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On the structural stability for a model of mixture of porous solids

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The present paper is dedicated to the structural stability of the linear model of a mixture of two porous solids. It is shown that the variation of the characteristic coefficients that describe the coupling of the various mechanical effects involved in the model in concern does not destroy its structure. This means that any small variation of these coefficients leads to small variations in the corresponding solutions of the associated initial boundary value problems. For this purpose, more mathematical estimates are presented describing precisely the continuous dependence of the solutions with respect to all external given data of the initial boundary value problem, as well as with respect to appropriate measures of the set of coupling parameters. This allows the conclusion that the model of the mixture of porous materials is consistent. In particular, it is believed that the estimates obtained in terms of structural stability are particularly meaningful with regard to the materials used in building contexts and the related decay phenomena.

KEYWORDS

convergence, energy bounds, mixture of porous solids, porous materials, structural stability

MSC CLASSIFICATION

74F20, 74H55, 74A60, 74E20

1 | INTRODUCTION

Structural stability provides important information on the behavior of solutions when some constitutive coefficients undergo alterations, especially in the sense of degradation or decay. It is well known that many materials used with important structural functions in building contexts, even of a complex nature, can undergo phenomena of alteration of their physical-chemical characteristics and therefore in terms of response to stress actions. We recall that such phenomena almost always have an environmental origin and, if not limited through appropriate actions, are capable of compromising the integrity of the single load-bearing element and also of the entire structural complex. In particular, it is worth mentioning the oxidation processes of iron and steel, the carbonation processes of cement (a very relevant aspect dealing with porous materials), the physical-chemical damages induced by acid rain, wind erosion, persistent states of humidity, and so forth. Therefore, the study of structural stability allows us to know—in a deterministic way—the link between the alterations of the constitutive coefficients of the material and the solutions of the elastodynamic problem, observing whether “small” constitutive variations correspond to equally “small” variations of the solutions (stability), or vice versa (instability).

The interest in the modeling of porous materials has been very high for decades—just think of application areas such as soil mechanics or petroleum industry—but today the importance of such field of investigation is still growing if we

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also include, for instance, the bioengineering applications. An extensive historically based review of porous material theories can be found in [1–4]. In order to describe the behavior of granular materials such as rocks, soils, or porous bodies in general, it is useful to refer to the theory of elastic materials with voids, that is, endowed of an elastic matrix and material-free interstices. For the description of this theory, whereby the bulk density is the product of the matrix material density field and the volume fraction field, as well as of related developments, the reader is referred to [5–7]. It is worth mentioning that, in particular, the linear theory for elastic materials with voids [7] applies to elastic bodies with small voids that are distributed throughout the material.

Structural stability and continuous dependence on the model itself are very important as highlighted in the books of Hirsch and Smale [8], Bellomo and Preziosi [9], Straughan [10, 11], and Flavin and Rionero [12]. Roughly speaking, they are related to the fact that a small change in the coefficients involved in the differential equations or small changes of boundary conditions or of the initial conditions result in a suitably small change in the solution of the associated initial boundary value problems. Within the field of elasticity, continuous dependence on modeling has been comprehensively analyzed by Knops and Payne [13, 14]. The issue of stability in porous media was presented extensively in Straughan's book [10].

We have to outline that the structural stability is an important topic in the mathematical literature, and moreover, the analyses of this subject in various fields of continuum mechanics may be found in many works like: Ames and Hughes [15], Castro et al. [16], Celebi and Kalantarov [17], Celik and Hoang [18], Chiriță et al. [19], Chiriță and Ciarletta [20], Eltayeb [21], Harfash [22], Hill et al. [23], Liu [24], Markowich et al. [25], Otani and Uchida [26], and Varsakelis and Papalexandris [27].

On the other hand, the mathematical modeling of natural phenomena continues with even more intensity. In this context appears also the mathematical modeling of mixtures, started by prestigious researchers such as Truesdell and Toupin [28], Kelly [29], Eringen and Ingram [30, 31], Green and Naghdi [32, 33], Müller [34], Dunwoody and Müller [35], and Bowen and Wiese [36], and it was based on the spatial description of the deformation of fluid or gas constituents. A different point of view, namely, through the use of a Lagrangian description, was introduced by Bedford and Stern [37], and further extensive reviews of the subject were given by Bowen [1], Atkin and Craine [38, 39], Bedford and Drumheller [3], and Rajagopal and Tao [40].

In a series of works, Ieșan [41–43] has developed models of mixtures based on a Lagrangian description and by taking into account the microporous structure of the material. Such a theory for binary mixtures of elastic solids involves the following independent constitutive variables: the displacement gradients, displacement fields, volume fractions, and volume fraction gradients. In the same line, a nonlinear theory is developed by Chiriță and Galeș [44] for a heat-conducting viscoelastic composite, which is modeled as a mixture consisting of a microstretch Kelvin-Voigt material and a microstretch elastic solid.

The present paper considers the theory of mixtures as it was developed by Ieșan [41] and contributes to the study of the well-posedness of the model itself. In fact, we show in a first phase the continuous dependence of the solutions with respect to the external data (initial and Dirichlet boundary data, as well as mass loads) for the associated initial boundary value problems. On this basis, we approach the problem of structural stability in the sense of the continuous dependence of the solutions with respect to the changes that may occur in the coupling constitutive coefficients involved in the model in concern. We have identified two sets of material coupling coefficients: (a) one that couples the classical mixture model with the microporosity of the material and (b) another that connects the two material constituents taken into account in the development of the model. For each of the two sets, we have established precise mathematical estimates that describe continuous dependence with respect to possible variations of their coefficients. We also show how the solution to the coupled system of mixture converges, in an appropriate measure, to the solution of the uncoupled systems as the coupling coefficients tend to zero.

The plan of the work is the following. Section 2 presents the basic system of differential equations describing the evolutionary behavior of the mixture of porous solids in the line described by Ieșan [41]. Section 3 introduces a class of auxiliary problems and presents appropriate estimates of their solutions in terms of the boundary given data. Section 4 treats the continuous dependence of solution with respect to all external given data (initial and boundary data as well as the external loads). Section 5 establishes appropriate estimates describing continuous dependence with respect to the variation of coupling coefficients. Section 6 shows that the solution of the coupled initial boundary value problem converges to the solution of the uncoupled initial boundary value problem when all the coupling coefficients tend to zero.

2 | BASIC MODEL OF THE MIXTURE OF POROUS SOLIDS

We consider a mixture of two interacting porous continua s_1 and s_2 . We assume that the body occupies at time $t = 0$ the region B of Euclidean three-dimensional space which is bounded by the piecewise smooth surface ∂B . The configuration of the body at time $t = 0$ is taken as the reference configuration. We refer the motion of each constituent to the reference configuration and a fixed system of rectangular Cartesian axes. We use vector and Cartesian tensor notation with Latin indices having the values 1, 2, 3. Greek indices are understood to range over the integers (1, 2), and the summation convention is not used for these indices. We suppose that ∂B is star-shaped with respect to the origin of the Cartesian coordinate system, and therefore, we have

$$x_k n_k \geq h_0 > 0, \quad |x_k t_k| \leq \delta_0 \text{ on } \partial B, \quad (1)$$

where h_0 and δ_0 are appropriate positive constants, x_k are the components of the vector $\mathbf{x}$, n_k are the components of unit outward normal vector $\mathbf{n}$ to ∂B , and t_k are the components of a unit tangent vector to ∂B .

According with the linear theory of mixtures developed by Ieşan [41], the basic equations are as follows:

1. Equations of motion:

$$t_{ji,j} - p_i + \rho_1^0 F_i^{(1)} = \rho_1^0 \ddot{u}_i, \quad s_{ji,j} + p_i + \rho_2^0 F_i^{(2)} = \rho_2^0 \ddot{w}_i, \quad (2)$$

2. Equilibrated equations of motion:

$$h_{i,i}^{(1)} + g^{(1)} + \rho_1^0 L^{(1)} = \rho_1^0 \kappa_1 \ddot{\varphi}, \quad h_{i,i}^{(2)} + g^{(2)} + \rho_2^0 L^{(2)} = \rho_2^0 \kappa_2 \ddot{\psi}, \quad (3)$$

3. Constitutive equations:

$$\begin{aligned} t_{ji} &= \tau_{ji} + a_{ijkl} u_{k,l} + b_{ijkl} w_{k,l} + (D_{ji} + M_{ji})\varphi + (E_{ji} + N_{ji})\psi, \\ s_{ji} &= \sigma_{ji} + b_{klij} u_{k,l} + d_{ijkl} w_{k,l} + M_{ij}\varphi + N_{ij}\psi, \\ p_i &= a_{ij} d_j + b_{ij} \varphi_{,j} + c_{ij} \psi_{,j}, \\ h_i^{(1)} &= \alpha_{ij} \varphi_{,j} + \beta_{ij} \psi_{,j} + b_{ji} d_j, \quad h_i^{(2)} = \beta_{ji} \varphi_{,j} + \gamma_{ij} \psi_{,j} + c_{ji} d_j, \\ g^{(1)} &= -C - (D_{ij} + M_{ij}) u_{j,i} - M_{ij} w_{i,j} - \xi \varphi - \tau \psi, \\ g^{(2)} &= -D - (E_{ij} + N_{ij}) u_{j,i} - N_{ij} w_{i,j} - \tau \varphi - \eta \psi, \end{aligned} \quad (4)$$

in $B \times (0, T)$. We specify that in the above equations, we have used the following notations: u_i and w_i represent the components of the displacement vector fields associated with the constituents s_1 and s_2 , respectively, and moreover, we have set $d_i = u_i - w_i$; φ and ψ are the volume fraction fields of the constituent s_1 and s_2 , respectively. The mass densities ρ_1^0 and ρ_2^0 and the equilibrated inertia coefficients κ_1 and κ_2 for the two constituents, as well as all the other constitutive coefficients, are prescribed functions of x , supposed to be as smooth as required in our subsequent analysis and satisfying the following symmetry relations:

$$a_{ijkl} = a_{klij}, \quad d_{ijkl} = d_{klij}, \quad \alpha_{ij} = \alpha_{ji}, \quad \gamma_{ij} = \gamma_{ji}, \quad a_{ij} = a_{ji}. \quad (5)$$

Furthermore, t_{ij} and s_{ij} are the components of the partial stresses associated with the two constituents s_1 and s_2 , respectively, while $h_i^{(a)}$ represents the components of the partial equilibrated stresses, $g^{(a)}$ are the intrinsic equilibrated body force, and p_i expresses the interactions of the two constituents; $F_i^{(a)}$ is the body force per unit mass acting on the constituent s_a , and $L^{(a)}$ is the extrinsic equilibrated body force.

It should be noted that in [41], the internal energy of the mixture per unit mass is considered as a second degree polynomial in the independent variables provided by the deformation measures. For this reason, the coefficients τ_{ij} , σ_{ij} , C and D appear in the constitutive equations described by (4) with the meaning of initial values for t_{ij} , s_{ij} , $-g^{(1)}$ and $-g^{(2)}$, respectively, in the unstrained state! Further, the group of coefficients a_{ijkl} , b_{ijkl} , d_{ijkl} , a_{ij} characterizes the interdependence between the deformations of the two constituents of the mixture, and in particular, b_{ijkl} describes the coupling between these two deformations. Then, the group of coefficients α_{ij} , β_{ij} , γ_{ij} , ξ , τ , η is characteristic considering the porous structure of the mixture, with β_{ij} and τ coupling coefficients of the porous deformation of the two constituents.

Finally, the group of coefficients $D_{ij}, E_{ij}, M_{ij}, N_{ij}, b_{ij}, c_{ij}$ describes the coupling between the effects of deformation with the porous ones!

Throughout this paper we will consider the initial boundary value problem $\mathcal{P}$ defined by the basic equations (2), (3), and (4) and the following boundary conditions:

$$u_i = u_i^*, w_i = w_i^*, \varphi = \varphi^*, \psi = \psi^* \text{ on } \partial B \times (0, T), \quad (6)$$

and the following initial conditions:

$$\begin{aligned} u_i(x, 0) &= u_i^{(0)}(x), \dot{u}_i(x, 0) = \dot{u}_i^{(0)}(x), w_i(x, 0) = w_i^{(0)}(x), \dot{w}_i(x, 0) = \dot{w}_i^{(0)}(x), \\ \varphi(x, 0) &= \varphi^{(0)}(x), \dot{\varphi}(x, 0) = \dot{\varphi}^{(0)}(x), \psi(x, 0) = \psi^{(0)}(x), \dot{\psi}(x, 0) = \dot{\psi}^{(0)}(x), \end{aligned} \quad (7)$$

for all $x \in B$.

By a solution of the initial boundary value problem $\mathcal{P}$, corresponding to the given data $\mathcal{D} = \{F_i^{(1)}, F_i^{(2)}, L^{(1)}, L^{(2)}; u_i^{(0)}, w_i^{(0)}, \dot{u}_i^{(0)}, \dot{w}_i^{(0)}, \varphi^{(0)}, \psi^{(0)}, \dot{\varphi}^{(0)}, \dot{\psi}^{(0)}; u_i^*, w_i^*, \varphi^*, \psi^*\}$, we mean the ordered array $U = \{u_i, w_r, \varphi, \psi\}$ defined on $B \times (0, T)$ with the properties that $u_i(x, t) \in C^{2,2}(B \times (0, T))$, $w_i(x, t) \in C^{2,2}(B \times (0, T))$, $\varphi(x, t) \in C^{2,2}(B \times (0, T))$ and $\psi(x, t) \in C^{2,2}(B \times (0, T))$ and which satisfy the field equations (2) to (4), the initial conditions (7) and the boundary conditions (6). Throughout this paper, we will assume the existence of such solution! This means that we do not deal with the existence and regularity of the solution of the initial boundary value problem.

The main problem considered in this work is that of the structural stability of the model of the considered mixture, more precisely that of the continuous dependence of the model with respect to the variation (measurement errors) of the characteristic coupling coefficients of the material. In this context, it will be useful to establish a quantitative estimate of the solution to the initial boundary value problem $\mathcal{P}$ in terms of the external data of this problem (initial and boundary data, as well as the mechanical loads).

For our subsequent analysis, we need to associate with the solution $U = \{u_i, w_r, \varphi, \psi\}$ of the initial boundary value problem $\mathcal{P}$ the internal energy $\mathcal{E}(U)$ defined by

$$\begin{aligned} 2\mathcal{E}(U) &= a_{ijkl}u_{i,j}u_{k,l} + 2b_{ijkl}u_{i,j}w_{k,l} + d_{ijkl}w_{i,j}w_{k,l} + 2(D_{ji} + M_{ji})u_{i,j}\varphi \\ &\quad + 2(E_{ji} + N_{ji})u_{i,j}\psi + 2M_{ij}w_{i,j}\varphi + 2N_{ij}w_{i,j}\psi + a_{ij}d_i d_j + \alpha_{ij}\varphi_{,i}\varphi_{,j} \\ &\quad + \gamma_{ij}\psi_{,i}\psi_{,j} + 2\beta_{ij}\varphi_{,i}\psi_{,j} + 2b_{ij}d_i\varphi_{,j} + 2c_{ij}d_i\psi_{,j} + \xi\varphi^2 + 2\tau\varphi\psi + \eta\psi^2, \end{aligned} \quad (8)$$

and we note that

$$\begin{aligned} \dot{\mathcal{E}}(U(t)) &= (t_{ji} - \tau_{ji})\dot{u}_{i,j} + (s_{ji} - \sigma_{ji})\dot{w}_{i,j} + h_i^{(1)}\dot{\varphi}_{,i} + h_i^{(2)}\dot{\psi}_{,i} \\ &\quad + p_i\dot{d}_i - (g^{(1)} + C)\dot{\varphi} - (g^{(2)} + D)\dot{\psi}, \end{aligned} \quad (9)$$

and

$$\begin{aligned} 2\mathcal{E}(U) &= (t_{ji} - \tau_{ji})u_{i,j} + (s_{ji} - \sigma_{ji})w_{i,j} + h_i^{(1)}\varphi_{,i} + h_i^{(2)}\psi_{,i} \\ &\quad + p_i d_i - (g^{(1)} + C)\varphi - (g^{(2)} + D)\psi. \end{aligned} \quad (10)$$

At this time, we recall that (see, e.g., Ieşan [41])

$$\varphi_1^0 > 0, \varphi_2^0 > 0, \kappa_1 > 0, \kappa_2 > 0, \tau \geq 0, \xi \geq 0. \quad (11)$$

Moreover, we assume that the internal energy is a positive definite quadratic form, that is,

$$\begin{aligned} \mu_m [u_{r,s}u_{r,s} + w_{r,s}w_{r,s} + \varphi^2 + \psi^2 + \chi_m d_r d_r + \pi_m (\varphi_{,r}\varphi_{,r} + \psi_{,r}\psi_{,r})] &\leq 2\mathcal{E}(U) \\ &\leq \mu_M [u_{r,s}u_{r,s} + w_{r,s}w_{r,s} + \varphi^2 + \psi^2 + \chi_M d_r d_r + \pi_M (\varphi_{,r}\varphi_{,r} + \psi_{,r}\psi_{,r})], \end{aligned} \quad (12)$$

where μ_m, χ_m and π_m and μ_M, χ_M and π_M are strictly positive constants. In the left side of (12), there is no summation upon subscript m .

To deal with the problem of the structural stability of the mixture model, we first need to establish some estimates regarding the continuous dependence of the solution of the initial boundary value problem $\mathcal{P}$ with respect to its data (initial and boundary data, as well as those given by load systems). If the boundary data of the problem $\mathcal{P}$ are zero, then the required estimates can be obtained much more simply than in the case of non-zero boundary data. This latter case requires the use of some auxiliary elliptic problems that allow estimates in terms of the non-zero boundary data.

3 | SOME AUXILIARY PROBLEMS AND RELATED ESTIMATES

For our next analysis, we need to use some elliptic boundary value problems associated with the region B with the boundary surface ∂B . In fact, we need to establish appropriate estimates of their solutions in terms of the boundary given data.

We therefore consider the Dirichlet boundary value problem $\mathcal{P}_0(\phi^*)$ in terms of the unknown ϕ , defined by

$$\begin{aligned}\Delta\phi &= 0 \text{ in } B, \\ \phi &= \phi^* \text{ on } \partial B,\end{aligned}\tag{13}$$

where Δ is the Laplace operator. We note that the existence of the solution ϕ of the problem $\mathcal{P}_0(\phi^*)$ is assured by means of the existence theory developed by Fichera [45].

To obtain convenient estimates regarding the solution ϕ in terms of ϕ^* , let us multiply relation (13) by $x_k\phi_{,k}$ to obtain the identity

$$0 = \int_B x_k\phi_{,k}\Delta\phi dv = \frac{1}{2} \int_B \phi_{,i}\phi_{,i} dv + \int_{\partial B} n_i x_k \phi_{,k} \phi_{,i} da - \frac{1}{2} \int_{\partial B} x_k n_k \phi_{,i} \phi_{,i} da.\tag{14}$$

On the boundary surface ∂B , we use the following decomposition $\nabla\phi = (\partial\phi/\partial n)\mathbf{n} + \nabla_t\phi\mathbf{t}$, where $\mathbf{n}$ is the outward unit normal vector to ∂B , $\mathbf{t}$ is a unit tangent vector to the boundary surface, ∇ is the gradient operator, $(\partial/\partial n)$ is the normal derivative and ∇_t is the tangential derivative. Then the identity (14) becomes

$$\begin{aligned}& \frac{1}{2} \int_B \phi_{,i}\phi_{,i} dv + \frac{1}{2} \int_{\partial B} x_k n_k \left(\frac{\partial\phi}{\partial n} \right)^2 da \\ &= - \int_{\partial B} x_k t_k \frac{\partial\phi}{\partial n} \nabla_t \phi da + \frac{1}{2} \int_{\partial B} x_k n_k (\nabla_t \phi)^2 da,\end{aligned}\tag{15}$$

so that, by means of the relation (1) and the Cauchy-Schwarz and arithmetic-geometric mean inequalities, we obtain the following estimate:

$$\frac{1}{2} \int_B \phi_{,i}\phi_{,i} dv + \frac{h_0}{4} \int_{\partial B} \left(\frac{\partial\phi}{\partial n} \right)^2 da \leq \frac{1}{2} \int_{\partial B} \left(x_k n_k + \frac{2\delta_0^2}{h_0} \right) (\nabla_t \phi)^2 da.\tag{16}$$

At this instant, we recall the Poincaré inequality:

$$\lambda_1 \int_B \phi^2 dv \leq \int_B \phi_{,r}\phi_{,r} dv + C_1 \int_{\partial B} \phi^{*2} da,\tag{17}$$

with λ_1 and C_1 positive constants. Consequently, we can see that the estimates (16) and (17) furnish a priori bounds for $\int_B \phi^2 dv$, $\int_B \phi_{,i}\phi_{,i} dv$ and $\int_{\partial B} (\partial\phi/\partial n)^2 da$ in terms of the given data ϕ^* on the boundary surface ∂B .

4 | CONTINUOUS DATA DEPENDENCE WITH RESPECT TO THE EXTERNAL GIVEN DATA FOR THE PROBLEM $\mathcal{P}$

Let us first consider the boundary value problem $\mathcal{P}_0(\dot{u}_i^*)$ and let us denote its solution by G_i . In view of the analysis of the above section, it results that

$$\frac{1}{2} \int_B G_{i,r} G_{i,r} dv + \frac{h_0}{4} \int_{\partial B} \frac{\partial G_i}{\partial n} \frac{\partial G_i}{\partial n} da \leq \frac{1}{2} \int_{\partial B} \left(x_k n_k + \frac{2\delta_0^2}{h_0} \right) (\nabla_t \dot{u}_i^*) (\nabla_t \dot{u}_i^*) da, \quad (18)$$

and

$$\lambda_1 \int_B G_i G_i dv \leq \int_B G_{i,r} G_{i,r} dv + C_1 \int_{\partial B} \dot{u}_i^* \dot{u}_i^* da. \quad (19)$$

Then we multiply the equation (2)₁ by $G_i - \dot{u}_i$ to get

$$\int_B \left[(t_{ji} - \tau_{ji})_{,j} - p_i - \varrho_1^0 \ddot{u}_i + \left(\varrho_1^0 F_i^{(1)} + \tau_{ji,j} \right) \right] (G_i - \dot{u}_i) dv = 0 \quad (20)$$

so that, by means of the divergence theorem and the boundary condition $G_i - \dot{u}_i = 0$ on ∂B , we get

$$\begin{aligned} \int_B \left[\varrho_1^0 \dot{u}_i \ddot{u}_i + (t_{ji} - \tau_{ji}) \dot{u}_{i,j} + p_i \dot{u}_i \right] dv &= \frac{d}{dt} \int_B \varrho_1^0 \dot{u}_i G_i dv - \int_B \varrho_1^0 \dot{u}_i \dot{G}_i dv \\ &+ \int_B \left[(t_{ji} - \tau_{ji}) G_{i,j} + p_i G_i \right] dv + \int_B \left(\varrho_1^0 F_i^{(1)} + \tau_{ji,j} \right) (\dot{u}_i - G_i) dv. \end{aligned} \quad (21)$$

Next, we consider the boundary value problem $\mathcal{P}_0(\dot{w}_i^*)$ and let us denote its solution by H_i . In view of the analysis of the above section, it results that

$$\frac{1}{2} \int_B H_{i,r} H_{i,r} dv + \frac{h_0}{4} \int_{\partial B} \frac{\partial H_i}{\partial n} \frac{\partial H_i}{\partial n} da \leq \frac{1}{2} \int_{\partial B} \left(x_k n_k + \frac{2\delta_0^2}{h_0} \right) (\nabla_t \dot{w}_i^*) (\nabla_t \dot{w}_i^*) da, \quad (22)$$

and

$$\lambda_1 \int_B H_i H_i dv \leq \int_B H_{i,r} H_{i,r} dv + C_1 \int_{\partial B} \dot{w}_i^* \dot{w}_i^* da. \quad (23)$$

Further, we multiply the equation (2)₂ by $H_i - \dot{w}_i$ to get

$$\int_B \left[(s_{ji} - \sigma_{ji})_{,j} + p_i - \varrho_2^0 \ddot{w}_i + \left(\varrho_2^0 F_i^{(2)} + \sigma_{ji,j} \right) \right] (H_i - \dot{w}_i) dv = 0 \quad (24)$$

so that, by means of the divergence theorem, we get

$$\begin{aligned} \int_B \left[\varrho_2^0 \dot{w}_i \ddot{w}_i + (s_{ji} - \sigma_{ji}) \dot{w}_{i,j} - p_i \dot{w}_i \right] dv &= \frac{d}{dt} \int_B \varrho_2^0 \dot{w}_i H_i dv - \int_B \varrho_2^0 \dot{w}_i \dot{H}_i dv \\ &+ \int_B \left[(s_{ji} - \sigma_{ji}) H_{i,j} - p_i H_i \right] dv + \int_B \left(\varrho_2^0 F_i^{(2)} + \sigma_{ji,j} \right) (\dot{w}_i - H_i) dv. \end{aligned} \quad (25)$$

Subsequently, we consider the boundary value problem $\mathcal{P}_0(\dot{\varphi}^*)$ and let us denote its solution by Φ . Therefore, we have

$$\frac{1}{2} \int_B \Phi_{,r} \Phi_{,r} dv + \frac{h_0}{4} \int_{\partial B} \left(\frac{\partial \Phi}{\partial n} \right)^2 da \leq \frac{1}{2} \int_{\partial B} \left(x_k n_k + \frac{2\delta_0^2}{h_0} \right) (\nabla_t \dot{\varphi}^*)^2 da, \quad (26)$$

and

$$\lambda_1 \int_B \Phi^2 dv \leq \int_B \Phi_{,r} \Phi_{,r} dv + C_1 \int_{\partial B} \dot{\Phi}^{*2} da. \quad (27)$$

Now, we multiply the equation (3)₁ by $\Phi - \dot{\Phi}$ to get

$$\int_B \left[h_{i,i}^{(1)} + (g^{(1)} + C) + (\rho_1^0 L^{(1)} - C) - \rho_1^0 \kappa_1 \ddot{\Phi} \right] (\Phi - \dot{\Phi}) dv = 0, \quad (28)$$

so that, we get

$$\begin{aligned} \int_B \left[\rho_1^0 \kappa_1 \dot{\Phi} \ddot{\Phi} + h_i^{(1)} \dot{\Phi}_{,i} - (g^{(1)} + C) \dot{\Phi} \right] dv &= \frac{d}{dt} \int_B \rho_1^0 \kappa_1 \dot{\Phi} \Phi dv - \int_B \rho_1^0 \kappa_1 \dot{\Phi} \dot{\Phi} dv \\ &+ \int_B \left[h_i^{(1)} \Phi_{,i} - (g^{(1)} + C) \Phi \right] dv + \int_B (\rho_1^0 L^{(1)} - C) (\dot{\Phi} - \Phi) dv. \end{aligned} \quad (29)$$

Finally, we consider the boundary value problem $\mathcal{P}_0(\dot{\Psi}^*)$ and denote its solution by Ψ and note that

$$\frac{1}{2} \int_B \Psi_{,r} \Psi_{,r} dv + \frac{h_0}{4} \int_{\partial B} \left(\frac{\partial \Psi}{\partial n} \right)^2 da \leq \frac{1}{2} \int_{\partial B} \left(x_k n_k + \frac{2\delta_0^2}{h_0} \right) (\nabla_t \dot{\Psi}^*)^2 da, \quad (30)$$

and

$$\lambda_1 \int_B \Psi^2 dv \leq \int_B \Psi_{,r} \Psi_{,r} dv + C_1 \int_{\partial B} \dot{\Psi}^{*2} da. \quad (31)$$

Further, we start with the identity

$$\int_B \left[h_{i,i}^{(2)} + (g^{(2)} + D) + (\rho_2^0 L^{(2)} - D) - \rho_2^0 \kappa_2 \ddot{\Psi} \right] (\Psi - \dot{\Psi}) dv = 0, \quad (32)$$

in order to get

$$\begin{aligned} \int_B \left[\rho_2^0 \kappa_2 \dot{\Psi} \ddot{\Psi} + h_i^{(2)} \dot{\Psi}_{,i} - (g^{(2)} + D) \dot{\Psi} \right] dv &= \frac{d}{dt} \int_B \rho_2^0 \kappa_2 \dot{\Psi} \Psi dv - \int_B \rho_2^0 \kappa_2 \dot{\Psi} \dot{\Psi} dv \\ &+ \int_B \left[h_i^{(2)} \Psi_{,i} - (g^{(2)} + D) \Psi \right] dv + \int_B (\rho_2^0 L^{(2)} - D) (\dot{\Psi} - \Psi) dv. \end{aligned} \quad (33)$$

The next step is to add the equations (21), (25), (29), and (33) and to use the relations (9) and (10) to see that

$$\begin{aligned} &\frac{d}{dt} \int_B \left[\frac{1}{2} (\rho_1^0 \dot{u}_i \dot{u}_i + \rho_2^0 \dot{w}_i \dot{w}_i + \rho_1^0 \kappa_1 \dot{\Phi}^2 + \rho_2^0 \kappa_2 \dot{\Psi}^2) + \mathcal{E}(U) \right] dv - \frac{d}{dt} \int_B (\rho_1^0 \dot{u}_i G_i + \rho_2^0 \dot{w}_i H_i + \rho_1^0 \kappa_1 \dot{\Phi} \Phi + \rho_2^0 \kappa_2 \dot{\Psi} \Psi) dv \\ &= - \int_B (\rho_1^0 \dot{u}_i \dot{G}_i + \rho_2^0 \dot{w}_i \dot{H}_i + \rho_1^0 \kappa_1 \dot{\Phi} \dot{\Phi} + \rho_2^0 \kappa_2 \dot{\Psi} \dot{\Psi}) dv \\ &+ \int_B \left[(t_{ji} - \tau_{ji}) G_{i,j} + (s_{ji} - \sigma_{ji}) H_{i,j} + h_i^{(1)} \Phi_{,i} + h_i^{(2)} \Psi_{,i} + p_i (G_i - H_i) - (g^{(1)} + C) \Phi - (g^{(2)} + D) \Psi \right] dv \\ &+ \int_B \left[(\rho_1^0 F_i^{(1)} + \tau_{ji,j}) \dot{u}_i + (\rho_2^0 F_i^{(2)} + \sigma_{ji,j}) \dot{w}_i + (\rho_1^0 L^{(1)} - C) \dot{\Phi} + (\rho_2^0 L^{(2)} - D) \dot{\Psi} \right] dv \\ &- \int_B \left[(\rho_1^0 F_i^{(1)} + \tau_{ji,j}) G_i + (\rho_2^0 F_i^{(2)} + \sigma_{ji,j}) H_i + (\rho_1^0 L^{(1)} - C) \Phi + (\rho_2^0 L^{(2)} - D) \Psi \right] dv. \end{aligned} \quad (34)$$

At this instant, we introduce the notations:

$$F(t) = \int_{B(t)} \left[\frac{1}{2} (\rho_1^0 \dot{u}_i \dot{u}_i + \rho_2^0 \dot{w}_i \dot{w}_i + \rho_1^0 \kappa_1 \dot{\phi}^2 + \rho_2^0 \kappa_2 \dot{\psi}^2) + \mathcal{E}(U) \right] dv, \quad (35)$$

$$I_1(t) = \int_{B(t)} \left[(t_{ji} - \tau_{ji}) G_{i,j} + (s_{ji} - \sigma_{ji}) H_{i,j} + h_i^{(1)} \Phi_{,i} + h_i^{(2)} \Psi_{,i} + p_i (G_i - H_i) - (g^{(1)} + C) \Phi - (g^{(2)} + D) \Psi \right] dv, \quad (36)$$

$$I_2(t) = \int_{B(t)} \left[(\rho_1^0 F_i^{(1)} + \tau_{ji,j}) \dot{u}_i + (\rho_2^0 F_i^{(2)} + \sigma_{ji,j}) \dot{w}_i + (\rho_1^0 L^{(1)} - C) \dot{\phi} + (\rho_2^0 L^{(2)} - D) \dot{\psi} \right] dv, \quad (37)$$

$$I_3(t) = - \int_{B(t)} \left[(\rho_1^0 F_i^{(1)} + \tau_{ji,j}) G_i + (\rho_2^0 F_i^{(2)} + \sigma_{ji,j}) H_i + (\rho_1^0 L^{(1)} - C) \Phi + (\rho_2^0 L^{(2)} - D) \Psi \right] dv, \quad (38)$$

where $\int_{B(t)}$ means that the integrand is evaluated at time t .

Then the identity (34) implies that

$$\begin{aligned} F(t) &= \int_{B(t)} (\rho_1^0 \dot{u}_i G_i + \rho_2^0 \dot{w}_i H_i + \rho_1^0 \kappa_1 \dot{\phi} \Phi + \rho_2^0 \kappa_2 \dot{\psi} \Psi) dv - \int_0^t \int_{B(s)} (\rho_1^0 \dot{u}_i \dot{G}_i + \rho_2^0 \dot{w}_i \dot{H}_i + \rho_1^0 \kappa_1 \dot{\phi} \dot{\Phi} + \rho_2^0 \kappa_2 \dot{\psi} \dot{\Psi}) dv ds \\ &\quad + \int_0^t I_1(s) ds + \int_0^t I_2(s) ds + \int_0^t I_3(s) ds + F(0) - \int_{B(0)} (\rho_1^0 \dot{u}_i G_i + \rho_2^0 \dot{w}_i H_i + \rho_1^0 \kappa_1 \dot{\phi} \Phi + \rho_2^0 \kappa_2 \dot{\psi} \Psi) dv. \end{aligned} \quad (39)$$

Our goal now is to obtain from the identity (39) an estimate of the solution $U = \{u_i, w_r, \phi, \psi\}$ of the initial boundary value problem $\mathcal{P}$ in terms of the given data. For this purpose, using the arithmetic-geometric mean inequality and the Cauchy-Schwarz inequality, we note that

$$\begin{aligned} &\int_{B(t)} (\rho_1^0 \dot{u}_i G_i + \rho_2^0 \dot{w}_i H_i + \rho_1^0 \kappa_1 \dot{\phi} \Phi + \rho_2^0 \kappa_2 \dot{\psi} \Psi) dv \\ &\leq \frac{1}{4} \int_{B(t)} (\rho_1^0 \dot{u}_i \dot{u}_i + \rho_2^0 \dot{w}_i \dot{w}_i + \rho_1^0 \kappa_1 \dot{\phi}^2 + \rho_2^0 \kappa_2 \dot{\psi}^2) dv \end{aligned} \quad (40)$$

$$\begin{aligned} &\quad + \max_{t \in [0, T]} \int_{B(t)} (\rho_1^0 G_i G_i + \rho_2^0 H_i H_i + \rho_1^0 \kappa_1 \Phi^2 + \rho_2^0 \kappa_2 \Psi^2) dv; \\ &\quad - \int_0^t \int_{B(s)} (\rho_1^0 \dot{u}_i \dot{G}_i + \rho_2^0 \dot{w}_i \dot{H}_i + \rho_1^0 \kappa_1 \dot{\phi} \dot{\Phi} + \rho_2^0 \kappa_2 \dot{\psi} \dot{\Psi}) dv ds \\ &\leq \int_0^t P(s) \left(\int_{B(s)} (\rho_1^0 \dot{u}_i \dot{u}_i + \rho_2^0 \dot{w}_i \dot{w}_i + \rho_1^0 \kappa_1 \dot{\phi}^2 + \rho_2^0 \kappa_2 \dot{\psi}^2) dv \right)^{1/2} ds \\ &\leq \int_0^t P(s) \sqrt{2F(s)} ds, \end{aligned} \quad (41)$$

where

$$P(t) = \left(\int_{B(t)} (\rho_1^0 \dot{G}_i \dot{G}_i + \rho_2^0 \dot{H}_i \dot{H}_i + \rho_1^0 \kappa_1 \dot{\Phi}^2 + \rho_2^0 \kappa_2 \dot{\Psi}^2) dv \right)^{1/2}. \quad (42)$$

Furthermore, from the relation (36), we have

$$\begin{aligned} |I_1(t)| &\leq Q(t) \left(\int_{B(t)} \left[(t_{ji} - \tau_{ji}) (t_{ji} - \tau_{ji}) + (s_{ji} - \sigma_{ji}) (s_{ji} - \sigma_{ji}) \right. \right. \\ &\quad \left. \left. + \frac{1}{\pi_m} (h_i^{(1)} h_i^{(1)} + h_i^{(2)} h_i^{(2)}) + \frac{1}{\chi_m} p_i p_i + (g^{(1)} + C)^2 + (g^{(2)} + D)^2 \right] dv \right)^{1/2} \\ &\leq Q(t) \sqrt{2\mu_M F(t)}, \end{aligned} \quad (43)$$

with

$$Q(t) = \left(\int_{B(t)} [G_{i,j}G_{i,j} + H_{i,j}H_{i,j} + \pi_m (\Phi_{,i}\Phi_{,i} + \Psi_{,i}\Psi_{,i}) + \chi_m (G_i - H_i)(G_i - H_i) + \Phi^2 + \Psi^2] dv \right)^{1/2}; \quad (44)$$

While from (37), we get

$$|I_2(t)| \leq R(t) \left(\int_{B(t)} (\rho_1^0 \dot{u}_i \dot{u}_i + \rho_2^0 \dot{w}_i \dot{w}_i + \rho_1^0 \kappa_1 \dot{\phi}^2 + \rho_2^0 \kappa_2 \dot{\psi}^2) dv \right)^{1/2} \leq R(t) \sqrt{2\mathcal{F}(t)}, \quad (45)$$

with

$$R(t) = \left(\int_{B(t)} \left[\frac{1}{\rho_1^0} (\rho_1^0 F_i^{(1)} + \tau_{ri,r}) (\rho_1^0 F_i^{(1)} + \tau_{si,s}) + \frac{1}{\rho_2^0} (\rho_2^0 F_i^{(2)} + \sigma_{ri,r}) (\rho_2^0 F_i^{(2)} + \sigma_{si,s}) + \frac{1}{\rho_1^0 \kappa_1} (\rho_1^0 L^{(1)} - C)^2 + \frac{1}{\rho_2^0 \kappa_2} (\rho_2^0 L^{(2)} - D)^2 \right] dv \right)^{1/2}. \quad (46)$$

Finally, we use the estimates (40), (41), (43), and (45) into identity (39) to obtain the following integral inequality:

$$\mathcal{F}(t) \leq \int_0^t \xi(s) \sqrt{\mathcal{F}(s)} ds + \mathcal{G}, \quad (47)$$

where

$$\begin{aligned} \xi(t) &= 2\sqrt{2} [P(t) + \sqrt{\mu_M} Q(t) + R(t)], \\ \mathcal{G} &= 2\mathcal{F}(0) + 2 \left| \int_{B(0)} (\rho_1^0 \dot{u}_i G_i + \rho_2^0 \dot{w}_i H_i + \rho_1^0 \kappa_1 \dot{\phi} \Phi + \rho_2^0 \kappa_2 \dot{\psi} \Psi) dv \right| \\ &\quad + 2 \max_{t \in [0, T]} \int_{B(t)} (\rho_1^0 G_i G_i + \rho_2^0 H_i H_i + \rho_1^0 \kappa_1 \Phi^2 + \rho_2^0 \kappa_2 \Psi^2) dv + 2 \int_0^T |I_3(t)| dt. \end{aligned} \quad (48)$$

We now handle the integral inequality (47) to obtain the sought continuous dependence estimate. In this connection, we introduce the function

$$\Lambda(t) = \sqrt{\int_0^t \xi(s) \sqrt{\mathcal{F}(s)} ds + \mathcal{G}}, \quad t \in [0, T], \quad (49)$$

and note that we have

$$\sqrt{\mathcal{F}(t)} \leq \Lambda(t) \quad \text{for all } t \in [0, T]. \quad (50)$$

Moreover, we get

$$2\Lambda(t)\dot{\Lambda}(t) = \xi(t)\sqrt{\mathcal{F}(t)} \leq \xi(t)\Lambda(t) \quad \text{for all } t \in [0, T], \quad (51)$$

so that we obtain

$$\Lambda(t) \leq \frac{1}{2} \int_0^t \xi(s) ds + \Lambda(0) \quad \text{for all } t \in [0, T]. \quad (52)$$

Consequently, from the relations (50) and (52), we obtain the expected continuous dependence estimate

$$\sqrt{\mathcal{F}(t)} \leq \frac{1}{2} \int_0^t \xi(s) ds + \sqrt{\mathcal{G}} \quad \text{for all } t \in [0, T]. \quad (53)$$

It is important to note that, based on the assumptions described in relations (11) and (12), it can be seen that $\mathcal{F}$ represents a measure of the solution $U = \{u_i, w_r, \varphi, \psi\}$ of the initial boundary value problem $\mathcal{P}$ in the sense that $\mathcal{F}(U(t)) \geq 0$ for all $U(t)$, and $\mathcal{F}(U(t)) = 0$ implies $U(t) = 0$. On the other hand, through relations (18), (19), (22), (23), (26), (27), (30), and (31) as well as (42), (44), (46), and (48), we can see that the right member of the estimate (53) represents a measure of all data of the initial boundary value problem $\mathcal{P}$.

It should be mentioned that when all external data are null, the estimate (53) becomes

$$\sqrt{\mathcal{F}(t)} \leq t\sqrt{2} \left(\int_B \left(\frac{1}{\phi_1^0} \tau_{ri,r} \tau_{si,s} + \frac{1}{\phi_2^0} \sigma_{ri,r} \sigma_{si,s} + \frac{1}{\phi_1^0 \kappa_1} C^2 + \frac{1}{\phi_2^0 \kappa_2} D^2 \right) dv \right)^{1/2}, \quad t \in [0, T] \quad (54)$$

and it expresses the continuous dependence of solution with respect to the set of characteristic coefficients $C_1 = \{\tau_{ji}, \sigma_{rs}, C, D\}$.

In view of the relations (8) and (35) and the hypotheses (11) and (12), we can see that the inequality (53) furnishes a priori estimates for the following quantities

$$\begin{aligned} & \int_B \dot{u}_i \dot{u}_i dv, \int_B \dot{w}_r \dot{w}_r dv, \int_B \dot{\varphi}^2 dv, \int_B \dot{\psi}^2 dv, \int_B u_{r,s} u_{r,s} dv, \int_B w_{r,s} w_{r,s} dv, \\ & \int_B \varphi^2 dv, \int_B \psi^2 dv, \int_B d_r d_r dv, \int_B \varphi_{,i} \varphi_{,i} dv, \int_B \psi_{,i} \psi_{,i} dv. \end{aligned} \quad (55)$$

Furthermore, it is worth to outline that the above procedure allows us to obtain a priori bounds for

$$\int_B \dot{u}_{r,s} \dot{u}_{r,s} dv, \int_B \dot{w}_{r,s} \dot{w}_{r,s} dv, \int_B \dot{\varphi}_{,i} \dot{\varphi}_{,i} dv, \int_B \dot{\psi}_{,i} \dot{\psi}_{,i} dv. \quad (56)$$

We have to outline that the estimates described by relations (55) and (56) will be useful in the next sections in connection with the evaluation of appropriate supply terms involving the variations of the coupling coefficients.

5 | CONTINUOUS DEPENDENCE ON THE COUPLING COEFFICIENTS

In this section we study the structural stability of the mathematical model of the mixture of poroelastic solids. One of the most important tasks in the study of the structural stability is to prove that the solutions of the initial boundary value problems depend continuously on the material characteristic coefficients, which may be subjected to measurement errors, perturbations in the mathematical modeling process, or even decay and degradation phenomena. The structural stability is related to the continuous dependence of solutions on changes in the model itself rather than on the given data. That means changes in coefficients in the constitutive equations and changes in the system of differential equations may be reflected physically by changes in the constitutive parameters (as, for example, the coefficients obtained in a small deformation superposed to a finite one). Moreover, the estimates of continuous dependence play a central role in obtaining numerical approximations to these kinds of problems.

The model in our study represents a coupling of the classical elastic mixtures with the consideration of the volume fraction microstructure. Therefore, we have the set of coefficients $C_2 = \{D_{ij}, E_{ij}, M_{ij}, N_{ij}, b_{ij}, c_{ij}\}$ coupling the mechanical deformations (of classical mixture model) with the poroelastic effects. We can also consider the model in question as a coupling of the two porous material constituents and hence we have the following set of characteristic coefficients $C_3 = \{b_{ijkl}, E_{ij} + N_{ij}, M_{ij}, \beta_{ij}, a_{ij}, b_{ij}, c_{ij}, \tau\}$.

5.1 | Continuous dependence on the set of coupling characteristic coefficients

$$C_2 = \{D_{ij}, E_{ij}, M_{ij}, N_{ij}, b_{ij}, c_{ij}\}$$

Throughout this section, we investigate the continuous dependence of the solution of the initial boundary value problem $\mathcal{P}$ in the situation when all the characteristic coefficients of the mixture remain unchanged, and only the coupling characteristic coefficients $C_2 = \{D_{ij}, E_{ij}, M_{ij}, N_{ij}, b_{ij}, c_{ij}\}$ undergo a variation. The set C_2 characterizes the coupling of the effects of the classical mixture with those of porosity.

Therefore, in what follows, we consider that $U^{(\alpha)} = \{u_i^{(\alpha)}, w_i^{(\alpha)}, \varphi^{(\alpha)}, \psi^{(\alpha)}\}$, $\alpha = 1, 2$, is the solution of the initial boundary value problem $\mathcal{P}$ corresponding to the same initial, boundary and body loads data, and to the same material characteristics excepting the coupling material parameters $C_2^{(\alpha)} = \{D_{ij}^{(\alpha)}, E_{ij}^{(\alpha)}, M_{ij}^{(\alpha)}, N_{ij}^{(\alpha)}, b_{ij}^{(\alpha)}, c_{ij}^{(\alpha)}\}$, $\alpha = 1, 2$.

If we set

$$\begin{aligned} U &= \{u_i, w_i, \varphi, \psi\} = U^{(1)} - U^{(2)} = \left\{ u_i^{(1)} - u_i^{(2)}, w_i^{(1)} - w_i^{(2)}, \varphi^{(1)} - \varphi^{(2)}, \psi^{(1)} - \psi^{(2)} \right\}, \\ C_2 &= \{D_{ij}, E_{ij}, M_{ij}, N_{ij}, b_{ij}, c_{ij}\} = C_2^{(1)} - C_2^{(2)} \\ &= \left\{ D_{ij}^{(1)} - D_{ij}^{(2)}, E_{ij}^{(1)} - E_{ij}^{(2)}, M_{ij}^{(1)} - M_{ij}^{(2)}, N_{ij}^{(1)} - N_{ij}^{(2)}, b_{ij}^{(1)} - b_{ij}^{(2)}, c_{ij}^{(1)} - c_{ij}^{(2)} \right\}, \\ d_i &= u_i - w_i, \end{aligned} \quad (57)$$

then the following initial conditions

$$\begin{aligned} u_i(x, 0) &= 0, \quad \dot{u}_i(x, 0) = 0, \quad w_i(x, 0) = 0, \quad \dot{w}_i(x, 0) = 0, \\ \varphi(x, 0) &= 0, \quad \dot{\varphi}(x, 0) = 0, \quad \psi(x, 0) = 0, \quad \dot{\psi}(x, 0) = 0, \quad \text{for all } x \in B, \end{aligned} \quad (58)$$

and the boundary conditions

$$u_i = 0, \quad w_i = 0, \quad \varphi = 0, \quad \psi = 0 \quad \text{on } \partial B \times (0, T), \quad (59)$$

are fulfilled.

The basic system of differential equations is

$$\begin{aligned} \rho_1^0 \ddot{u}_i - \left[a_{ijkl} u_{k,l} + b_{ijkl} w_{k,l} + \left(D_{ji}^{(1)} + M_{ji}^{(1)} \right) \varphi + \left(E_{ji}^{(1)} + N_{ji}^{(1)} \right) \psi \right]_{,j} \\ + a_{ij} d_j + b_{ij}^{(1)} \varphi_{,j} + c_{ij}^{(1)} \psi_{,j} = \rho_1^0 \tilde{F}_i^{(1)}, \end{aligned} \quad (60)$$

$$\rho_2^0 \ddot{w}_i - \left(b_{klij} u_{k,l} + d_{ijkl} w_{k,l} + M_{ij}^{(1)} \varphi + N_{ij}^{(1)} \psi \right)_{,j} - a_{ij} d_j - b_{ij}^{(1)} \varphi_{,j} - c_{ij}^{(1)} \psi_{,j} = \rho_2^0 \tilde{F}_i^{(2)}, \quad (61)$$

$$\rho_1^0 \kappa_1 \ddot{\varphi} - \left(\alpha_{ij} \varphi_{,j} + \beta_{ij} \psi_{,j} + b_{ji}^{(1)} d_j \right)_{,i} + \left(D_{ji}^{(1)} + M_{ji}^{(1)} \right) u_{i,j} + M_{ij}^{(1)} w_{i,j} + \xi \varphi + \tau \psi = \rho_1^0 \kappa_1 \tilde{L}^{(1)}, \quad (62)$$

$$\rho_2^0 \kappa_2 \ddot{\psi} - \left(\beta_{ji} \varphi_{,j} + \gamma_{ij} \psi_{,j} + c_{ji}^{(1)} d_j \right)_{,i} + \left(E_{ji}^{(1)} + N_{ji}^{(1)} \right) u_{i,j} + N_{ij}^{(1)} w_{i,j} + \tau \varphi + \eta \psi = \rho_2^0 \kappa_2 \tilde{L}^{(2)}, \quad (63)$$

with

$$\rho_1^0 \tilde{F}_i^{(1)} = \left[\left(D_{ji} + M_{ji} \right) \varphi^{(2)} + \left(E_{ji} + N_{ji} \right) \psi^{(2)} \right]_{,j} - b_{ij} \varphi_{,j}^{(2)} - c_{ij} \psi_{,j}^{(2)}, \quad (64)$$

$$\rho_2^0 \tilde{F}_i^{(2)} = \left(M_{ij} \varphi^{(2)} + N_{ij} \psi^{(2)} \right)_{,j} + b_{ij} \varphi_{,j}^{(2)} + c_{ij} \psi_{,j}^{(2)}, \quad (65)$$

$$\rho_1^0 \kappa_1 \tilde{L}^{(1)} = \left(b_{ji} d_j^{(2)} \right)_{,i} - \left(D_{ji} + M_{ji} \right) u_{i,j}^{(2)} - M_{ij} w_{i,j}^{(2)}, \quad (66)$$

$$\rho_2^0 \kappa_2 \tilde{L}^{(2)} = \left(c_{ji} d_j^{(2)} \right)_{,i} - \left(E_{ji} + N_{ji} \right) u_{i,j}^{(2)} - N_{ij} w_{i,j}^{(2)}. \quad (67)$$

We multiply equation (60) by $\dot{u}_i$, then multiply equation (61) by $\dot{w}_i$, multiply equation (62) by $\dot{\varphi}$, multiply equation (63) by $\dot{\psi}$, and then add the results and integrate over B and finally use the boundary conditions (59), to find

$$\dot{F}_1(t) = \int_{B(t)} \left(\rho_1^0 \tilde{F}_i^{(1)} \dot{u}_i + \rho_2^0 \tilde{F}_i^{(2)} \dot{w}_i + \rho_1^0 \kappa_1 \tilde{L}^{(1)} \dot{\varphi} + \rho_2^0 \kappa_2 \tilde{L}^{(2)} \dot{\psi} \right) dv, \quad (68)$$

where

$$\begin{aligned} \mathcal{F}_1(t) = & \frac{1}{2} \int_{B(t)} \left[\left(\varrho_1^0 \dot{u}_i \dot{u}_i + \varrho_2^0 \dot{w}_i \dot{w}_i + \varrho_1^0 \kappa_1 \dot{\varphi}^2 + \varrho_2^0 \kappa_2 \dot{\psi}^2 \right) + a_{ijkl} u_{i,j} u_{k,l} + 2b_{ijkl} u_{i,j} w_{k,l} \right. \\ & + d_{ijkl} w_{i,j} w_{k,l} + 2 \left(D_{ji}^{(1)} + M_{ji}^{(1)} \right) u_{i,j} \varphi + 2 \left(E_{ji}^{(1)} + N_{ji}^{(1)} \right) u_{i,j} \psi + 2M_{ij}^{(1)} w_{i,j} \varphi \\ & + 2N_{ij}^{(1)} w_{i,j} \psi + a_{ij} d_i d_j + \alpha_{ij} \varphi_{,i} \varphi_{,j} + \gamma_{ij} \psi_{,i} \psi_{,j} + 2\beta_{ij} \varphi_{,i} \psi_{,j} + 2b_{ij}^{(1)} d_i \varphi_{,j} \\ & \left. + 2c_{ij}^{(1)} d_i \psi_{,j} + \xi \varphi^2 + 2\tau \varphi \psi + \eta \psi^2 \right] dv. \end{aligned} \quad (69)$$

Now, we integrate (68) with respect to time variable on $(0, t)$ and take into consideration the null initial conditions (58), in order to obtain the following identity

$$\mathcal{F}_1(t) = \int_0^t \int_{B(s)} \left(\varrho_1^0 \tilde{F}_i^{(1)} \dot{u}_i + \varrho_2^0 \tilde{F}_i^{(2)} \dot{w}_i + \varrho_1^0 \kappa_1 \tilde{L}^{(1)} \dot{\varphi} + \varrho_2^0 \kappa_2 \tilde{L}^{(2)} \dot{\psi} \right) dv ds. \quad (70)$$

By applying the Cauchy-Schwarz inequality in (70), we obtain

$$\mathcal{F}_1(t) \leq \int_0^t \left(\int_{B(s)} \left(\varrho_1^0 \dot{u}_i \dot{u}_i + \varrho_2^0 \dot{w}_i \dot{w}_i + \varrho_1^0 \kappa_1 \dot{\varphi}^2 + \varrho_2^0 \kappa_2 \dot{\psi}^2 \right) dv \right)^{1/2} \mathcal{K}(s) ds, \quad (71)$$

and hence, by (69), we have

$$\mathcal{F}_1(t) \leq \int_0^t \sqrt{2\mathcal{F}_1(s)} \mathcal{K}(s) ds, \quad (72)$$

where

$$\mathcal{K}(t) = \left(\int_{B(t)} \left(\varrho_1^0 \tilde{F}_i^{(1)} \tilde{F}_i^{(1)} + \varrho_2^0 \tilde{F}_i^{(2)} \tilde{F}_i^{(2)} + \varrho_1^0 \kappa_1 \tilde{L}^{(1)} \tilde{L}^{(1)} + \varrho_2^0 \kappa_2 \tilde{L}^{(2)} \tilde{L}^{(2)} \right) dv \right)^{1/2}. \quad (73)$$

Furthermore, we can treat the inequality by the same procedure like that for (47), to obtain

$$\sqrt{\mathcal{F}_1(t)} \leq \frac{1}{\sqrt{2}} \int_0^t \mathcal{K}(s) ds, \quad t \geq 0. \quad (74)$$

It is a straightforward way to show that, by means of Cauchy-Schwarz and arithmetic-geometric mean inequalities and by using the estimates of type described by relations (55) and (56), as well as the relations (64)–(67), (73), together with (8) and (12), there exists a computable constant K to have

$$\mathcal{K}(t) \leq K \mathcal{H}_2(C_2) \int_{B(t)} \mathcal{E}(U^{(2)}) dv, \quad (75)$$

with $\mathcal{H}_2(C_2) = \text{meas}_B \{C_2\}$ an appropriate measure of the coupling parameters. Then, recalling (35), the relation (74) implies

$$\sqrt{\mathcal{F}_1(t)} \leq \left(\frac{K}{\sqrt{2}} \int_0^t \mathcal{F}(U^{(2)}) ds \right) \mathcal{H}_2(C_2), \quad t \geq 0. \quad (76)$$

In view of the analysis of the Section 4, namely, the estimate (53), this last estimate mathematically shows that a small variation of the material coupling coefficients in the set C_2 produces a sufficiently small variation of the corresponding solutions.

5.2 | Continuous dependence on the set of coupling characteristic coefficients

$$C_3 = \{b_{ijkl}, E_{ij} + N_{ij}, M_{ij}, \beta_{ij}, a_{ij}, b_{ij}, c_{ij}, \tau\}$$

We proceed now to investigate the continuous dependence of the solution of the initial boundary value problem $\mathcal{P}$ in the situation when all the characteristic coefficients of the mixture remain unchanged, excepting the coupling characteristic coefficients $C_3 = \{b_{ijkl}, E_{ij} + N_{ij}, M_{ij}, \beta_{ij}, a_{ij}, b_{ij}, c_{ij}, \tau\}$ which undergo a variation. This set of coefficients represents the coupling between the two constituent materials of the mixture.

Let $U^{(\alpha)} = \{u_i^{(\alpha)}, w_r^{(\alpha)}, \varphi^{(\alpha)}, \psi^{(\alpha)}\}$, $\alpha = 1, 2$, be the solution of the initial boundary value problem $\mathcal{P}$ corresponding to the same initial, boundary and body loads data, and to the same material characteristics excepting the coupling material parameters $C_3^{(\alpha)} = \{b_{ijkl}^{(\alpha)}, E_{ij}^{(\alpha)} + N_{ij}^{(\alpha)}, M_{ij}^{(\alpha)}, \beta_{ij}^{(\alpha)}, a_{ij}^{(\alpha)}, b_{ij}^{(\alpha)}, c_{ij}^{(\alpha)}, \tau^{(\alpha)}\}$, $\alpha = 1, 2$.

Further, we set

$$\begin{aligned} U &= \{u_i, w_r, \varphi, \psi\} = \left\{ u_i^{(1)} - u_i^{(2)}, w_i^{(1)} - w_i^{(2)}, \varphi^{(1)} - \varphi^{(2)}, \psi^{(1)} - \psi^{(2)} \right\}, d_i = u_i - w_i, \\ C_3 &= \{b_{ijkl}, E_{ij} + N_{ij}, M_{ij}, \beta_{ij}, a_{ij}, b_{ij}, c_{ij}, \tau\} = C_3^{(1)} - C_3^{(2)} = \left\{ b_{ijkl}^{(1)} - b_{ijkl}^{(2)}, E_{ij}^{(1)} + N_{ij}^{(1)} \right. \\ &\quad \left. - E_{ij}^{(2)} - N_{ij}^{(2)}, M_{ij}^{(1)} - M_{ij}^{(2)}, \beta_{ij}^{(1)} - \beta_{ij}^{(2)}, a_{ij}^{(1)} - a_{ij}^{(2)}, b_{ij}^{(1)} - b_{ij}^{(2)}, c_{ij}^{(1)} - c_{ij}^{(2)}, \tau^{(1)} - \tau^{(2)} \right\}, \end{aligned} \quad (77)$$

and therefore, we have the following null initial conditions

$$\begin{aligned} u_i(x, 0) &= 0, \dot{u}_i(x, 0) = 0, w_i(x, 0) = 0, \dot{w}_i(x, 0) = 0, \\ \varphi(x, 0) &= 0, \dot{\varphi}(x, 0) = 0, \psi(x, 0) = 0, \dot{\psi}(x, 0) = 0, \text{ for all } x \in B, \end{aligned} \quad (78)$$

and null boundary conditions:

$$u_i = 0, w_i = 0, \varphi = 0, \psi = 0 \text{ on } \partial B \times (0, T). \quad (79)$$

Then the basic system of differential equations is

$$\rho_1^0 \ddot{u}_i - \left[a_{ijkl} u_{k,l} + b_{ijkl} w_{k,l} + (D_{ji} + M_{ji}) \varphi + (E_{ji}^{(1)} + N_{ji}^{(1)}) \psi \right]_{,j} + a_{ij}^{(1)} d_j + b_{ij}^{(1)} \varphi_{,j} + c_{ij}^{(1)} \psi_{,j} = \rho_1^0 \tilde{G}_i^{(1)}, \quad (80)$$

$$\rho_2^0 \ddot{w}_i - \left(b_{klij}^{(1)} u_{k,l} + d_{ijkl} w_{k,l} + M_{ij}^{(1)} \varphi + N_{ij}^{(1)} \psi \right)_{,j} - a_{ij}^{(1)} d_j - b_{ij}^{(1)} \varphi_{,j} - c_{ij}^{(1)} \psi_{,j} = \rho_2^0 \tilde{G}_i^{(2)}, \quad (81)$$

$$\rho_1^0 \kappa_1 \ddot{\varphi} - \left(a_{ij} \varphi_{,j} + \beta_{ij}^{(1)} \psi_{,j} + b_{ji}^{(1)} d_j \right)_{,i} + (D_{ji} + M_{ji}) u_{i,j} + M_{ij}^{(1)} w_{i,j} + \xi \varphi + \tau^{(1)} \psi = \rho_1^0 \kappa_1 \tilde{M}^{(1)}, \quad (82)$$

$$\rho_2^0 \kappa_2 \ddot{\psi} - \left(\beta_{ji}^{(1)} \varphi_{,j} + \gamma_{ij} \psi_{,j} + c_{ji}^{(1)} d_j \right)_{,i} + (E_{ji}^{(1)} + N_{ji}^{(1)}) u_{i,j} + N_{ij} w_{i,j} + \tau^{(1)} \varphi + \eta \psi = \rho_2^0 \kappa_2 \tilde{M}^{(2)}, \quad (83)$$

where

$$\rho_1^0 \tilde{G}_i^{(1)} = \left[b_{ijkl} w_{k,l}^{(2)} + (E_{ji} + N_{ji}) \psi^{(2)} \right]_{,j} - a_{ij} d_j^{(2)} - b_{ij} \varphi_{,j}^{(2)} - c_{ij} \psi_{,j}^{(2)}, \quad (84)$$

$$\rho_2^0 \tilde{G}_i^{(2)} = \left(b_{klij} u_{k,l}^{(2)} + M_{ij} \varphi^{(2)} \right)_{,j} + a_{ij} d_j^{(2)} + b_{ij} \varphi_{,j}^{(2)} + c_{ij} \psi_{,j}^{(2)}, \quad (85)$$

$$\rho_1^0 \kappa_1 \tilde{M}^{(1)} = \left(\beta_{ij} \psi_{,j}^{(2)} + b_{ji} d_j^{(2)} \right)_{,i} - M_{ij} w_{i,j}^{(2)} - \tau \psi^{(2)}, \quad (86)$$

$$\rho_2^0 \kappa_2 \tilde{M}^{(2)} = \left(\beta_{ji} \varphi_{,j}^{(2)} + c_{ji} d_j^{(2)} \right)_{,i} - (E_{ji} + N_{ji}) u_{i,j}^{(2)} - \tau \varphi^{(2)}. \quad (87)$$

On the basis of the relations (78) to (87), we can obtain the following identity:

$$F_2(t) = \int_0^t \int_{B(s)} \left(\rho_1^0 \tilde{G}_i^{(1)} \dot{u}_i + \rho_2^0 \tilde{G}_i^{(2)} \dot{w}_i + \rho_1^0 \kappa_1 \tilde{M}^{(1)} \dot{\varphi} + \rho_2^0 \kappa_2 \tilde{M}^{(2)} \dot{\psi} \right) dv ds, \quad (88)$$

where

$$\begin{aligned} F_2(t) = & \frac{1}{2} \int_{B(t)} \left[\left(\rho_1^0 \dot{u}_i \dot{u}_i + \rho_2^0 \dot{w}_i \dot{w}_i + \rho_1^0 \kappa_1 \dot{\varphi}^2 + \rho_2^0 \kappa_2 \dot{\psi}^2 \right) + a_{ijkl} u_{i,j} u_{k,l} + 2b_{ijkl}^{(1)} u_{i,j} w_{k,l} \right. \\ & + d_{ijkl} w_{i,j} w_{k,l} + 2(D_{ji} + M_{ji}) u_{i,j} \varphi + 2(E_{ji}^{(1)} + N_{ji}^{(1)}) u_{i,j} \psi + 2M_{ij}^{(1)} w_{i,j} \varphi \\ & + 2N_{ij} w_{i,j} \psi + a_{ij}^{(1)} d_i d_j + \alpha_{ij} \varphi_{,i} \varphi_{,j} + \gamma_{ij} \psi_{,i} \psi_{,j} + 2\beta_{ij}^{(1)} \varphi_{,i} \psi_{,j} + 2b_{ij}^{(1)} d_i \varphi_{,j} \\ & \left. + 2c_{ij}^{(1)} d_i \psi_{,j} + \xi \varphi^2 + 2\tau^{(1)} \varphi \psi + \eta \psi^2 \right] dv. \end{aligned} \quad (89)$$

Following the analysis of the previous section, we arrive at the following estimate

$$\sqrt{F_2(t)} \leq \left(\frac{K}{\sqrt{2}} \int_0^t F(U^{(2)}) ds \right) \mathcal{H}_3(C_3), \quad t \geq 0, \quad (90)$$

where $\mathcal{H}_3(C_3) = \text{meas}_B \{C_3\}$ is an appropriate measure of the coupling parameters included into C_3 .

6 | CONVERGENCE RESULT

We investigate here how the solution of the basic initial boundary value problem behaves as the coupling coefficients tend to zero. In this connection, we take in Section 5.1 $U^{(1)} = \{u_i^{(1)}, w_r^{(1)}, \varphi^{(1)}, \psi^{(1)}\}$ to be the solution of the coupled initial boundary value problem $\mathcal{P}$ corresponding to the same initial, boundary, and body loads data and to the same material characteristics excepting the coupling material parameters $C_2^{(1)} = \{D_{ij}^{(1)}, E_{ij}^{(1)}, M_{ij}^{(1)}, N_{ij}^{(1)}, b_{ij}^{(1)}, c_{ij}^{(1)}\}$. While we take $U^{(2)} = \{u_i^{(2)}, w_r^{(2)}, \varphi^{(2)}, \psi^{(2)}\}$ to be the solution of the uncoupled initial boundary value problem $\mathcal{P}$, that is with the same initial, boundary and body loads data, and the same material characteristics, but with the null coupling coefficients $C_2^{(2)} = \{D_{ij}^{(2)}, E_{ij}^{(2)}, M_{ij}^{(2)}, N_{ij}^{(2)}, b_{ij}^{(2)}, c_{ij}^{(2)}\} = 0$. Consequently, we have $C_2 = C_2^{(1)} - C_2^{(2)} = C_2^{(1)}$.

We are interested in how the solution $U^{(1)} = \{u_i^{(1)}, w_r^{(1)}, \varphi^{(1)}, \psi^{(1)}\}$ of the coupled initial boundary value problem $\mathcal{P}$ behaves when the coupling coefficients in the set $C_2^{(1)} = \{D_{ij}^{(1)}, E_{ij}^{(1)}, M_{ij}^{(1)}, N_{ij}^{(1)}, b_{ij}^{(1)}, c_{ij}^{(1)}\}$ tend to zero.

To this end, we note that, according to the results of Section 5.1, for the difference solution $U = U^{(1)} - U^{(2)} = \{u_i^{(1)} - u_i^{(2)}, w_r^{(1)} - w_r^{(2)}, \varphi^{(1)} - \varphi^{(2)}, \psi^{(1)} - \psi^{(2)}\}$ the estimate (76) is valid, which in the present context is read as

$$\sqrt{F_1(t)} \leq \left(\frac{K}{\sqrt{2}} \int_0^T F(U^{(2)}) ds \right) \mathcal{H}_2(C_2^{(1)}), \quad t \geq 0, \quad (91)$$

where $F_1(t)$, as given by the relation (69), represents a measure of the difference solution $U = U^{(1)} - U^{(2)}$, and $\mathcal{H}_2(C_2^{(1)})$ represents an appropriate measure of the coupling coefficients given in the set $C_2^{(1)} = \{D_{ij}^{(1)}, E_{ij}^{(1)}, M_{ij}^{(1)}, N_{ij}^{(1)}, b_{ij}^{(1)}, c_{ij}^{(1)}\}$, and, in view of the relations (8), (35), and (53), the term $\int_0^T F(U^{(2)}) ds$ is bounded in terms of the external given data of the uncoupled initial boundary value problem $\mathcal{P}$.

It follows clearly from the estimate (91) that, when $\mathcal{H}_2(C_2^{(1)})$ tends to zero, $F_1(t)$ tends to zero for any $t \in (0, T)$. That means, the solution $U^{(1)} = \{u_i^{(1)}, w_r^{(1)}, \varphi^{(1)}, \psi^{(1)}\}$ of the coupled initial boundary value problem $\mathcal{P}$ converges to $U^{(2)} = \{u_i^{(2)}, w_r^{(2)}, \varphi^{(2)}, \psi^{(2)}\}$, the solution of the uncoupled initial boundary value problem $\mathcal{P}$, when all the coupling coefficients in the set $C_2^{(1)} = \{D_{ij}^{(1)}, E_{ij}^{(1)}, M_{ij}^{(1)}, N_{ij}^{(1)}, b_{ij}^{(1)}, c_{ij}^{(1)}\}$ tend to zero. We have to recall that, when the coupling coefficients $C_2^{(1)}$ tend to zero, the effects of the mixture deformation measured from the reference configuration are decoupled, and the two constituents deform independently of each other.

7 | FINAL CONCLUSIONS

Our analysis in this paper demonstrates, under minimal assumptions on the material characteristics, that the porous mixture model, as developed by Ieşan [41], is consistent in the sense that:

- (i) the estimate (53) provides a continuous dependence result for the solution of the initial boundary value problem $\mathcal{P}$ in terms of the external given data;
- (ii) the estimates (76) and (90) furnish continuous dependence of solution with respect to the two sets of coupling parameters C_2 and C_3 , respectively. This shows that suitably small variations of the coupling coefficients do not destroy the well definiteness of the mixture model, that is the structural stability of the system is established;
- (iii) the estimate (91) shows that the solution $U^{(1)} = \{u_i^{(1)}, w_r^{(1)}, \varphi^{(1)}, \psi^{(1)}\}$ of the coupled initial boundary value problem $\mathcal{P}$ converges to $U^{(2)} = \{u_i^{(2)}, w_r^{(2)}, \varphi^{(2)}, \psi^{(2)}\}$, the solution of the uncoupled initial boundary value problem $\mathcal{P}$, when all the coupling coefficients in the set $C_2^{(1)} = \{D_{ij}^{(1)}, E_{ij}^{(1)}, M_{ij}^{(1)}, N_{ij}^{(1)}, b_{ij}^{(1)}, c_{ij}^{(1)}\}$ tend to zero.

On the other hand, all the estimates of continuous dependence presented in the paper can be very useful in experimental measurements of both external data and material coefficients, because the propagation of a small error in a measurement of data internal or external to the mixture, does not lead to chaos.

Finally, the authors think that their estimates can be very useful in the numerical treatment of the initial boundary value problems associated with the porous mixtures. This is because, for instance, a possible numerical approximation in the treatment of the problem leads to a solution sufficiently close to its exact value. More generally, the structural stability estimates obtained can be particularly helpful in areas such as construction with regard to important degradation and decay phenomena, as better detailed in the Introduction.

AUTHOR CONTRIBUTIONS

Julia de Castro Motta: Investigation; writing—review and editing; validation. **Vittorio Zampoli:** Writing—review and editing; investigation; validation. **Stan Chiriță:** Methodology; writing—original draft; investigation. **Michele Ciarletta:** Conceptualization; supervision.

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CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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