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Stan Chiriţă

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Some characteristic properties of the solutions in the three-phase-lag heat conduction

Stan Chiriţă

Faculty of Mathematics, Al. I. Cuza University of Iaşi, Iaşi, Romania

ABSTRACT
In this paper we consider the three-phase-lag model of heat conduction that involves second-order effects in phase lag of the heat flux vector. This model leads to a fourth-order in time equation of Moore–Gibson–Thompson type. We use the thermodynamic restrictions derived from the compatibility of the constitutive equation with the Second Law of Thermodynamics to study the properties of the solutions of the initial boundary value problems associated with the model in concern. In this connection we establish a series of well-posedness results concerning the related solutions like: uniqueness, continuous data dependence, exponentially stability or domain of influence. Furthermore, based on the thermodynamic restrictions, we show that the thermal model in question admits damped in time propagating waves as well as exponentially decaying standing modes. We also show that when the thermodynamic restrictions are not fulfilled, then wave solutions appear that cause the energy blows up as time goes to infinity.

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1. Introduction

Motivated mainly by the fact that the use of classical Fourier's law leads to an infinite signal speed paradox, several other constitutive relations for the heat flux have been considered, cf. Chandrasekharaiah [1], Hetnarski and Ignaczak [2, 3], Straughan [4] or Tzou [5]. Among these is the Maxwell-Cattaneo-Vernotte's law (see e.g. [6–8]) that included the heat flux, its time derivative and its phase lag. As a consequence, the thermal model based on such a constitutive equation leads to a third-order in time equation in terms of temperature variation, known as the Moore-Gibson-Thompson equation (Cf. Moore and Gibson [9] and Jordan [10])

$$\tau\dddot{u} + \ddot{u} - k\Delta\dot{u} - k^{*}\Delta u = 0, \tag{1}$$

where $\tau > 0$ is a relaxation time, Δ is the Laplace operator and k and k^{*} are positive parameters under restriction $k > \tau k^{*}$. This equation has attracted much attention in recent years in relation to the mathematical study of problems of uniqueness, continuous data dependence, exponential decay of energy or domain of influence: Kaltenbacher et al. [11], Kaltenbacher et al. [12], Marchand et al. [13], Conejer et al. [14], Lasiecka and Wang [15, 16], Dell'Oro et al. [17], Dell'Oro and Pata [18], Pellicer and Said-Houari [19], Jangid and Mukhopadhyay [20], Chen and Ikehata [21], Ostoja-Starzewski and Quintanilla [22], Fernández et al. [23], Abouelregal et al. [24].

Later, the Roy Choudhuri [25] proposed a constitutive law which involved three different phase delays in the heat flux vector, the temperature gradient and the gradient of thermal

CONTACT Stan Chiriţă schirita@uaic.ro Faculty of Mathematics, Al. I. Cuza University of Iaşi, Iaşi, Romania.

displacement. The restrictions imposed by the compatibility of this constitutive equation with the Second Law of Thermodynamics, relating the relaxation times and the other thermal coefficients involved, were determined by Chiriţă et al. [26]. It must be said that the three-phase-lag model has been subjected to intense research in recent years, either from the point of view of the well-posedness of the model, as well as from the point of view of treating some practical problems in medicine or heat transfer. We mention here the results obtained by Quintanilla and Racke [27], Akbarzadeh et al. [28], Chiriţă et al. [26], D'Apice et al. [29], Biswas et al. [30], Chiriţă [31], Zhang et al. [32], Kumari and Singh [33] and Singh et al. [34].

A generalization of equation (1) was proposed and studied independently by Quintanilla and Racke [27] and by Dell'Oro and Pata [35] in the following form

$$\frac{\tau_q^2}{2}\ \rho c_v \ddddot{T} + \tau_q\ \rho c_v \dddot{T} + \rho c_v \ddot{T} = k^* \Delta T + \tau_\nu^* \Delta \dot{T} + k\tau_T \Delta \ddot{T}, \tag{2}$$

where the coefficients appearing are positive constant material parameters and $\tau_\nu^* = k^*\tau_\nu + k$. The authors obtain conditions on the material parameters to guarantee the exponential stability of solutions. As we will show below, the constitutive law proposed by Roy Choudhuri [25], in combination with the usual heat equation, leads to a fourth-order in time differential equation in terms of the thermal displacement, that generalizes the Moore-Gibson-Thompson equation (1) as well as that described by (2), the latter being obtained as a particular case of a homogeneous and isotropic rigid conductor.

Our analysis in this paper is dedicated to studying the characteristic properties of the three-phase-lag model proposed by Roy Choudhuri [25], with second-order terms involved in the heat flux. In this sense, we use the thermodynamic restrictions imposed by the compatibility of the constitutive equation with the Second Law of Thermodynamics in order to study well-posedness of the model by establishing some results regarding the uniqueness of the solutions as well as their continuous data dependence. We also establish a domain of influence of the given data of the initial boundary value problem, a result that shows that outside the domain of influence no thermal activity is felt, in other words we obtain an upper bound of the speed of propagation of the effects of the given data.

Furthermore, under congruent restrictions with those of thermodynamics, we study the types of waves possible in a homogeneous and isotropic rigid conductor and show that there can be waves damped in time or standing modes that decrease exponentially with increasing time. We also show that when the thermodynamic restrictions are not fulfilled, some wave solutions can appear that cause the energy blows up when time increases to infinity. Moreover, we show that any solution of the model in question, which represents the effect of some initial conditions, is exponentially stable.

The plan of the work is the following. *Section 2* presents the basic system of differential equations describing the evolutionary behavior of the heat flux and of the thermal displacement in the line described by Roy Choudhuri [25]. The associated fourth-order in time differential equation in terms of thermal displacement is explicitly written. *Section 3* formulates the initial boundary value problem associated with the model in concern and then describe its auxiliary version that will be useful in the future analysis. A law of conservation of energy is established and which introduces a measure of the solution in terms of thermal displacement. *Section 4* is dedicated to the well-posedness of the model: there is established a uniqueness theorem and a continuous data dependence result under restrictions congruent with those imposed by Second Law. We also used the Lagrange identity and the logarithmic convexity methods to obtain the uniqueness under mild restrictions upon the material characteristics. *Section 5* is related to the class of waves propagating in a three-phase-lag model. *Section 6* considers the general model as proposed by Roy Choudhuri [25] and establishes the exponential decay in time of the solutions, that is the exponential stability. *Section 7* presents the domain of influence theorem for a semi-infinite cylinder.

2. Three-phase-lag heat conducting model for a rigid conductor

Starting from the Green-Naghdi model [36] and the Tzou model [37], Roy Choudhuri [25] proposed the following constitutive equation for the heat flux vector

$$q_i(x, t+\tau_q) = -\Big[k_{ij}(x)T_{,j}(x, t+\tau_T) + k_{ij}^*(x)\alpha_{,j}(x, t+\tau_\alpha)\Big], \tag{3}$$

where q_i are the components of the heat flux vector, α is the thermal displacement, $T=\dot{\alpha}$ represents the temperature variation from the constant reference temperature $T_0>0$, k_{ij} are the components of the conductivity tensor and k_{ij}^* are the components of the conductivity rate tensor; moreover, t is the time variable, x is the spatial variable, while τ_q, τ_T and τ_α are the phase-lags (or delay times) of the heat flux vector, of the temperature gradient, and of thermal displacement gradient, respectively. In agreement with the Roy Choudhuri's interpretation, the equation (3) means that a temperature gradient and a thermal displacement gradient imposed across a volume element at times $t+\tau_T$ and $t+\tau_\alpha$, respectively, result in a heat flux flowing at a different time $t+\tau_q$. However, the constitutive equation (3) does not formulate any restriction regarding the three relaxation times and under this general form it does not seem possible to solve its compatibility with the Second Law of Thermodynamics in order to determine the restrictions on the three relaxation times and the constitutive thermal coefficients. To solve this situation, Roy Choudhuri proposes the following generalized heat conduction law valid at a point x at time t

$$\begin{aligned} &q_i(x,t) + \tau_q\dot{q}_i(x,t) + \frac{\tau_q^2}{2}\ddot{q}_i(x,t) = -k_{ij}^*(x)\alpha_{,j}(x,t) \\ &-\Big(k_{ij}(x) + \tau_\alpha k_{ij}^*(x)\Big)\dot{\alpha}_{,j}(x,t) - \tau_T k_{ij}(x)\ddot{\alpha}_{,j}(x,t). \end{aligned} \tag{4}$$

Equation (4) serves as a generalized constitutive heat conduction law in which the elastic deformation term is ignored. The compatibility of the three-phase-lag constitutive equation (4) with the Second Law of Thermodynamics was studied by Chiriță et al. [26] and it requires that the following tensors

$$\zeta_{ij} = k_{ij} + (\tau_\alpha - \tau_q)k_{ij}^*, \quad \varkappa_{ij} = \tau_T\ k_{ij} - \frac{\tau_q}{2}\ \Big(k_{ij} + \tau_\alpha\ k_{ij}^*\Big), \tag{5}$$

to be positive semi-definite.

Throughout this paper we consider the three-phase-lags model for a rigid conductor as proposed by Roy Choudhuri [25] based on the constitutive equation (4) and the well-known energy conservation equation

$$-q_{i,i}(x,t) + r(x,t) = c(x)\ddot{\alpha}(x,t), \tag{6}$$

where $r(x,t)$ represents the heat source acting per unit volume and $c(x)$ is the specific heat per unit volume. Under the thermodynamic restrictions just described in (5), some results about the continuous dependence of the solutions with respect to the given initial data and to the supply term are established for the related initial boundary value problems in [26].

In terms of the thermal displacement α, the above three-phase-lags model for a rigid conductor is based upon the following differential equation

$$\begin{aligned} &\frac{1}{2}\ c\tau_q^2\ddddot{\alpha} + c\tau_q\dddot{\alpha} + c\ddot{\alpha} - \Big(k_{ij}^*\alpha_{,j}\Big)_{,i} - \Big[\Big(k_{ij} + \tau_\alpha k_{ij}^*\Big)\dot{\alpha}_{,j}\Big]_{,i} \\ &-\tau_T\big(k_{ij}\ddot{\alpha}_{,j}\big)_{,i} = 0, \end{aligned} \tag{7}$$

where we assumed the vanishing of the heat source and, moreover, the dependence on the independent variables x and t was suppressed, but implicitly understood.

For an isotropic and homogeneous rigid thermal conductor, when $k_{ij} = k\delta_{ij}$ and $k^*_{ij} = k^*\delta_{ij}$, the equation (7) becomes

$$\frac{1}{2}\ c\tau_q^2\ \ddddot{\alpha} + c\tau_q\dddot{\alpha} + c\ddot{\alpha} - k^*\ \Delta\alpha - (k + \tau_\alpha k^*)\Delta\ \dot{\alpha} - \tau_T k\Delta\ \ddot{\alpha} = 0, \tag{8}$$

where Δ is the Laplace operator. Moreover, the thermodynamic restrictions described by (5) can be read as

$$\zeta = k + (\tau_\alpha - \tau_q)\ k^* \geq 0, \quad \varkappa = \tau_T\ k - \frac{\tau_q}{2}\ (k + \tau_\alpha\ k^*) \geq 0. \tag{9}$$

Equation (8) was proposed and studied by Quintanilla and Racke [27] in relation to the exponential stability of solutions under suitable Dirichlet boundary and initial conditions. An equation of like that described by (8) was obtained by Dell'Oro and Pata [35] by means of a Moore-Gibson-Thompson equation with memory in the presence of an exponential kernel. Under homogeneous Dirichlet boundary condition, there are established some stability properties of the related solution semigroup and a necessary and sufficient condition for exponential stability is obtained, in terms of the values of certain stability numbers depending on the strictly positive coefficients involved.

3. Formulation of the initial boundary value problem and its auxiliary form

Throughout this paper we shall assume that a bounded three-dimensional region B is filled by an inhomogeneous and anisotropic conductor material with three-phase-lag times. We denote by ∂B the boundary surface of B and assume that it is sufficiently regular to allow application of the divergence theorem. Throughout this paper we consider the initial boundary value problem $\mathcal{P}$ defined by the heat equation (6), the constitutive equation (4), the initial conditions

$$\begin{aligned} &\alpha(x,0) = 0, \quad \dot{\alpha}(x,0) = \dot{\alpha}^0(x), \\ &q_i(x,0) = q_i^0(x), \quad \dot{q}_i(x,0) = \dot{q}_i^0(x), \quad \text{for all} \quad x \in B, \end{aligned} \tag{10}$$

and the following boundary conditions

$$\begin{aligned} &\alpha(x,t) = \Theta(x,t) \quad on \quad \Sigma_1 \times (0,\infty), \\ &q_i(x,t)n_i = Q(x,t) \quad on \quad \Sigma_2 \times (0,\infty). \end{aligned} \tag{11}$$

Here $\dot{\alpha}^0(x)$, $q_i^0(x)$ and $\dot{q}_i^0(x)$, as well as $\Theta(x,t)$ and $Q(x,t)$ are prescribed smooth functions. Moreover, n_i are the components of the outward normal vector to ∂B and Σ_1 and Σ_2 are subsets of the boundary ∂B so that $\bar{\Sigma}_1 \cup \Sigma_2 = \partial B$ and $\Sigma_1 \cap \Sigma_2 = \emptyset$. The initial conditions (10) and the boundary conditions (11) are presented in terms of the thermal displacement α only for mathematical reasons, but they can be easily expressed in terms of the temperature variation T (a fundamental physical quantity that can be measured by experiments) through the relation $\alpha(x,t) = \int_0^t T(x,s)ds$, that is, we have: $\dot{\alpha}^0(x) = T(x,0)$ for $x \in B$ and $\Theta(x,t) = \int_0^t T(x,s)ds$ for $(x,t) \in \Sigma_1 \times (0,\infty)$.

By a solution of the initial boundary value problem $\mathcal{P}$, corresponding to the given data $\mathcal{D} = \{r; \dot{\alpha}^0, q_i^0, \dot{q}_i^0; \Theta, Q\}$ we mean the ordered array $\mathcal{S} = \{\alpha, q_i\}$ defined on $B \times (0,\infty)$ with the properties that $\alpha(x,t) \in C^{2,2}(B \times (0,\infty))$, $q_i(x,t) \in C^{1,2}(B \times (0,\infty))$ and which satisfy the field equations (4) and (6), the initial conditions (10) and the boundary conditions (11). Throughout this paper we will assume the existence of such a solution! This means that we do not deal with the existence and regularity of the solutions of the initial boundary value problem.

Our further analysis requires to introduce an auxiliary initial boundary value problem $\tilde{\mathcal{P}}$ associated with the problem in question $\mathcal{P}$. In this sense we introduce the notations

$$\tilde{q}_i = q_i + \tau_q \dot{q}_i + \frac{\tau_q^2}{2}\ \ddot{q}_i, \quad \tilde{\alpha} = \alpha + \tau_q \dot{\alpha} + \frac{\tau_q^2}{2}\ \ddot{\alpha}, \quad \tilde{r} = r + \tau_q \dot{r} + \frac{\tau_q^2}{2}\ \ddot{r}, \tag{12}$$

and note that the heat equation (6) implies

$$-\tilde{q}_{i,i} + \tilde{r} = c\ddot{\tilde{\alpha}} \quad \text{in} \quad B \times (0, \infty), \tag{13}$$

while the constitutive equation (4) becomes

$$\tilde{q}_i = -k_{ij}^* \alpha_{,j} - \left(k_{ij} + \tau_\alpha k_{ij}^*\right)\dot{\alpha}_{,j} - \tau_T k_{ij} \ddot{\alpha}_{,j} \quad \text{in} \quad B \times (0, \infty). \tag{14}$$

Furthermore, in view of the initial conditions (10) and the heat equation (6), we have

$$\begin{aligned} &\tilde{\alpha}(x, 0) = \tau_q\ \dot{\alpha}^0(x) + \frac{\tau_q^2}{2}\ \ddot{\alpha}(x, 0), \\ &\dot{\tilde{\alpha}}(x, 0) = \dot{\alpha}^0(x) + \tau_q\ \ddot{\alpha}(x, 0) + \frac{\tau_q^2}{2}\ \dddot{\alpha}(x, 0) \quad \text{for all} \quad x \in B, \end{aligned} \tag{15}$$

while the boundary conditions (11) furnishes

$$\begin{aligned} &\tilde{\alpha}(x, t) = \tilde{\Theta}(x, t) \quad on \quad \Sigma_1 \times (0, \infty), \\ &\tilde{q}_i(x, t) n_i = \tilde{Q}(x, t) \quad on \quad \Sigma_2 \times (0, \infty), \end{aligned} \tag{16}$$

where $\ddot{\alpha}(x, 0)$ and $\dddot{\alpha}(x, 0)$ are calculated by means of the heat equation (6) as

$$\ddot{\alpha}(x, 0) = \frac{1}{c}\left[r(x, 0) - q_{i,i}^0(x)\right], \quad \dddot{\alpha}(x, 0) = \frac{1}{c}\left[\dot{r}(x, 0) - \dot{q}_{i,i}^0(x)\right], \tag{17}$$

and

$$\tilde{\Theta} = \Theta + \tau_q \dot{\Theta} + \frac{\tau_q^2}{2}\ \ddot{\Theta}, \quad \tilde{Q} = Q + \tau_q \dot{Q} + \frac{\tau_q^2}{2}\ \ddot{Q}. \tag{18}$$

For future convenience we write the constitutive equation (14) in the following form

$$\tilde{q}_i = -k_{ij}^* \tilde{\alpha}_{,j} - \zeta_{ij}\left(\dot{\alpha}_{,j} + \frac{\tau_q}{2}\ \ddot{\alpha}_{,j}\right) - \varkappa_{ij} \ddot{\alpha}_{,j} \quad \text{in} \quad B \times (0, \infty). \tag{19}$$

Moreover, we note that the relations (12) and (19) furnish the following identity

$$\begin{aligned} -\tilde{q}_i(t)\dot{\tilde{\alpha}}_{,i}(t) = &\frac{1}{2}\ \frac{\partial}{\partial t}\ \Bigg\{k_{ij}^* \tilde{\alpha}_{,i}(t)\tilde{\alpha}_{,j}(t) + \varkappa_{ij}\dot{\alpha}_{,i}(t)\dot{\alpha}_{,j}(t) \\ &+\tau_q \zeta_{ij}\left(\dot{\alpha}_{,i}(t) + \frac{\tau_q}{2}\ \ddot{\alpha}_{,i}(t)\right)\left(\dot{\alpha}_{,j}(t) + \frac{\tau_q}{2}\ \ddot{\alpha}_{,j}(t)\right) + \frac{\tau_q}{2}\ [\zeta_{ij}\dot{\alpha}_{,i}(t)\dot{\alpha}_{,j}(t) \\ &+\tau_q\ \varkappa_{ij}\ \ddot{\alpha}_{,i}(t)\ddot{\alpha}_{,j}(t)]\Bigg\} + \left[\zeta_{ij}\dot{\alpha}_{,i}(t)\dot{\alpha}_{,j}(t) + \tau_q\ \varkappa_{ij}\ \ddot{\alpha}_{,i}(t)\ddot{\alpha}_{,j}(t)\right]. \end{aligned} \tag{20}$$

In our further analysis we will need some of the following hypotheses upon the characteristic material coefficients

(H1): the specific heat per unit volume is strictly positive in B, that is

$$c(x) > 0 \quad \text{for all} \quad x \in B; \tag{21}$$

(H2): k_{ij}^* is a positive semi-definite tensor as

$$k_{rs}^*(x)\xi_r\xi_s \geq 0 \quad \text{for all} \quad (\xi_1, \xi_2, \xi_3) \quad \text{and for all} \quad x \in B; \tag{22}$$

(H3): $\zeta_{ij} = k_{ij} + (\tau_\alpha - \tau_q)k_{ij}^*$ is a positive semi-definite tensor as

$$\zeta_{rs}(x)\xi_r\xi_s \geq 0 \quad \text{for all} \quad (\xi_1, \xi_2, \xi_3) \quad \text{and for all} \quad x \in B; \tag{23}$$

(H4): $\varkappa_{ij} = \tau_T\ k_{ij} - \frac{\tau_q}{2}(k_{ij} + \tau_\alpha\ k^*_{ij})$ is a positive semi-definite tensor, that is

$$\varkappa_{rs}(x)\xi_r\xi_s \geq 0 \quad \text{for all} \quad (\xi_1, \xi_2, \xi_3) \quad \text{and for all} \quad x \in B. \tag{24}$$

The last two hypothesis proves that the Second Law of Thermodynamics is fulfilled as results from the relation (5). The first hypothesis proves that a genuine dynamic thermal situation is considered. While the hypothesis (H2) represents an extension of hypothesis (H3) for the conductivity rate tensor.

We can now establish a basic identity regarding the solutions of the initial boundary value problem $\tilde{\mathcal{P}}$ and thereby for those of the initial boundary value problem $\mathcal{P}$. For this purpose, we note that the heat equation (13), the divergence theorem and the boundary condition (11) provide

$$\int_B c\dot{\tilde{\alpha}}(t)\ddot{\tilde{\alpha}}(t)dv = \int_B \tilde{r}(t)\dot{\tilde{\alpha}}(t)dv - \int_{\partial B} \tilde{q}_i(t)n_i\dot{\tilde{\alpha}}(t)da + \int_B \tilde{q}_i(t)\dot{\tilde{\alpha}}_{,i}(t)dv, \tag{25}$$

and so, by using the relation (20), we obtain the following identity

$$\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t \int_B \tilde{r}(s)\dot{\tilde{\alpha}}(s)dvds - \int_0^t \int_{\partial B} \tilde{q}_i(s)n_i\dot{\tilde{\alpha}}(s)dads, \tag{26}$$

where

$$\mathcal{E}(t) = E_1(t) + \int_0^t \int_B \left[\zeta_{ij}\dot{\alpha}_{,i}(s)\dot{\alpha}_{,j}(s) + \tau_q\ \varkappa_{ij}\ddot{\alpha}_{,i}(s)\ddot{\alpha}_{,j}(s)\right]dvds, \tag{27}$$

and

$$\begin{aligned} E_1(t) =& \frac{1}{2}\int_B \Big\{ c\dot{\tilde{\alpha}}(t)^2 + k^*_{ij}\tilde{\alpha}_{,i}(t)\tilde{\alpha}_{,j}(t) + \varkappa_{ij}\dot{\alpha}_{,i}(t)\dot{\alpha}_{,j}(t) \\ &+\tau_q\zeta_{ij}\left(\dot{\alpha}_{,i}(t) + \frac{\tau_q}{2}\ \ddot{\alpha}_{,i}(t)\right)\left(\dot{\alpha}_{,j}(t) + \frac{\tau_q}{2}\ \ddot{\alpha}_{,j}(t)\right) \\ &+\frac{\tau_q}{2}\ \left[\zeta_{ij}\dot{\alpha}_{,i}(t)\dot{\alpha}_{,j}(t) + \tau_q\ \varkappa_{ij}\ddot{\alpha}_{,i}(t)\ddot{\alpha}_{,j}(t)\right]\Big\}dv. \end{aligned} \tag{28}$$

By virtue of our hypotheses (H1) to (H4) we can see that

$$\mathcal{E}(t) \geq 0 \quad \text{for all} \quad t > 0. \tag{29}$$

Furthermore, under the hypothesis of null initial conditions, it can be seen that $\mathcal{E}(t)$ can be considered a measure of the solution $\mathcal{S} = \{\alpha, q_i\}$ in the sense that if $\mathcal{E}(t) = 0$ for all $t > 0$, then $\mathcal{S}(x,t) = \{\alpha(x,t), q_i(x,t)\} = 0$ for any $x \in B$ and any $t > 0$. In fact, we see that in (27) and (28) we have a sum of positive terms equal to zero and therefore each of the terms must be zero. Thus, we see that if $\mathcal{E}(t) = 0$ then it results that $\dot{\tilde{\alpha}}(t) = 0$, that is

$$\dot{\alpha}(x,t) + \tau_q\ \ddot{\alpha}(x,t) + \frac{\tau_q^2}{2}\ \dddot{\alpha}(x,t) = 0, \quad \text{for all} \quad x \in B, \quad t > 0, \tag{30}$$

which, integrated with respect to t under zero initial conditions for α, provides

$$\alpha(x,t) = 0 \quad \text{for all} \quad x \in B, \quad t > 0. \tag{31}$$

Then, by taking into account this last relationship in the constitutive equation (4) we deduce

$$q_i(x,t) + \tau_q\ \dot{q}_i(x,t) + \frac{\tau_q^2}{2}\ \ddot{q}_i(x,t) = 0 \quad \text{for all} \quad x \in B, \quad t > 0, \tag{32}$$

which, integrated with respect to t under the initial conditions $q_i(x,0) = 0$ and $\dot{q}_i(x,0) = 0$, furnishes

$$q_i(x,t) = 0 \quad \text{for all} \quad x \in B, \quad t > 0. \tag{33}$$

Thus, it results that $\mathcal{S}(x,t) = 0$ and hence we can conclude that $\mathcal{E}(t)$ is a measure of the solution $\mathcal{S} = \{\alpha, q_i\}$.

4. Well-posedness results

In this section, we deal with issues related to the well-setting of the related initial boundary value problems: uniqueness of solutions and their continuous dependence with respect to the given data, all under some assumptions on the thermal characteristics that are congruent with the established thermodynamic restrictions.

We will start with the following uniqueness result.

Theorem 1. *(**Uniqueness of solution**) Suppose the hypotheses (H1) to (H4) to be fulfilled. Then the initial boundary value problem $\mathcal{P}$ admits at most one solution.*

Proof. Having a linear problem, proving the uniqueness of the solution is equivalent to proving that the initial boundary value problem $\mathcal{P}$ with null given data $\mathcal{D} = \{0; 0, 0, 0, 0; 0, 0\}$, admits only the trivial solution $\mathcal{S} = \{\alpha, q_i\} = 0$. In this connection we note that from relations (15) to (17), we have

$$\tilde{\alpha}(x,0) = 0, \quad \dot{\tilde{\alpha}}(x,0) = 0, \quad \dot{\alpha}(x,0) = 0, \quad \ddot{\alpha}(x,0) = 0 \quad \text{for all} \quad x \in B, \tag{34}$$

and hence, from (27) and (28), it follows that

$$\mathcal{E}(0) = 0. \tag{35}$$

Moreover, the relations (12) and (18) give

$$\begin{aligned} &\tilde{\Theta}(x,t) = 0 \quad \text{on} \quad \Sigma_1 \times (0,\infty), \quad \tilde{Q}(x,t) = 0 \quad \text{on} \quad \Sigma_2 \times (0,\infty), \\ &\tilde{r}(x,t) = 0 \quad \text{for all} \quad (x,t) \in B \times (0,\infty). \end{aligned} \tag{36}$$

Consequently, the identity (26) becomes

$$\mathcal{E}(t) = 0 \quad \text{for all} \quad t > 0, \tag{37}$$

that is $\mathcal{S}(x,t) = 0$ for all $x \in B$ and $t > 0$ and so the proof is complete. □

Theorem 2. *(**Continuous dependence**) Suppose the hypotheses (H1) to (H4) to be fulfilled. Let $\mathcal{S} = \{\alpha, q_i\}$ be a solution of the initial boundary value problem $\mathcal{P}$ corresponding to the given data $\mathcal{D} = \{r; \dot{\alpha}^0, q_i^0, \dot{q}_i^0; 0, 0\}$. Then the following estimate holds true*

$$\sqrt{\mathcal{E}(t)} \leq \sqrt{\mathcal{E}(0)} + \frac{1}{\sqrt{2}} \int_0^t \left(\int_B \frac{1}{c} \, \tilde{r}(s)^2 dv \right)^{1/2} ds, \quad \text{for all} \quad t > 0. \tag{38}$$

Proof. Under the given data of the problem, the identity (26) is written as

$$\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t \int_B \tilde{r}(s) \dot{\tilde{\alpha}}(s) dv ds, \tag{39}$$

and therefore, by means of the Schwarz inequality, we find

$$\mathcal{E}(t) \leq \mathcal{E}(0) + \int_0^t g(s) \left(\int_B c \, \dot{\tilde{\alpha}}(s)^2 dv \right)^{1/2} ds, \tag{40}$$

where

$$g(t) = \left(\int_B \frac{1}{c} \, \tilde{r}(t)^2 dv \right)^{1/2}. \tag{41}$$

Further, we use the relation (27) and (28) into (40) to deduce

$$\mathcal{E}(t) \le \mathcal{E}(0) + \int_0^t g(s)\sqrt{2\mathcal{E}(s)}ds. \tag{42}$$

Now we set

$$\Psi(t) = \sqrt{\mathcal{E}(0) + \int_0^t g(s)\sqrt{2\mathcal{E}(s)}ds}, \tag{43}$$

and note that

$$\dot{\Psi}(t) \le \frac{1}{\sqrt{2}} \, g(t). \tag{44}$$

Concluding, an integration with respect to time variable in (44) and by taking into account that $\Psi(0) = \sqrt{\mathcal{E}(0)}$ and $\sqrt{\mathcal{E}(t)} \le \Psi(t)$, we are led to the estimate (38) and the proof is complete. □

Herewith, we present a theorem of the uniqueness of the solution under milder assumptions on the thermal coefficients and the three relaxation times. In fact, we can avoid the hypothesis (H2) either by means of the Lagrange identity method or by the logarithmic convexity method.

Theorem 3. (Uniqueness under mild assumptions) Suppose the hypotheses (H1), (H3) and (H4) to be fulfilled. Then the initial boundary value problem $\mathcal{P}$ admits at most one solution.

Proof. **1: Lagrange identity method (see, e.g. Brun** [38]**, Rionero and Chiriță** [39]). We start with the following identity

$$\begin{aligned} &\frac{\partial}{\partial z}\left[\tilde{\alpha}(t+z)\dot{\tilde{\alpha}}(t-z) + \tilde{\alpha}(t-z)\dot{\tilde{\alpha}}(t+z)\right] \\ &= \tilde{\alpha}(t-z)\ddot{\tilde{\alpha}}(t+z) - \tilde{\alpha}(t+z)\ddot{\tilde{\alpha}}(t-z), \end{aligned} \tag{45}$$

and then an integration over z on $[0, t]$, followed by the use of the heat equation (13) and the zero given data, gives

$$2\int_B c\tilde{\alpha}(t)\dot{\tilde{\alpha}}(t)dv = \int_0^t \int_B \left[\tilde{\alpha}(t-z)\tilde{q}_{i,i}(t+z) - \tilde{\alpha}(t+z)\tilde{q}_{i,i}(t-z)\right]dvdz. \tag{46}$$

Then we use the divergence theorem combined with the null data on the boundary to obtain

$$2\int_B c\tilde{\alpha}(t)\dot{\tilde{\alpha}}(t)dv = \int_0^t \int_B \left[\tilde{q}_i(t-z)\tilde{\alpha}_{,i}(t+z) - \tilde{q}_i(t+z)\tilde{\alpha}_{,i}(t-z)\right]dvdz. \tag{47}$$

Furthermore, we use the relation (12) and the constitutive equation (19) to get

$$\begin{aligned} &\tilde{q}_i(t-z)\tilde{\alpha}_{,i}(t+z) - \tilde{q}_i(t+z)\tilde{\alpha}_{,i}(t-z) \\ &= \frac{\partial}{\partial z}\left\{\zeta_{ij}\alpha_{,i}(t-z)\alpha_{,j}(t+z) + \left(\varkappa_{ij} + \frac{\tau_q}{2}\,\zeta_{ij}\right)\alpha_{,i}(t-z)\dot{\alpha}_{,j}(t+z)\right. \\ &\left. + \left(\varkappa_{ij} + \frac{\tau_q}{2}\,\zeta_{ij}\right)\alpha_{,i}(t+z)\dot{\alpha}_{,j}(t-z) + \tau_q\varkappa_{ij}\dot{\alpha}_{,i}(t-z)\dot{\alpha}_{,j}(t+z)\right\}. \end{aligned} \tag{48}$$

Now, we use (48) into relation (47) and then by an integration combined with null given data we deduce

$$\int_B \left[c\tilde{\alpha}(t)^2 + \left(\varkappa_{ij} + \frac{\tau_q}{2}\ \zeta_{ij}\right)\alpha_{,i}(t)\alpha_{,j}(t)\right]dv + \int_0^t\int_B \left[\zeta_{ij}\alpha_{,i}(s)\alpha_{,j}(s) + \tau_q\varkappa_{ij}\dot{\alpha}_{,i}(s)\dot{\alpha}_{,j}(s)\right]dvds = 0. \tag{49}$$

As can be seen, by virtue of the constitutive hypotheses (H1), (H3) and (H4), in (49) we have a sum of positive terms and this can be equal to zero only if each term is vanishing. In particular, we deduce that

$$c(x)\tilde{\alpha}(x,t)^2 = 0 \quad \text{in} \quad B\times(0,\infty). \tag{50}$$

Since $c(x) > 0$ in B, from (50) it follows that the relation (31) holds true and therefore it can be used to prove that $\mathcal{S} = \{\alpha, q_i\} = 0$ and the proof is completed.

2: Logarithmic convexity method (see e.g. Knops and Payne [40]**).** Guided by the above proof we introduce now the following function

$$G(t) = \int_B \left[c\tilde{\alpha}(t)^2 + \left(\varkappa_{ij} + \frac{\tau_q}{2}\ \zeta_{ij}\right)\alpha_{,i}(t)\alpha_{,j}(t)\right]dv + \int_0^t\int_B \left[\zeta_{ij}\alpha_{,i}(s)\alpha_{,j}(s) + \tau_q\varkappa_{ij}\dot{\alpha}_{,i}(s)\dot{\alpha}_{,j}(s)\right]dvds, \tag{51}$$

and note that

$$\dot{G}(t) = 2\int_B \left[c\tilde{\alpha}(t)\dot{\tilde{\alpha}}(t) + \left(\varkappa_{ij} + \frac{\tau_q}{2}\ \zeta_{ij}\right)\alpha_{,i}(t)\dot{\alpha}_{,j}(t)\right]dv + 2\int_0^t\int_B \left[\zeta_{ij}\alpha_{,i}(s)\dot{\alpha}_{,j}(s) + \tau_q\varkappa_{ij}\dot{\alpha}_{,i}(s)\ddot{\alpha}_{,j}(s)\right]dvds, \tag{52}$$

and moreover,

$$\ddot{G}(t) = 2\int_B \left[c\dot{\tilde{\alpha}}(t)^2 + \left(\varkappa_{ij} + \frac{\tau_q}{2}\ \zeta_{ij}\right)\dot{\alpha}_{,i}(t)\dot{\alpha}_{,j}(t) + c\tilde{\alpha}(t)\ddot{\tilde{\alpha}}(t) + \left(\varkappa_{ij} + \frac{\tau_q}{2}\ \zeta_{ij}\right)\alpha_{,i}(t)\ddot{\alpha}_{,j}(t) + \zeta_{ij}\alpha_{,i}(t)\dot{\alpha}_{,j}(t) + \tau_q\varkappa_{ij}\dot{\alpha}_{,i}(t)\ddot{\alpha}_{,j}(t)\right\}dv. \tag{53}$$

Furthermore, in view of the basic equations (13) and (19) and by using the null given data, we have

$$\int_B c\tilde{\alpha}(t)\ddot{\tilde{\alpha}}(t)dv = -\int_B\left[\tau_q\zeta_{ij}\left(\dot{\alpha}_{,i}(t) + \frac{\tau_q}{2}\ \ddot{\alpha}_{,i}(t)\right)\left(\dot{\alpha}_{,j}(t) + \frac{\tau_q}{2}\ \ddot{\alpha}_{,j}(t)\right) + k^*_{ij}\tilde{\alpha}_{,i}(t)\tilde{\alpha}_{,j}(t) + \frac{\tau_q^2}{2}\ \varkappa_{ij}\ddot{\alpha}_{,i}(t)\ddot{\alpha}_{,j}(t) + \zeta_{ij}\alpha_{,i}(t)\dot{\alpha}_{,j}(t) + \left(\varkappa_{ij} + \frac{\tau_q}{2}\ \zeta_{ij}\right)\alpha_{,i}(t)\ddot{\alpha}_{,j}(t) + \tau_q\varkappa_{ij}\dot{\alpha}_{,i}(t)\ddot{\alpha}_{,j}(t)\right]dv. \tag{54}$$

Thus, relations (53) and (54) furnish

$$\ddot{G}(t) = 2\int_B \left[c\dot{\tilde{\alpha}}(t)^2 + \left(\varkappa_{ij} + \frac{\tau_q}{2}\ \zeta_{ij}\right)\dot{\alpha}_{,i}(t)\dot{\alpha}_{,j}(t) - k^*_{ij}\tilde{\alpha}_{,i}(t)\tilde{\alpha}_{,j}(t) - \tau_q\zeta_{ij}\left(\dot{\alpha}_{,i}(t) + \frac{\tau_q}{2}\ \ddot{\alpha}_{,i}(t)\right)\left(\dot{\alpha}_{,j}(t) + \frac{\tau_q}{2}\ \ddot{\alpha}_{,j}(t)\right) - \frac{\tau_q^2}{2}\ \varkappa_{ij}\ddot{\alpha}_{,i}(t)\ddot{\alpha}_{,j}(t)\right]dv. \tag{55}$$

Finally, the conservation law $\mathcal{E}(t) = 0$ gives

$$-\int_B \left[k^*_{ij}\tilde{\alpha}_{,i}(t)\tilde{\alpha}_{,j}(t) + \tau_q\zeta_{ij}\left(\dot{\alpha}_{,i}(t) + \frac{\tau_q}{2}\ \ddot{\alpha}_{,i}(t)\right)\left(\dot{\alpha}_{,j}(t) + \frac{\tau_q}{2}\ \ddot{\alpha}_{,j}(t)\right) + \frac{\tau_q^2}{2}\ \varkappa_{ij}\ddot{\alpha}_{,i}(t)\ddot{\alpha}_{,j}(t)\right] dv = \int_B \left[c\dot{\tilde{\alpha}}(t)^2 + \left(\varkappa_{ij} + \frac{\tau_q}{2}\ \zeta_{ij}\right)\dot{\alpha}_{,i}(t)\dot{\alpha}_{,j}(t)\right] dv + 2\int_0^t \int_B \left[\zeta_{ij}\dot{\alpha}_{,i}(s)\dot{\alpha}_{,j}(s) + \tau_q\varkappa_{ij}\ddot{\alpha}_{,i}(s)\ddot{\alpha}_{,j}(s)\right] dvds, \tag{56}$$

so that the relation (55) becomes

$$\ddot{G}(t) = 4\left\{ \int_B \left[c\dot{\tilde{\alpha}}(t)^2 + \left(\varkappa_{ij} + \frac{\tau_q}{2}\ \zeta_{ij}\right)\dot{\alpha}_{,i}(t)\dot{\alpha}_{,j}(t)\right] dv + \int_0^t \int_B \left[\zeta_{ij}\dot{\alpha}_{,i}(s)\dot{\alpha}_{,j}(s) + \tau_q\varkappa_{ij}\ddot{\alpha}_{,i}(s)\ddot{\alpha}_{,j}(s)\right] dvds \right\}. \tag{57}$$

Based on the Cauchy-Schwarz inequality, from relations (51), (52) and (57) we deduce that

$$G(t)\ddot{G}(t) - \dot{G}(t)^2 \geq 0, \quad \text{for all} \quad t > 0, \tag{58}$$

that proves that $\ln[G(t)]$ is a convex function on $(0, \infty)$. Thus, we conclude that

$$G(t) = 0, \tag{59}$$

and the proof follows like the above Proof 1. □

Remark. The Lagrange identity method and the logarithmic convexity method also allow the study of continuous dependence with respect to the given data, under the same assumptions (H1), (H3) and (H4), following, for example, the analysis carried out in the works by Rionero and Chiriţă [34] and Knops and Payne [35]. However, giving up the hypothesis (H2) requires appropriate restrictions on the solutions under discussion!

5. Wave solutions

In this section we will study possible waves that can propagate in a rigid thermal conductor with triple-phase-lags. To simplify the mathematical calculations we will consider the case of an isotropic and homogeneous material conductor where $k_{ij} = k\delta_{ij}$ and $k^*_{ij} = k^*\delta_{ij}$. According with the hypotheses (H1) to (H4) we assume now that

$$c > 0, \quad k^* > 0, \quad \zeta = k + (\tau_\alpha - \tau_q)k^* > 0, \quad \varkappa = \tau_T k - \frac{\tau_q}{2}\ (k + \tau_\alpha k^*) > 0. \tag{60}$$

We try to find wave solutions of the basic equations in the form of a wave propagating in the direction of x_1 axis as

$$\alpha(x_1, t) = \text{Re}\{A\ e^{i\chi(x_1 - vt)}\}, \tag{61}$$

where $i = \sqrt{-1}$ is the imaginary unit, $\text{Re}\{\cdot\}$ is the real part, $\chi > 0$ is the real wave number and A is a complex nonzero number. Further, x_1 is the spatial coordinate in the propagation direction and v is a complex parameter so that $\text{Re}(v) \geq 0$ will represent the wave speed and $\text{Im}(v) \leq 0$ will be related to the rate of decaying in time. We must note that for $\text{Re}(v) > 0$ there is a genuine wave, while for $\text{Re}(v) = 0$ there is a standing mode. Moreover, when $\text{Im}(v) < 0$ there is the phenomenon of damping in time, while for $\text{Im}(v) = 0$ there is an undamped in time wave.

When we replace the expression (61) in the equation (8) we are led to the following algebraic equation for determining parameter v:

$$\frac{1}{2}\ c\tau_q^2\chi^2\ \nu^4 + ic\tau_q\chi\ \nu^3 - \left(c + \tau_T k\chi^2\right)\nu^2 - i\chi(k + \tau_\alpha\ k^*)\nu + k^* = 0, \tag{62}$$

which can be written in terms of the parameter

$$\omega = -i\ \nu, \tag{63}$$

as the following four-degree algebraic equation with positive coefficients

$$P(\omega) \equiv \omega^4 + a_3\omega^3 + a_2\omega^2 + a_1\omega + a_0 = 0, \tag{64}$$

where

$$\begin{aligned} a_3 &= \frac{2}{\tau_q\chi}, \quad a_2 = \frac{2}{c\tau_q^2\chi^2}\left(c + \tau_T k\chi^2\right), \\ a_1 &= \frac{2}{c\tau_q^2\chi}(k + \tau_\alpha\ k^*), \quad a_0 = \frac{2k^*}{c\tau_q^2\chi^2}. \end{aligned} \tag{65}$$

According to the Routh-Hurwitz criterion the polynomial $P(\omega)$ will have all the roots in the open left half-plane if and only if

$$a_3a_2a_1 - a_1^2 - a_3^2a_0 > 0. \tag{66}$$

In view of relation (65), we have

$$a_3a_2a_1 - a_1^2 - a_3^2a_0 = \frac{8}{c^2\tau_q^5\chi^4}\left[c\zeta + \chi^2(k + \tau_\alpha\ k^*)\varkappa\right], \tag{67}$$

that is strictly positive based on the hypotheses described by the relationship (60). Therefore, the fourth-degree polynomial $P(\omega)$ has all four roots with negative real part.

For a generic root $\omega = -\nu \pm i\eta$, with $\nu > 0$, $\eta > 0$, of the polynomial $P(\omega)$, we are led to wave solutions of the form

$$\alpha(x_1, t) = \text{Re}\{A\ \ e^{i\chi(x_1 - \eta t)}\}e^{-\chi\nu t}, \tag{68}$$

when a genuine complex root is considered, or

$$\alpha(x_1, t) = \text{Re}\{A\ \ e^{i\chi x_1}\}e^{-\chi\nu t}, \tag{69}$$

when there is a real root.

Concluding, we can see that the restrictions described in (60) show that one can have genuine wave solutions whose amplitude decreases exponentially in time (in the case of a complex root) or wave solutions in form of standing mode exponentially decaying in time (when there is a negative real root).

It is important to note that if the thermodynamic restrictions (9) are not fulfilled (i.e., $\zeta < 0$ or $\varkappa < 0$) then inequality (66) can no longer be fulfilled and then one of the solutions presented in (68) or (69) contains a term of the form $e^{\chi\nu t}$ with $\nu > 0$ which becomes infinite when time increases to infinity. In such a case the energy blows up as time goes to infinity. We are then led to unrealistic wave solutions, with infinite energy, which expresses exponential instability.

6. Exponential stability

Guided by the results of the previous section as well as by the exponential decrease present in the classic theory of heat conduction based on Fourier's law, we think that this exponential decrease in time could also be present in the three-phase-lag model in the study. With this in our mind, just to simplify the reasoning, we suppose a homogeneous material and then we proceed to establish the exponential decay in time result.

To develop our analysis in the remainder of this paper we need to strengthen our previous hypotheses (H1) to (H4) in the sense that:

($\tilde{H}1$): the specific heat per unit volume is strictly positive, that is

$$c > 0; \tag{70}$$

($\tilde{H}2$): k^*_{ij} is a positive definite tensor as

$$k^*_{rs}\xi_r\xi_s \geq k^*_m\xi_i\xi_i \quad \text{for all} \quad (\xi_1, \xi_2, \xi_3) \neq 0; \tag{71}$$

($\tilde{H}3$): $\zeta_{ij} = k_{ij} + (\tau_\alpha - \tau_q)k^*_{ij}$ is a positive definite tensor as

$$\zeta_{rs}\xi_r\xi_s \geq \zeta_m\xi_i\xi_i \quad \text{for all} \quad (\xi_1, \xi_2, \xi_3) \neq 0; \tag{72}$$

($\tilde{H}4$): $\varkappa_{ij} = \tau_T \ k_{ij} - \frac{\tau_q}{2} \ (k_{ij} + \tau_\alpha \ k^*_{ij})$ is a positive definite tensor, that is

$$\varkappa_{rs}\xi_r\xi_s \geq \varkappa_m\xi_i\xi_i \quad \text{for all} \quad (\xi_1, \xi_2, \xi_3) \neq 0, \tag{73}$$

where k^*_m, ζ_m and $\varkappa_m$ are the smallest eigenvalues of the tensors k^*_{ij}, ζ_{ij} and $\varkappa_{ij}$, respectively.

Theorem 4. *(**Exponential stability**) Suppose the hypotheses ($\tilde{H}1$) to ($\tilde{H}4$) to be fulfilled. Also we assume that meas(Σ_1) $\neq 0$. Let $\mathcal{S} = \{\alpha, q_i\}$ be a solution of the initial boundary value problem $\mathcal{P}$ corresponding to the given data $\mathcal{D} = \{0; \dot{\alpha}^0, q^0_i, \dot{q}^0_i; 0, 0\}$. Then $\mathcal{S} = \{\alpha, q_i\}$ is exponentially stable.*

Proof. Since *meas*(Σ_1) $\neq 0$ and by using the boundary condition $u = 0$ on Σ_1, we will have occasion to use the Poincaré inequality

$$\int_B u_{,i}u_{,i}dv \geq \lambda \int_B u^2 dv, \tag{74}$$

where λ is the minimal eigenvalue of the (negative) Laplace operator in the space $W^{1,2}_0(B)$. In this connection we note that the identity (26) implies

$$\frac{dE_1}{dt}(t) = -\int_B \left[\zeta_{ij}\dot{\alpha}_{,i}(t)\dot{\alpha}_{,j}(t) + \tau_q\varkappa_{ij}\ddot{\alpha}_{,i}(t)\ddot{\alpha}_{,j}(t)\right]dv, \tag{75}$$

so that, by means of the inequalities (72), (73) and (74), we deduce

$$\frac{dE_1}{dt}(t) \leq -\lambda\int_B \left[\zeta_m\dot{\alpha}(t)^2 + \tau_q\varkappa_m\ddot{\alpha}(t)^2\right]dv, \tag{76}$$

and therefore, we have

$$\begin{aligned}\frac{dE_1}{dt}(t) \leq &-\frac{1}{2}\int_B \left[\zeta_m\dot{\alpha}_{,i}(t)\dot{\alpha}_{,i}(t) + \tau_q\varkappa_m\ddot{\alpha}_{,i}(t)\ddot{\alpha}_{,i}(t)\right]dv \\ &-\frac{\lambda}{2}\int_B \left[\zeta_m\dot{\alpha}(t)^2 + \tau_q\varkappa_m\ddot{\alpha}(t)^2\right]dv.\end{aligned} \tag{77}$$

Moreover, from the relations (28), (71) to (74), we have

$$\begin{aligned}E_1(t) \geq &\frac{1}{2} \int_B \Bigg\{c\dot{\tilde{\alpha}}(t)^2 + k^*_m\lambda\tilde{\alpha}(t)^2 + \lambda\tau_q\zeta_m\left(\dot{\alpha}(t) + \frac{\tau_q}{2} \ \ddot{\alpha}(t)\right)^2 \\ &+\lambda\varkappa_m\dot{\alpha}(t)^2 + \frac{\tau_q}{2}\left[\zeta_m\dot{\alpha}_{,i}(t)\dot{\alpha}_{,i}(t) + \tau_q\varkappa_m\ddot{\alpha}_{,i}(t)\ddot{\alpha}_{,i}(t)\right]\Bigg\}dv,\end{aligned} \tag{78}$$

and therefore $E_1(t)$ appears like a measure $L_2(B)$ for $\{\alpha, \alpha_{,i}, \dot{\alpha}, \ddot{\alpha}, \dot{\alpha}_{,i}, \ddot{\alpha}_{,i}, \dddot{\alpha}\}$ and hence there exist the computable constants $c_m > 0$ and $c_M > 0$, so that

$$0 < c_m \ meas\{\alpha, \alpha_{,i}, \dot{\alpha}, \ddot{\alpha}, \dot{\alpha}_{,i}, \ddot{\alpha}_{,i}, \dddot{\alpha}\} \leq E_1(t) \leq c_M \ meas\{\alpha, \alpha_{,i}, \dot{\alpha}, \ddot{\alpha}, \dot{\alpha}_{,i}, \ddot{\alpha}_{,i}, \dddot{\alpha}\}. \tag{79}$$

At this point it is useful to remember that, in the situation when the three relaxation times could be negligible, then the two relations (28) and (75) could be coupled to give the differential inequality

$$\frac{dE_1^0}{dt}(t) \leq -\frac{2\lambda k_m}{c} \ E_1^0(t), \tag{80}$$

which integrated implies the following exponential decay estimate

$$E_1^0(t) \leq E_1^0(0)e^{(-2\lambda k_m t)/c}, \tag{81}$$

that is the exponential decay estimate corresponding to the classical heat equation based on the Fourier law for the heat flux. Here $E_1^0(t)$ is the expression of $E_1(t)$ when the three relaxation times are neglected.

Returning to the model considered by us, it can be seen from the relations (77) and (78) that the terms $\alpha(t)^2$ and $\dddot{\alpha}(t)^3$ are missing in the second member of the relation (77). To fix this inconvenience we have to use other identities found through the basic equations to help add the missing terms.

Consequently, if we multiply the heat equation (13), with null heat source, by α and $\dddot{\alpha}$, respectively, and then we use the relations (12) and (14), we obtain the following identities

$$\begin{aligned}\frac{dE_2}{dt}(t) = \int_B \Big[&c\dot{\alpha}(t)^2 + c\tau_q\dot{\alpha}(t)\ddot{\alpha}(t) + \tau_T k_{ij}\dot{\alpha}_{,i}(t)\dot{\alpha}_{,j}(t)\\ &+\left(c\tau_q^2/2\right) \ \dot{\alpha}(t)\dddot{\alpha}(t) - k_{ij}^*\alpha_{,i}(t)\alpha_{,j}(t)\Big]dv,\end{aligned} \tag{82}$$

$$\frac{dE_3}{dt}(t) = \int_B \left[k_{ij}^*\dot{\alpha}_{,i}(t)\ddot{\alpha}_{,j}(t) + \left(k_{ij} + \tau_\alpha k_{ij}^*\right)\ddot{\alpha}_{,i}(t)\ddot{\alpha}_{,j}(t) - c\tau_q \ \dddot{\alpha}(t)^2\right]dv, \tag{83}$$

where

$$E_2(t) = \int_B \left[c\alpha(t)\dot{\alpha}(t) + \tau_T k_{ij}\alpha_{,i}(t)\dot{\alpha}_{,j}(t) + (1/2) \ \left(k_{ij} + \tau_\alpha k_{ij}^*\right)\alpha_{,i}(t)\alpha_{,j}(t)\right]dv, \tag{84}$$

and

$$\begin{aligned}E_3(t) =\frac{1}{2} \int_B \Big[&c\ddot{\alpha}(t)^2 + (1/2)c\tau_q^2\dddot{\alpha}(t)^2 + \tau_T k_{ij}\ddot{\alpha}_{,i}(t)\ddot{\alpha}_{,j}(t)\\ &+2\left(k_{ij} + \tau_\alpha \ k_{ij}^*\right)\dot{\alpha}_{,i}(t)\ddot{\alpha}_{,j}(t) + 2k_{ij}^*\alpha_{,i}(t)\ddot{\alpha}_{,j}(t)\Big]dv.\end{aligned} \tag{85}$$

Therefore, if we set

$$F_1(t) = E_2(t) + \tau_q^3 E_3(t), \tag{86}$$

then we have

$$\begin{aligned}\frac{dF_1}{dt}(t) = \int_B \Big[&(9c/8)\dot{\alpha}(t)^2 + c\tau_q\dot{\alpha}(t)\ddot{\alpha}(t) + \tau_T k_{ij}\dot{\alpha}_{,i}(t)\dot{\alpha}_{,j}(t)\\ &+\tau_q^3 \ k_{ij}^*\dot{\alpha}_{,i}(t)\ddot{\alpha}_{,j}(t) + \tau_q^3\left(k_{ij} + \tau_\alpha \ k_{ij}^*\right)\ddot{\alpha}_{,i}(t)\ddot{\alpha}_{,j}(t)\\ &-(c/8)\left(2\tau_q^2\dddot{\alpha}(t) - \dot{\alpha}(t)\right)^2 - k_{ij}^*\alpha_{,i}(t)\alpha_{,j}(t) - (c\tau_q^4/2)\dddot{\alpha}(t)^2\Big]dv,\end{aligned} \tag{87}$$

Now we use the Cauchy-Schwarz and arithmetic-geometric mean inequalities in order to obtain the estimate

$$\begin{aligned}\frac{dF_1}{dt}(t) \leq \int_B \Bigg[&\frac{13c}{8} \ \dot{\alpha}(t)^2 + \frac{c\tau_q^2}{2} \ \ddot{\alpha}(t)^2 + K_1\tau_q\zeta_m\dot{\alpha}_{,i}(t)\dot{\alpha}_{,i}(t)\\ &+K_2\tau_q^2\varkappa_m\ddot{\alpha}_{,i}(t)\ddot{\alpha}_{,i}(t)\Bigg]dv - \int_B \left[k_{ij}^*\alpha_{,i}(t)\alpha_{,j}(t) + (c\tau_q^4/2)\dddot{\alpha}(t)^2\right]dv,\end{aligned} \tag{88}$$

where

$$K_1 = \frac{1}{\tau_q \zeta_m} \left[\left(\tau_T k_{rs} + \frac{\tau_q^2}{2} k_{rs}^* \right) \left(\tau_T k_{rs} + \frac{\tau_q^2}{2} k_{rs}^* \right) \right]^{1/2},$$
$$K_2 = \frac{\tau_q}{\varkappa_m} \left(\left[k_{rs} + \left(\tau_\alpha + \frac{\tau_q}{2} \right) k_{rs}^* \right] \left[k_{rs} + \left(\tau_\alpha + \frac{\tau_q}{2} \right) k_{rs}^* \right] \right)^{1/2}. \tag{89}$$

Furthermore, we want to dominate the first integral term from relation (88) with the help of relation (77). For this purpose we choose the non-dimensional parameter $\delta > 0$ and then we introduce the following function

$$\mathcal{F}(t) = \tau_q E_1(t) + \delta F_1(t), \tag{90}$$

so that, by means of the relations (77) and (88), we have

$$\frac{d\mathcal{F}}{dt}(t) \le -\frac{1}{2} \int_B \left[\gamma_1 c \dot{\alpha}(t)^2 + \gamma_2 c \tau_q^2 \ddot{\alpha}(t)^2 \right] dv - \frac{1}{2} \int_B [\gamma_3 \tau_q \zeta_m \dot{\alpha}_{,i}(t) \dot{\alpha}_{,i}(t)$$
$$+ \gamma_4 \tau_q^2 \varkappa_m \ddot{\alpha}_{,i}(t) \ddot{\alpha}_{,i}(t)] dv - \frac{\delta}{2} \int_B \left[2 k_{ij}^* \alpha_{,i}(t) \alpha_{,j}(t) + c \tau_q^4 \dddot{\alpha}(t)^2 \right] dv, \tag{91}$$

where

$$\gamma_1 = \frac{\lambda \tau_q \zeta_m}{c} - \frac{13\delta}{4}, \quad \gamma_2 = \frac{\lambda \varkappa_m}{c} - \delta, \quad \gamma_3 = 1 - 2\delta K_1, \quad \gamma_4 = 1 - 2\delta K_2. \tag{92}$$

At this moment, we choose the parameter δ so small in such a way as to make positive the four coefficients γ_1 to γ_4, that is, we put

$$0 < \delta < \delta_m, \quad \delta_m = \min \left(\frac{4\lambda \tau_q \zeta_m}{13c}, \frac{\lambda \kappa_m}{c}, \frac{1}{2K_1}, \frac{1}{2K_2} \right), \tag{93}$$

so that there is a positive computable constant f_m in order to have

$$\frac{d\mathcal{F}}{dt}(t) \le -f_m \ meas\{\alpha, \alpha_{,i}, \dot{\alpha}, \ddot{\alpha}, \dot{\alpha}_{,i}, \ddot{\alpha}_{,i}, \dddot{\alpha}\}. \tag{94}$$

Next, we present an evaluation of the expression $F_1(t)$. In this sense, we use relations (84) to (86) to write

$$F_1(t) = \int_B \Bigg[\frac{c}{2\tau_q} \left(\alpha(t) + \tau_q \dot{\alpha}(t) \right)^2 + \frac{1}{2} k_{ij} (\alpha_{,i}(t) + \tau_T \dot{\alpha}_{,i}(t)) (\alpha_{,j}(t) + \tau_T \dot{\alpha}_{,j}(t))$$
$$+ \frac{c \tau_q^3}{2} \ddot{\alpha}(t)^2 + \frac{c \tau_q^5}{4} \dddot{\alpha}(t)^2 + \frac{\tau_q^2}{2} \left(k_{ij} + \tau_\alpha k_{ij}^* \right) \left(\dot{\alpha}_{,i}(t) + \tau_q \ddot{\alpha}_{,i}(t) \right) \left(\dot{\alpha}_{,j}(t) + \tau_q \ddot{\alpha}_{,j}(t) \right)$$
$$+ \tau_q k_{ij}^* \left(\varepsilon \alpha_{,i}(t) + \frac{1}{2\varepsilon} \tau_q^2 \ddot{\alpha}_{,i}(t) \right) \left(\varepsilon \alpha_{,j}(t) + \frac{1}{2\varepsilon} \tau_q^2 \ddot{\alpha}_{,j}(t) \right) + \frac{1}{2} \tau_T \tau_q^3 k_{ij} \ddot{\alpha}_{,i}(t) \ddot{\alpha}_{,j}(t) \Bigg] dv$$
$$- \int_B \Bigg[\frac{c}{2\tau_q} \alpha(t)^2 + \frac{c \tau_q}{2} \dot{\alpha}(t)^2 + \frac{1}{2} \left(\tau_T^2 k_{ij} + \tau_q^2 \left(k_{ij} + \tau_\alpha k_{ij}^* \right) \right) \dot{\alpha}_{,i}(t) \dot{\alpha}_{,j}(t)$$
$$+ \left(\tau_q \left(k_{ij} + \tau_\alpha k_{ij}^* \right) + \frac{\tau_q^2}{2\varepsilon^2} k_{ij}^* \right) \ddot{\alpha}_{,i}(t) \ddot{\alpha}_{,j}(t) \Bigg] dv + \left(\frac{\tau_\alpha}{2} - \tau_q \varepsilon^2 \right) \int_B k_{ij}^* \alpha_i(t) \alpha_{,j}(t) dv, \tag{95}$$

where ε is a positive parameter at our hand. Now we choose ε so that

$$\varepsilon = \sqrt{\frac{\tau_\alpha}{2\tau_q}}, \tag{96}$$

and note that

$$F_1(t) \geq -\int_B \left[\frac{c}{2\tau_q} \alpha(t)^2 + \frac{c\tau_q}{2} \dot{\ddot{\alpha}}(t)^2 + \frac{1}{2} \left(\tau_T^2 k_{ij} + \tau_q^2 \left(k_{ij} + \tau_\alpha k_{ij}^* \right) \right) \dot{\alpha}_{,i}(t) \dot{\alpha}_{,j}(t) \right. \\ \left. + \left(\tau_q \left(k_{ij} + \tau_\alpha k_{ij}^* \right) + \frac{\tau_q^2}{2\varepsilon^2} k_{ij}^* \right) \ddot{\alpha}_{,i}(t) \ddot{\alpha}_{,j}(t) \right] dv \tag{97}$$

and

$$F_1(t) \leq \int_B \left[\frac{c}{2\tau_q} \left(\alpha(t) + \tau_q \dot{\ddot{\alpha}}(t) \right)^2 + \frac{1}{2} k_{ij} (\alpha_{,i}(t) + \tau_T \dot{\alpha}_{,i}(t)) (\alpha_{,j}(t) + \tau_T \dot{\alpha}_{,j}(t)) \right. \\ + \frac{c\tau_q^3}{2} \ddot{\alpha}(t)^2 + \frac{c\tau_q^5}{4} \dddot{\alpha}(t)^2 + \frac{\tau_q^2}{2} \left(k_{ij} + \tau_\alpha k_{ij}^* \right) \left(\dot{\alpha}_{,i}(t) + \tau_q \ddot{\alpha}_{,i}(t) \right) \left(\dot{\alpha}_{,j}(t) + \tau_q \ddot{\alpha}_{,j}(t) \right) \\ \left. + \tau_q k_{ij}^* \left(\varepsilon \alpha_{,i}(t) + \frac{1}{2\varepsilon} \tau_q^2 \ddot{\alpha}_{,i}(t) \right) \left(\varepsilon \alpha_{,j}(t) + \frac{1}{2\varepsilon} \tau_q^2 \ddot{\alpha}_{,j}(t) \right) + \frac{1}{2} \tau_T \tau_q^3 k_{ij} \ddot{\alpha}_{,i}(t) \ddot{\alpha}_{,j}(t) \right] dv. \tag{98}$$

It can be seen from (97) that one can determine a positive constant h_m so that

$$F_1(t) \geq -h_m meas\{\alpha, \alpha_{,i}, \dot{\alpha}, \ddot{\alpha}, \dot{\alpha}_{,i}, \ddot{\alpha}_{,i}, \dddot{\alpha}\}, \tag{99}$$

and therefore, by means of relation (79), we will have

$$\mathcal{F}(t) = \tau_q E_1(t) + \delta F_1(t) \geq \left(\tau_q c_m - \delta h_m \right) meas\{\alpha, \alpha_{,i}, \dot{\alpha}, \ddot{\alpha}, \dot{\alpha}_{,i}, \ddot{\alpha}_{,i}, \dddot{\alpha}\} > 0, \tag{100}$$

provided we choose the parameter δ so that

$$0 < \delta < \delta_m^*, \quad \delta_m^* = \frac{\tau_q c_m}{h_m}. \tag{101}$$

On the other side, by using the Cauchy-Schwarz and arithmetic-geometric mean inequalities and the relation (98), we can get a positive computable constant C so that

$$F_1(t) \leq C meas\{\alpha, \alpha_{,i}, \dot{\alpha}, \ddot{\alpha}, \dot{\alpha}_{,i}, \ddot{\alpha}_{,i}, \dddot{\alpha}\}, \tag{102}$$

and hence we have

$$\mathcal{F}(t) = \tau_q E_1(t) + \delta F_1(t) \leq \left(\tau_q c_M + \delta C \right) meas\{\alpha, \alpha_{,i}, \dot{\alpha}, \ddot{\alpha}, \dot{\alpha}_{,i}, \ddot{\alpha}_{,i}, \dddot{\alpha}\}. \tag{103}$$

Consequently, from the relations (94) and (103), we get

$$\frac{d\mathcal{F}}{dt}(t) \leq -\frac{f_m}{\tau_q c_M + \delta C} \mathcal{F}(t), \tag{104}$$

provided $0 < \delta < \min\{\delta_m, \delta_m^*\}$. The differential inequality (104) shows the expected exponential decrease in time of the solution like in (81). □

7. Domain of influence

Throughout this section we shall assume that a semi-infinite cylindrical region $B = D \times (0, \infty)$ is filled by a homogeneous and anisotropic conductor material with three-phase-lag times. We choose a Cartesian coordinate system $Ox_1x_2x_3$ in such a way that the base of the cylinder is contained in the plane $x_3 = 0$, and the axis Ox_3 is parallel to the generators of the cylindrical surface.

Throughout this section we consider the initial boundary value problem $\mathcal{P}_C$ defined by the heat equation (6), with null heat source, the constitutive equation (4), the initial conditions

$$\alpha(x,0)=0,\quad \dot{\alpha}(x,0)=0,\quad q_i(x,0)=0,\quad \dot{q}_i(x,0)=0,\quad \text{for all}\quad x\in B, \tag{105}$$

and the following boundary conditions

$$\begin{aligned} &q_\rho(x_1,x_2,x_3,t)n_\rho=0 \quad \text{for all} \quad (x_1,x_2,x_3)\in[\partial D\times(0,\infty)],\\ &q_3(x_1,x_2,0,t)=g(x_1,x_2,t) \quad \text{for all} \quad (x_1,x_2)\in D_0 \quad \text{and} \quad t\in(0,\infty). \end{aligned} \tag{106}$$

Here n_ρ are the components of the outward normal to the lateral surface of the cylinder and $g(x_1,x_2,t)$ is a prescribed smooth function, while D_0 is the base section of the cylinder. Throughout this section we are interested in how the solution of the initial boundary value problem $\mathcal{P}_C$ behaves with respect to the distance x_3 at the loaded base $x_3=0$. In this sense, we want to identify appropriate measures associated with the solution $\mathcal{S}=\{\alpha,q_i\}$ of the problem in question $\mathcal{P}_C$ that describe its behavior in terms of the distance x_3 to the base acted by the specified load $g(x_1,x_2,t)$.

We try to study our problem by using an associated "measure" of the solution $\mathcal{S}=\{\alpha,q_i\}$ like

$$H(x_3,t)=\int_0^t\int_{D_{x_3}} \tilde{q}_3(z)\dot{\tilde{\alpha}}(z)dadz,\quad x_3>0,\quad t>0, \tag{107}$$

where D_{x_3} is the transverse section of the cylinder with the plane $x_3=$ constant. We note that

$$\frac{\partial H}{\partial t}(x_3,t)=\int_{D_{x_3}} \tilde{q}_3(t)\dot{\tilde{\alpha}}(t)da,\quad x_3>0,\quad t>0, \tag{108}$$

and moreover,

$$\frac{\partial H}{\partial x_3}(x_3,t)=\int_0^t\int_{D_{x_3}} \left[\tilde{q}_{3,3}(z)\dot{\tilde{\alpha}}(z)+\tilde{q}_3(z)\dot{\tilde{\alpha}}_{,3}(z)\right]dadz. \tag{109}$$

Furthermore, on the basis of the equation (13) (with null heat source), the lateral boundary condition (106) and the divergence theorem we obtain

$$\frac{\partial H}{\partial x_3}(x_3,t)=-\int_0^t\int_{D_{x_3}} c\dot{\tilde{\alpha}}(z)\ddot{\tilde{\alpha}}(z)dadz+\int_0^t\int_{D_{x_3}} \tilde{q}_i(z)\dot{\tilde{\alpha}}_{,i}(z)dadz. \tag{110}$$

The relation (20), when substituted in relation (110), coupled with the use of the initial conditions (105), provides

$$\begin{aligned} -\frac{\partial H}{\partial x_3}(x_3,t)=&\frac{1}{2}\int_{D_{x_3}}\Big\{c\dot{\tilde{\alpha}}(t)^2+k_{ij}^*\tilde{\alpha}_{,i}(t)\tilde{\alpha}_{,j}(t)\\ &+\tau_q\zeta_{ij}\left(\dot{\alpha}_{,i}(t)+\frac{\tau_q}{2}\ \ddot{\alpha}_{,i}(t)\right)\left(\dot{\alpha}_{,j}(t)+\frac{\tau_q}{2}\ \ddot{\alpha}_{,j}(t)\right)\\ &+\varkappa_{ij}\dot{\alpha}_{,i}(t)\dot{\alpha}_{,j}(t)+\frac{\tau_q}{2}\ \left[\zeta_{ij}\dot{\alpha}_{,i}(t)\dot{\alpha}_{,j}(t)+\tau_q\varkappa_{ij}\ddot{\alpha}_{,i}(t)\ddot{\alpha}_{,j}(t)\right]\Big\}da\\ &+\textstyle\int_0^t\int_{D_{x_3}}\ \left[\zeta_{ij}\dot{\alpha}_{,i}(z)\dot{\alpha}_{,j}(z)+\tau_q\varkappa_{ij}\ddot{\alpha}_{,i}(z)\ddot{\alpha}_{,j}(z)\right]dadz. \end{aligned} \tag{111}$$

Consequently, in view of our hypotheses $(\tilde{H}1)$ to $(\tilde{H}4)$, we deduce

$$\begin{aligned} -\frac{\partial H}{\partial x_3}(x_3,t)\geq&\frac{1}{2}\int_{D_{x_3}}\Big[c\dot{\tilde{\alpha}}(t)^2+k_m^*\tilde{\alpha}_{,i}(t)\tilde{\alpha}_{,i}(t)+\frac{\tau_q^2}{2}\ \kappa_m\ddot{\alpha}_{,i}(t)\ddot{\alpha}_{,i}(t)\\ &+\tau_q\zeta_m\left(\dot{\alpha}_{,i}(t)+\frac{\tau_q}{2}\ \ddot{\alpha}_{,i}(t)\right)\left(\dot{\alpha}_{,i}(t)+\frac{\tau_q}{2}\ \ddot{\alpha}_{,i}(t)\right)\Big]da\geq 0, \end{aligned} \tag{112}$$

and hence $H(x_3,t)$ is a non-increasing function with respect to x_3 for all $t>0$. As we will see later, this last inequality suggests that $H(x_3,t)$ can lead to a measure of the solution $\mathcal{S}=\{\alpha,q_i\}$ of our initial boundary value problem $\mathcal{P}_C$.

Further, we use the Cauchy-Schwarz and the arithmetic-geometric mean inequalities into relation (108) in order to obtain

$$\left|\frac{\partial H}{\partial t}\right|(x_3,t)\le\frac{1}{2}\int_{D_{x_3}}\left[\varepsilon c\dot{\tilde{\alpha}}(t)^2+\frac{1}{\varepsilon c}\ \tilde{q}_3(t)^2\right]da, \tag{113}$$

for any positive parameter ε. Now from the constitutive relation (19) we deduce

$$\begin{aligned}|\tilde{q}_3|(t)\le{}&(k^*_{3r}k^*_{3r})^{1/2}[\tilde{\alpha}_{,i}(t)\tilde{\alpha}_{,i}(t)]^{1/2}+(\varkappa_{3r}\varkappa_{3r})^{1/2}[\ddot{\alpha}_{,i}(t)\ddot{\alpha}_{,i}(t)]^{1/2}\\&+(\zeta_{3r}\zeta_{3r})^{1/2}\left[\left(\dot{\alpha}_{,i}(t)+\tfrac{\tau_q}{2}\ \ddot{\alpha}_{,i}(t)\right)\left(\dot{\alpha}_{,i}(t)+\tfrac{\tau_q}{2}\ \ddot{\alpha}_{,i}(t)\right)\right]^{1/2},\end{aligned}\tag{114}$$

and hence we get

$$\begin{aligned}\tilde{q}_3(t)^2\le 3M^2\Bigg[&k^*_m\tilde{\alpha}_{,i}(t)\tilde{\alpha}_{,i}(t)+\tau_q\zeta_m\left(\dot{\alpha}_{,i}(t)+\frac{\tau_q}{2}\ \ddot{\alpha}_{,i}(t)\right)\times\\&\times\left(\dot{\alpha}_{,i}(t)+\frac{\tau_q}{2}\ \ddot{\alpha}_{,i}(t)\right)+\frac{\tau_q^2}{2}\ \varkappa_m\ddot{\alpha}_{,i}(t)\ddot{\alpha}_{,i}(t)\Bigg],\end{aligned}\tag{115}$$

where

$$M=\max\left(\left(\frac{k^*_{3r}k^*_{3r}}{k^*_m}\right)^{1/2},\left(\frac{2\varkappa_{3r}\varkappa_{3r}}{\tau_q^2\varkappa_m}\right)^{1/2},\left(\frac{\zeta_{3r}\zeta_{3r}}{\tau_q\zeta_m}\right)^{1/2}\right).\tag{116}$$

Therefore, the relations (113) and (115) lead to

$$\begin{aligned}\left|\frac{\partial H}{\partial t}\right|(x_3,t)\le\frac{1}{2}\int_{D_{x_3}}&\left\{\varepsilon c\dot{\tilde{\alpha}}(t)^2+\frac{3M^2}{\varepsilon c}\left[k^*_m\tilde{\alpha}_{,i}(t)\tilde{\alpha}_{,i}(t)\right.\right.\\&\left.\left.+\tau_q\zeta_m\left(\dot{\alpha}_{,i}(t)+\frac{\tau_q}{2}\ \ddot{\alpha}_{,i}(t)\right)\left(\dot{\alpha}_{,i}(t)+\frac{\tau_q}{2}\ \ddot{\alpha}_{,i}(t)\right)+\frac{\tau_q^2}{2}\ \varkappa_m\ddot{\alpha}_{,i}(t)\ddot{\alpha}_{,i}(t)\right]\right\}da,\end{aligned}\tag{117}$$

from which, putting

$$\varepsilon=M\sqrt{\frac{3}{c}},\tag{118}$$

and by using the estimate (112), we get

$$\frac{1}{\varepsilon}\ \left|\frac{\partial H}{\partial t}\right|(x_3,t)+\frac{\partial H}{\partial x_3}(x_3,t)\le 0,\quad\text{for all}\quad x_3>0,t>0.\tag{119}$$

This last inequality is equivalent to the following differential inequalities

$$\frac{1}{\varepsilon}\ \frac{\partial H}{\partial t}(x_3,t)+\frac{\partial H}{\partial x_3}(x_3,t)\le 0\quad\text{for all}\quad x_3>0,t>0,\tag{120}$$

and

$$-\frac{1}{\varepsilon}\ \frac{\partial H}{\partial t}(x_3,t)+\frac{\partial H}{\partial x_3}(x_3,t)\le 0\quad\text{for all}\quad x_3>0,t>0.\tag{121}$$

Let us first choose $t_0>0$ and $x_3^0\ge\varepsilon t_0$. If we set $t=t_0+(x_3-x_3^0)/\varepsilon$ in (120) it results

$$\frac{d}{dx_3}\left[H\left(x_3,t_0+\frac{x_3-x_3^0}{\varepsilon}\right)\right]\le 0,\tag{122}$$

and hence $H(x_3, t_0 + (x_3 - x_3^0)/\varepsilon)$ is a non-increasing function with respect to x_3. Thus, if we recall that $0 \leq x_3^0 - \varepsilon t_0 \leq x_3^0$, it results

$$H(x_3^0, t_0) \leq H(x_3^0 - \varepsilon t_0, 0) = 0. \tag{123}$$

Further, we set $t = t_0 - (x_3 - x_3^0)/\varepsilon$ in (121) so that it follows

$$\frac{d}{dx_3}\left[H\left(x_3, t_0 - \frac{x_3 - x_3^0}{\varepsilon}\right)\right] \leq 0, \tag{124}$$

and hence $H(x_3, t_0 - (x_3 - x_3^0)/\varepsilon)$ is a non-increasing function with respect to x_3. Since $x_3^0 \leq x_3^0 + \varepsilon t_0$ it results

$$H(x_3^0, t_0) \geq H(x_3^0 + \varepsilon t_0, 0) = 0. \tag{125}$$

Consequently, from the relations (123) and (125), we deduce that

$$H(\infty, t_0) = \lim_{x_3 \to \infty} H(x_3, t_0) = 0 \quad \text{for all} \quad t_0 > 0, \tag{126}$$

and hence, by an integration of the relation (112) over (x_3, ∞), we obtain

$$\begin{aligned} H(x_3, t) \geq \frac{1}{2} \int_{B_{x_3}} & \left[c\dot{\tilde{\alpha}}(t)^2 + k_m^* \tilde{\alpha}_{,i}(t)\tilde{\alpha}_{,i}(t) + \frac{\tau_q^2}{2} \varkappa_m \ddot{\alpha}_{,i}(t)\ddot{\alpha}_{,i}(t) \right. \\ & \left. + \tau_q \zeta_m \left(\dot{\alpha}_{,i}(t) + \frac{\tau_q}{2} \ddot{\alpha}_{,i}(t)\right)\left(\dot{\alpha}_{,i}(t) + \frac{\tau_q}{2} \ddot{\alpha}_{,i}(t)\right)\right] dv \geq 0, \end{aligned} \tag{127}$$

for all $x_3 > 0$, $t > 0$ and $B_{x_3} \equiv D \times (x_3, \infty)$. Thus, $H(x_3, t)$ appears like a measure of the solution $\mathcal{S} = \{\alpha, q_i\}$ of our initial boundary value problem $\mathcal{P}_C$.

Finally, we set $x_3 = \varepsilon t$ in (120) to obtain

$$\frac{d}{dt}[H(\varepsilon t, t)] \leq 0, \tag{128}$$

so that $H(\varepsilon t, t)$ is a non-increasing function with respect to t. Thus, we deduce

$$H(\varepsilon t, t) \leq H(0, 0) = 0. \tag{129}$$

Since $H(x_3, t)$ is a non-increasing function with respect x_3, it follows that for $x_3 \geq \varepsilon t$ we will have

$$H(x_3, t) \leq H(\varepsilon t, t) \leq 0, \tag{130}$$

which in conjunction with (127) proves

$$H(x_3, t) = 0 \quad \text{for all} \quad x_3 \geq \varepsilon t, t > 0. \tag{131}$$

In view of the relations (127) and (131), we deduce that

$$\dot{\tilde{\alpha}}(x_1, x_2, x_3, t) = 0, \quad (x_1, x_2) \in D_{x_3}, \quad x_3 \geq \varepsilon t, t > 0, \tag{132}$$

which integrated under zero initial conditions gives

$$\alpha(x_1, x_2, x_3, t) = 0, \quad (x_1, x_2) \in D_{x_3}, \quad x_3 \geq \varepsilon t, t > 0. \tag{133}$$

If we substitute this last relation in the constitutive equation (4), we obtain

$$q_i + \tau_q \dot{q}_i + \frac{\tau_q^2}{2} \ddot{q}_i = 0, \quad (x_1, x_2) \in D_{x_3}, \quad x_3 \geq \varepsilon t, t > 0, \tag{134}$$

which furnishes $q_i(x_1, x_2, x_3, t) = 0$ for all $(x_1, x_2) \in D_{x_3}$, $x_3 \geq \varepsilon t$ and $t > 0$. Thus, we have established the following result

Theorem 5. *(**Domain of influence**) Suppose the hypotheses $(\tilde{H}1)$ to $(\tilde{H}4)$ to be fulfilled. Let $\mathcal{S} = \{\alpha, q_i\}$ be a solution of the initial boundary value problem $\mathcal{P}_C$ corresponding to the given data $\mathcal{D} = \{0; 0, 0, 0; 0, g\}$. Then the following domain of influence result holds true*

$$\mathcal{S}(x_1, x_2, x_3, t) = 0, \quad (x_1, x_2) \in D_{x_3}, \quad x_3 \geq \varepsilon t, t > 0, \tag{135}$$

that is, in the part of the cylinder located at a distance $x_3 \geq \varepsilon t, \quad t > 0$, from the base of the cylinder, all thermal activity stops, that is, the prescribed action on the base of the cylinder is not felt there.

8. Final comments

Our present analysis establishes a series of results that highlight characteristic properties of the three-phase-lag model proposed by Roy Choudhury [25], as: uniqueness, continuous data dependence, domain of influence, damped in time wave solutions, standing modes, exponential stability. It generalizes, improves and completes some existing results in the specialized literature, but also presents new information on the model. Thus, we have:

(i) the results described in the manuscript refer to the solution (α, q_i) of the system of equations (4) and (6), while the results in Refs. [27] and [35] refer to the stability of solutions α of the differential equation (8). This means quite different boundary and initial conditions in the two ways of approaching the considered thermal model;

(ii) we have to note that the prescription of the initial conditions (10) in this manuscript cannot be made equivalent to the initial conditions (2.2) in Ref. [27];

(iii) the present model leads to the fourth-order in time equation (7), which for a homogeneous and isotropic rigid conductor takes the form of the equation (8) considered by Quintanilla and Racke [27] and by Dell'Oro and Pata [35]. Thus, the results presented in our work, regarding exponential stability of solutions, also apply to these particular cases, although here we also include, as normally, the boundary conditions in terms of the heat flux, and, moreover, the initial conditions also include the values of the heat flux vector;

(iv) unlike the results presented in [27] and [35], the restrictions under which our results are obtained have a clear mechanical meaning, that of compatibility of the constitutive equation (4) with the Second Law of Thermodynamics;

(v) our results regarding uniqueness and continuous data dependence improve those described in [26] and [29]. By using the Lagrange identity method (see, for example, Brun [38] and Rionero and Chiriţă [39]), as well as that of logarithmic convexity method (see, e.g. Knops and Payne [40]), allows to remove the hypothesis regarding the semi-positive definiteness of the conductivity rate tensor;

(vi) unlike Fourier's classical law for heat flux, in Ray Choudhuri's model it is possible to have both standing modes and damped in time waves;

(vii) Theorem 5 provides an upper bound of the propagation speed of the excitation effects on the base of the cylinder, along its generators. Although the results presented in section 7 refer to a particular domain, here a cylinder, they can be obtained for any domain, as presented in [41].

Disclosure statement

The author declares that there is no conflict of interest.

ORCID

Stan Chiriță http://orcid.org/0000-0001-7560-0367

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