} \equiv D \times (x_3, \infty)$. Thus, $F_T(x_3, t)$ appears like a measure of the solution $\mathcal{S} = \{\alpha, q_i\}$ of our initial boundary value problem $\mathcal{P}$.

Finally, we set $x_3 = \varepsilon t$ in (41) to obtain

$$\frac{d}{dt}[F_T(\varepsilon t, t)] \leq 0, \tag{49}$$

so that $F_T(\varepsilon t, t)$ is a nonincreasing function with respect to $t$. Thus, we deduce

$$F_T(\varepsilon t, t) \leq F_T(0, 0) = 0. \tag{50}$$

Since $F_T(x_3, t)$ is a nonincreasing function with respect $x_3$, it follows that for $x_3 \geq \varepsilon t$ we will have

$$F_T(x_3, t) \leq F_T(\varepsilon t, t) \leq 0, \tag{51}$$

which in conjunction with (48) proves

$$F_T(x_3, t) = 0 \quad \text{for all } x_3 \geq \varepsilon t,\ t > 0. \tag{52}$$

In view of the relations (48) and (52), we deduce that

$$\dot{\tilde{\alpha}}(x_1, x_2, x_3, t) = \dot{\alpha}(x_1, x_2, x_3, t) + \tau_q \ddot{\alpha}(x_1, x_2, x_3, t) = 0, \quad (x_1, x_2) \in D_{x_3},\ x_3 \geq \varepsilon t,\ t > 0, \tag{53}$$

which integrated under zero initial conditions gives

$$\alpha(x_1,x_2,x_3,t)=0, \quad (x_1,x_2)\in D_{x_3},\ x_3\geq \varepsilon t,\ t>0. \tag{54}$$

If we substitute this last relation in the constitutive equation (5), we obtain

$$q_i+\tau_q\dot{q}_i=0, \quad (x_1,x_2)\in D_{x_3},\ x_3\geq \varepsilon t,\ t>0, \tag{55}$$

which furnishes $q_i(x_1,x_2,x_3,t)=0$ for all $(x_1,x_2)\in D_{x_3},\ x_3\geq\varepsilon t$ and $t>0$. Thus, there is the following domain of influence result:

$$\mathcal{S}(x_1,x_2,x_3,t)=0, \quad (x_1,x_2)\in D_{x_3},\ x_3\geq \varepsilon t,\ t>0. \tag{56}$$

## 6. Model with phase-lag of the temperature gradient: exponential decay result

We return to the initially considered general thermal model (when $\tau_T>0$) and notice that it is no longer possible to estimate $\partial F/\partial t$ in terms of the first integral in the second member of the inequality (32), and therefore, it is no longer possible to get a result of the domain of influence type like that in the previous section. This is due to the presence of the term $\ddot{\alpha}_{,i}$ in the constitutive equation (5) appeared as a consequence of taking into account the phase-lag of the temperature gradient. For this reason we will follow the path described by Chiriţă [3] and Chiriţă and Ciarletta [4] for the parabolic equations of classical linear thermoelasticity.

**6.1.** ***Spatial decaying result with exponent depending on time.*** By means of the Cauchy–Schwarz and the arithmetic-geometric means inequalities, from (26) we obtain

$$|F(x_3,t)|\leq\frac{1}{2}\int_0^t\int_0^s\int_{D_{x_3}}\Big[\varepsilon_1 c\dot{\tilde{\alpha}}(z)^2+\frac{1}{\varepsilon_1 c}\tilde{q}_3(z)^2\Big]\,da\,dz\,ds, \tag{57}$$

for every positive parameter $\varepsilon_1$. But, the relation (17) implies

$$\tilde{q}_3^2\leq(1+\epsilon)\widetilde{Q}_3^2+\Big(1+\frac{1}{\epsilon}\Big)\widetilde{R}_3^2, \tag{58}$$

for every positive parameter $\epsilon$. Thus, by means of the estimates (20) and (21), we obtain

$$\tilde{q}_3^2\leq(1+\epsilon)2M_1^2\big[k_m^*\tilde{\alpha}_{,i}\tilde{\alpha}_{,i}+(\tau_q\kappa_m+\tau_T k_m)\dot{\alpha}_{,i}\dot{\alpha}_{,i}\big]+\Big(1+\frac{1}{\epsilon}\Big)\frac{\tau_T}{\tau_q}M_2^2(\tau_q\tau_T k_m\ddot{\alpha}_{,i}\ddot{\alpha}_{,i}), \tag{59}$$

where

$$M_1=\max\bigg(\Big(\frac{k_{3r}^*k_{3r}^*}{k_m^*}\Big)^{1/2},\Big(\frac{k_{3r}k_{3r}}{\tau_q\kappa_m+\tau_T k_m}\Big)^{1/2}\bigg),\quad M_2=\Big(\frac{k_{3r}k_{3r}}{k_m}\Big)^{1/2}. \tag{60}$$

Therefore, from (57) and (59), we deduce

$$\begin{aligned}|F(x_3,t)|\leq\frac{1}{2}\int_0^t\int_0^s\int_{D_{x_3}}\Big\{\varepsilon_1 c\dot{\tilde{\alpha}}(z)^2+\frac{2M_1^2(1+\epsilon)}{\varepsilon_1 c}\big[k_m^*\tilde{\alpha}_{,i}(z)\tilde{\alpha}_{,i}(z)+(\tau_T k_m+\tau_q\kappa_m)\dot{\alpha}_{,i}(z)\dot{\alpha}_{,i}(z)\big]\Big\}\,da\,dz\,ds\\ +\frac{\tau_T(1+\epsilon)M_2^2}{2\varepsilon_1 c\epsilon\tau_q}\int_0^t\int_0^s\int_{D_{x_3}}\tau_q\tau_T k_m\ddot{\alpha}_{,i}(z)\ddot{\alpha}_{,i}(z)\,da\,dz\,ds,\end{aligned} \tag{61}$$

so that, by setting

$$\varepsilon_1 = M_1\sqrt{\frac{2(1+\epsilon)}{c}}, \tag{62}$$

we get

$$|F(x_3,t)| \le \frac{\varepsilon_1 t}{2}\int_0^t\int_{D_{x_3}}\left[c\dot{\tilde{\alpha}}(z)^2 + k_m^*\tilde{\alpha}_{,i}(z)\tilde{\alpha}_{,i}(z) + (\tau_q\kappa_m + \tau_T k_m)\dot{\alpha}_{,i}(z)\dot{\alpha}_{,i}(z)\right]da\,dz \\ + \frac{\tau_T(1+\epsilon)M_2^2}{2\varepsilon_1 c\epsilon\tau_q}\int_0^t\int_0^s\int_{D_{x_3}}\tau_q\tau_T k_m\ddot{\alpha}_{,i}(z)\ddot{\alpha}_{,i}(z)\,da\,dz\,ds. \tag{63}$$

Now we equate the coefficients of the two integral terms in (63)

$$\varepsilon_1 t = \frac{\tau_T(1+\epsilon)M_2^2}{2\epsilon\varepsilon_1 c\tau_q}, \tag{64}$$

that is we set

$$\epsilon = \frac{\tau_T M_2^2}{4\tau_q M_1^2 t}. \tag{65}$$

Consequently, from (62) and (65) we have

$$\varepsilon_1 = \frac{M_3(t)}{\sqrt{t}}, \quad M_3(t) = \sqrt{\frac{\tau_T M_2^2 + 4\tau_q M_1^2 t}{2\tau_q c}} \tag{66}$$

and, therefore, by means of relations (32) and (63) we obtain

$$\frac{1}{M_3(t)\sqrt{t}}|F(x_3,t)| + \frac{\partial F}{\partial x_3}(x_3,t) \le 0, \quad \text{for all } x_3 > 0,\ t > 0. \tag{67}$$

In order to discuss the consequences of the differential inequality (67), we recall that (32) proves that $F(x_3,t)$ is a nonincreasing function with respect to $x_3$ on $(0,\infty)$ and therefore, there is one of the following situations: (i) let $F(x_3,t) > 0$ for any $x_3 \in (0,\infty)$ and for any $t > 0$, or (ii) there is a $x_3^* > 0$ such that $F(x_3,t) < 0$ for any $x_3 > x_3^*$ and for any $t > 0$.

Let us consider the point (i). Then the differential inequality (67) becomes

$$\frac{\partial}{\partial x_3}\left[\exp\left(\frac{x_3}{M_3(t)\sqrt{t}}\right)F(x_3,t)\right] \le 0, \quad \text{for all } x_3 > 0,\ t > 0, \tag{68}$$

and hence we have

$$0 \le F(x_3,t) \le F(0,t)\exp\left(-\frac{x_3}{M_3(t)\sqrt{t}}\right), \quad \text{for all } x_3 > 0,\ t > 0, \tag{69}$$

that is an exponential decaying Saint-Venant's estimate. With this in mind, we can integrate with respect to $x_3$ variable over $(x_3,\infty)$ in (32) to obtain

$$F(x_3,t) \ge \frac{1}{2}\int_0^t\int_{B_{x_3}}\left[c\dot{\tilde{\alpha}}(s)^2 + k_m^*\tilde{\alpha}_{,i}(s)\tilde{\alpha}_{,i}(s) + (\tau_q\kappa_m + \tau_T k_m)\dot{\alpha}_{,i}(s)\dot{\alpha}_{,i}(s)\right]dv\,ds \\ + \int_0^t\int_0^s\int_{B_{x_3}}\left[\kappa_m\dot{\alpha}_{,i}(z)\dot{\alpha}_{,i}(z) + \tau_q\tau_T k_m\ddot{\alpha}_{,i}(z)\ddot{\alpha}_{,i}(z)\right]dv\,dz\,ds \ge 0, \\ \text{for all} \quad x_3 > 0, t > 0, \tag{70}$$

a relation showing that $F(x_3, t)$ can be considered like a measure of the solution $\mathcal{S} = \{\alpha, q_i\}$ of our initial boundary value problem $\mathcal{P}$.

Let us now consider the point (ii). Then the differential inequality (67) implies

$$\frac{\partial}{\partial x_3}\left[\exp\left(-\frac{x_3}{M_3(t)\sqrt{t}}\right)F(x_3,t)\right] \leq 0, \quad \text{for all } x_3 > x_3^*,\ t > 0, \tag{71}$$

and hence we get

$$-F(x_3,t) \geq -F(x_3^*,t)\exp\left(\frac{x_3 - x_3^*}{M_3(t)\sqrt{t}}\right) \geq 0, \quad \text{for all } x_3 > x_3^*,\ t > 0, \tag{72}$$

when we can conclude that there is a solution with infinite energy.

**6.2.** ***Spatial decaying result with exponent independent of time.*** Let us fix a parameter $\lambda > 0$ and let us define the following function

$$G(x_3,t) = \int_0^t \int_{D_{x_3}} e^{-\lambda z} \tilde{q}_3(z)\dot{\tilde{\alpha}}(z)\, da\, dz, \tag{73}$$

so that we obtain

$$\begin{aligned}-\frac{\partial G}{\partial x_3}(x_3,t) = &\int_{D_{x_3}} \tfrac{1}{2}e^{-\lambda t}\left[c\dot{\tilde{\alpha}}(t)^2 + k_{ij}^*\tilde{\alpha}_{,i}(t)\tilde{\alpha}_{,j}(t) + (\tau_T k_{ij} + \tau_q\kappa_{ij})\dot{\alpha}_{,i}(t)\dot{\alpha}_{,j}(t)\right] da \\ &+ \int_0^t \int_{D_{x_3}} e^{-\lambda z}\Big\{\kappa_{ij}\dot{\alpha}_{,i}(z)\dot{\alpha}_{,j}(z) + \tau_q\tau_T k_{ij}\ddot{\alpha}_{,i}(z)\ddot{\alpha}_{,j}(z) \\ &\qquad + \frac{\lambda}{2}\left[c\dot{\tilde{\alpha}}(z)^2 + k_{ij}^*\tilde{\alpha}_{,i}(z)\tilde{\alpha}_{,j}(z) + (\tau_T k_{ij} + \tau_q\kappa_{ij})\dot{\alpha}_{,i}(z)\dot{\alpha}_{,j}(z)\right]\Big\}\, da\, dz.\end{aligned} \tag{74}$$

In view of our hypotheses (H1) to (H4), from (74) we get

$$\begin{aligned}-\frac{\partial G}{\partial x_3}(x_3,t) \geq &\int_{D_{x_3}} \tfrac{1}{2}e^{-\lambda t}\left[c\dot{\tilde{\alpha}}(t)^2 + k_m^*\tilde{\alpha}_{,i}(t)\tilde{\alpha}_{,i}(t) + (\tau_T k_m + \tau_q\kappa_m)\dot{\alpha}_{,i}(t)\dot{\alpha}_{,i}(t)\right] da \\ &+ \int_0^t \int_{D_{x_3}} e^{-\lambda z}\Big\{\frac{\lambda}{2}\left[c\dot{\tilde{\alpha}}(z)^2 + k_m^*\tilde{\alpha}_{,i}(z)\tilde{\alpha}_{,i}(z)\right] + \left[\kappa_m + \frac{\lambda}{2}(\tau_T k_m + \tau_q\kappa_m)\right]\dot{\alpha}_{,i}(z)\dot{\alpha}_{,i}(z) \\ &\qquad + \tau_q\tau_T k_m\ddot{\alpha}_{,i}(z)\ddot{\alpha}_{,i}(z)\Big\}\, da\, dz \geq 0, \quad \text{for all } x_3 > 0,\ t > 0,\end{aligned} \tag{75}$$

and hence $G(x_3, t)$ is a nonincreasing function with respect to $x_3$ on $(0, \infty)$ for all $t > 0$.

In the present context we use relations (20), (21) and (58) to obtain the convenient estimate

$$\tilde{q}_3^2 \leq 2(1+\delta)M_3^2\Big\{\frac{\lambda}{2}k_m^*\tilde{\alpha}_{,i}\tilde{\alpha}_{,i} + \left[\kappa_m + \frac{\lambda}{2}(\tau_T k_m + \tau_q\kappa_m)\right]\dot{\alpha}_{,i}\dot{\alpha}_{,i}\Big\} + \left(1 + \frac{1}{\delta}\right)\frac{\tau_T M_2^2}{\tau_q}(\tau_q\tau_T k_m\ddot{\alpha}_{,i}\ddot{\alpha}_{,i}), \tag{76}$$

for any positive parameter $\delta$ and with $M_3$ given by

$$M_3 = \max\left(\left(\frac{2k_{3r}^*k_{3r}^*}{\lambda k_m^*}\right)^{1/2}, \left(\frac{2\kappa_{3r}\kappa_{3r}}{2\kappa_m + \lambda(\tau_T k_m + \tau_q\kappa_m)}\right)^{1/2}\right). \tag{77}$$

Therefore, we have

$$
\begin{aligned}
|G(x_3,t)| &\le \frac{1}{2}\int_0^t\int_{D_{x_3}} e^{-\lambda z}\Big[\varepsilon_3\lambda c\dot{\tilde{\alpha}}(z)^2+\frac{1}{\lambda\varepsilon_3 c}\tilde{q}_3(z)^2\Big]\,da\,dz\\
&\le \int_0^t\int_{D_{x_3}} e^{-\lambda z}\Big\{\varepsilon_3\Big(\frac{\lambda c}{2}\dot{\tilde{\alpha}}(z)^2\Big)\\
&\qquad +\frac{(1+\delta)M_3^2}{\varepsilon_3\lambda c}\Big[\frac{\lambda}{2}k_m^*\tilde{\alpha}_{,i}(z)\tilde{\alpha}_{,i}(z)+\Big(\kappa_m+\frac{\lambda}{2}(\tau_q\kappa_m+\tau_T k_m)\Big)\dot{\alpha}_{,i}(z)\dot{\alpha}_{,i}(z)\Big]\\
&\qquad +\frac{\tau_T(1+\delta)M_2^2}{2\varepsilon_3\delta\lambda c\tau_q}(\tau_q\tau_T k_m\ddot{\alpha}_{,i}(z)\ddot{\alpha}_{,i}(z))\Big\}\,da\,dz, \qquad (78)
\end{aligned}
$$

for any positive parameter $\varepsilon_3$. Then we set

$$
\varepsilon_3=M_3\sqrt{\frac{1+\delta}{\lambda c}},\qquad \delta=\frac{\tau_T M_2^2}{2\tau_q M_3^2}, \qquad (79)
$$

so that, by means of relation (75), we obtain the following differential inequality

$$
|G(x_3,t)|+\varepsilon_3\frac{\partial G}{\partial x_3}(x_3,t)\le 0,\qquad \text{for all } x_3>0,\ t>0. \qquad (80)
$$

This last inequality can be treated like the differential inequality (67) to obtain (i) let $G(x_3,t)>0$ for any $x_3\in(0,\infty)$ and for any $t>0$, or (ii) there is a $x_3^*>0$ such that $G(x_3,t)<0$ for any $x_3>x_3^*$ and for any $t>0$. In the case (i) it results that

$$
0\le G(x_3,t)\le G(0,t)e^{-x_3/\varepsilon_3},\qquad \text{for all } x_3>0,\ t>0, \qquad (81)
$$

that is an exponential decaying Saint-Venant's estimate, while in the case (ii) there is the estimate

$$
-G(x_3,t)\ge -G(x_3^*,t)e^{(x_3-x_3^*)/\varepsilon_3}\ge 0,\qquad \text{for all } x_3>x_3^*,\ t>0, \qquad (82)
$$

when we can conclude that there is a solution with infinite energy.

**Remark.** For a cylinder of finite length, be it $L$, under the boundary condition $q_3(x_1,x_2,L,t)=0$ on the upper end, it results that $F(L,t)=0$ or $G(L,t)=0$; a situation in which we obtain only the exponential decaying estimates (69) or (81). It can be seen from our analysis that the estimates in question remain valid whether $\tau_T$ is negligible or not! This means that they can also be used in conjunction with the result of the domain of influence expressed by the relation (56)!

## 7. Final comments

Our study highlights some important aspects in the analysis of the mechanical model in concern, which can be summarized as follows:

(a) For a homogeneous and isotropic rigid thermal conductor, when $k_{ij}=k\delta_{ij}$, $k_{ij}^*=k^*\delta_{ij}$, equation (7) becomes

$$
c\tau_q\dddot{\alpha}+c\ddot{\alpha}-k^*\Delta\alpha-(k+\tau_\alpha k^*)\Delta\dot{\alpha}-\tau_T k\Delta\ddot{\alpha}=0, \qquad (83)
$$

and the restrictions concerning the compatibility of the constitutive equation with the second law of thermodynamics are (i) $k \geq 0$, (ii) $k + (\tau_\alpha - \tau_q)k^* \geq 0$. If we set $c = 1$, $\tau_\alpha = 0$ and $\tau_T = 0$ in (83) we obtain the Moore–Gibson–Thompson equation (1). Consequently, the results concerning the domain of influence reported in Section 5 can be applied also for equation (1), already treated by Ostoja-Starzewski and Quintanilla [19] under homogeneous Dirichlet boundary condition on the lateral surface.

However, when the base of the cylinder is excited for a longer time, then the domain of influence theorem described by (56) is less informal because it says nothing about the behavior of the solution inside this domain. Estimates are then needed to describe the behavior of the solution in this part of the cylinder and these estimates are of the (81) type, which, as we said, are also valid in the situation when $\tau_T$ is negligible. This aspect was not discussed in [19].

(b) Our analysis shows that by taking into consideration all three relaxation times we are led to a situation totally different from that used to obtain the Moore–Gibson–Thompson equation. In essence, the main position in this consideration is given by the phase-lag of the temperature gradient, on whose presence or absence depends the way of describing the spatial behavior of the solution.

(c) In fact, when the relaxation time $\tau_T$ is absent, our analysis provides the result of the domain of influence described by the relation (56). While when its presence is consistent, then it is no longer possible to establish the domain of influence, but it is possible to establish some exponential decrease estimates of Saint-Venant type. Thus, instead of the domain of influence, we establish exponential decay estimate with a time-dependent exponent as described in (69), in such a way that when short time elapses a rapid exponential decreasing of the excited base effects is described.

(d) For long time flows, neither the domain of influence described by (56), nor the spatial estimation (69) provide valuable information about how the effects of the excited base of the cylinder are felt along the generators. This is because (i) either the interior of the influence domain becomes large enough and the domain described by the inequality $x_3 > \varepsilon t$ is no longer significant and (ii) the exponential decrease described by (69) is too slow for long times. For both of these situations, it is convenient to use the estimate described by the relation (81).

(e) Our study takes into account both the lateral boundary data for the thermal displacement and the lateral boundary data regarding the heat flux, that is, we study both the problem of the cylinder for which the lateral surface can have free heat exchanges with the outside, as well as that with thermally insulated lateral surface.

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