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On a three-phase-lag heat conduction model for rigid conductor

Stan Chiriţă
Alexandru Ioan Cuza University of Iaşi, Iaşi, Romania

Ciro D'Apice
Dipartimento di Scienze Aziendali – Management & Innovation Systems (DISA-MIS), University of Salerno, Fisciano, Italy

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Abstract
We study a thermal model associated with a heat-conducting material based on a three-phase-lag constitutive equation for the heat flux, a model that leads to a Moore–Gibson–Thompson type equation for the thermal displacement. We are researching the compatibility of the three-phase-lag constitutive equation in concern with the second law of thermodynamics, thus discovering restrictions to be imposed on the involved thermal coefficients. On this basis, we manage to obtain the well-posedness problem of the model as the uniqueness of the solutions and their continuous dependence on the given data. Finally, we show that such a model not only allows the propagation of damped in time waves but also exponentially decaying in time thermal standing mode waves. We also show that if the thermodynamic restrictions are not fulfilled, then we can be led to instability. Through the present treatment of the thermal model in question, we obtain important information on the associated Moore–Gibson–Thompson type equation for the thermal displacement.

Keywords
Three-phase-lag heat conduction, thermodynamic compatibility, well-posed problems, Moore–Gibson–Thompson type equation for the thermal displacement, instability of solutions

1. Introduction

Jordan [1] considered the propagation of finite-amplitude acoustic waves in fluids that exhibit both viscosity and thermal relaxation described by the Maxwell–Cattaneo law, which is a well-known generalization of the Fourier law that includes the effects of thermal inertia. The Maxwell–Cattaneo law, when combined with the other basic equations of the model, leads to a nonlinear third-order in time partial differential equation (see, for example, Jordan [1] and Kaltenbacher et al. [2]):

$$\tau \dddot{\psi} + \ddot{\psi} - c^2 \Delta \psi - b \Delta \dot{\psi} = \frac{d}{dt}\left(\frac{1}{c^2}\frac{B}{2A}\dot{\psi}^2 + \psi_{,i}\psi_{,i} \right), \tag{1}$$

Corresponding author:
Stan Chiriţă, Alexandru Ioan Cuza University of Iaşi, Iaşi 700506, Romania
Email: schirita@uaic.ro

where Δ is the Laplace operator, τ is a relaxation time, and the other coefficients are some positive parameters. A linearized version is written in the more general abstract form:

$$\tau \dddot{u} + a\ddot{u} + c^2 \mathcal{A}u + b\mathcal{A}\dot{u} = 0, \tag{2}$$

(with $\mathcal{A}$ being a self-adjoint positive operator and a, b, and c being the positive parameters); it was subjected to an intense research regarding the well-posedness and stability of its solutions (see, for example, Kaltenbacher et al. [2], Lasiecka and Wang [3, 4], Pellicer and Said-Houari [5], and Chen and Ikehata [6]). It was observed by Quintanilla [7] that the Moore–Gibson–Thompson equation appears in the generalized thermoelastic theories based upon the introduction of the thermal displacement (see Green–Naghdi's model [8–10]) and using the Maxwell–Cattaneo constitutive equation for the heat flux vector. Besides, there is a recent, rich, and intense research activity of the third-order in time differential equations, either separately or coupled with the elastic deformations of medium, leading to valuable results concerning the well-posedness: Quintanilla [7], Pellicer and Quintanilla [11], Conti et al. [12], Ostoja-Starzewski and Quintanilla [13], Bazarra et al. [14], Fernandez and Quintanilla [15, 16], and Fernandez et al. [17].

All these studies suffer from an inconvenience, namely, that they prescribe initial and boundary data in terms of temperature (or thermal displacement), instead of including mixed data for temperature (or thermal displacement) and heat flux vector. That is why initial boundary value problems with important physical significance are excluded from their research. Also, by leaving the original thermomechanical model, the thermodynamic restrictions on the thermal coefficients involved are not known.

In this work, we eliminate all the shortcomings listed above. For this purpose, we consider the three-phase-lag constitutive equation for the heat flux vector as proposed by Roy Choudhuri [18] and then we determine the conditions on the coefficients involved in order to have compatibility with the second law of thermodynamics. Then, we formulate and establish the well-posedness for the associated initial boundary value problems. In this sense, we use the classical conservation law, as well as the Lagrange identity method and the logarithmic convexity method. Finally, we show that such a model not only allows the propagation of damped in time harmonic waves but also exponentially decaying in time thermal standing mode waves. We also show that if the thermodynamic restrictions are not fulfilled, then we can be led to instability. Through the present treatment of the thermal model in question, we obtain important information regarding well-posedness and instability on the associated Moore–Gibson–Thompson type equation for the thermal displacement.

The plan of the work is the following. Section 2 presents the basic system of differential equations describing the evolutionary behavior of the heat flux and of the thermal displacement in the line described by Roy Choudhuri [18]. The associated third-order in time differential equation in terms of thermal displacement is explicitly written. Section 3 establishes the explicit restrictions on the thermal coefficients deduced from the compatibility of the constitutive law with the second law of thermodynamics. Section 4 formulates the initial boundary value problems associated with the model in concern and then describes a modified version of them that will be useful in the future analysis. A conservation law of energy identity is also presented. Section 5 is dedicated to the study of uniqueness and continuous data dependence problems under common assumptions on the thermal coefficients. The Lagrange identity and logarithmic convexity methods are also used to establish uniqueness result under mild assumptions on the thermal coefficients. Section 6 discusses possible thermal wave solutions that can satisfy the basic equations. It is shown that if the thermodynamic restrictions are not fulfilled, then we can be led to unbounded wave solutions when time goes to infinity and so we have instability.

2. Three-phase-lag heat conducting model for a rigid conductor

Tzou [19] has proposed a two-phase-lag model by generalizing the Fourier law of heat conduction $q_i(x,t) = -k_{ij}T_{,j}(x,t)$ in the following form:

$$q_i(x, t + \tau_q) = -k_{ij}(x)T_{,j}(x, t + \tau_T), \tag{3}$$

where q_i are the components of the heat flux vector, T is the variation of the temperature from the constant reference temperature $T_0 > 0$, k_{ij} are the components of the thermal conductivity tensor, and the delay time $\tau_q > 0$ is the phase-lag of the heat flux and the delay time $\tau_T > 0$ is the phase-lag of the temperature gradient.

On the other side, Green and Naghdi [8–10] have introduced the thermal displacement by $\alpha(x,t) = \dot{T}(x,t)$ and they have proposed a heat conduction law as:

$$q_i(x,t) = -\left[k_{ij}(x)T_{,j}(x,t) + k^*_{ij}(x)\alpha_{,j}(x,t)\right], \tag{4}$$

where k_{ij}^* are the components of the conductivity rate tensor.

Roy Choudhuri [18] combined the two above models and he proposed the three-phase-lags for the heat flux vector q_i, i.e., he considers the following generalized constitutive equation for heat conduction in order to describe the lagging behavior:

$$q_i(x, t + \tau_q) = -\left[k_{ij}(x)T_{,j}(x, t + \tau_T) + k_{ij}^*(x)\alpha_{,j}(x, t + \tau_\alpha)\right], \tag{5}$$

where $\tau_\alpha > 0$ is the phase-lag of the thermal displacement gradient.

Furthermore, by taking the Taylor series expansion of equation (5) up to the first-order terms in τ_q, τ_T, and τ_α, Roy Choudhuri proposes the following generalized heat conduction law valid at a point x at time t:

$$q_i(x, t) + \tau_q \dot{q}_i(x, t) = -k_{ij}^*(x)\alpha_{,j}(x, t) - \left(k_{ij}(x) + \tau_\alpha k_{ij}^*(x)\right)\dot{\alpha}_{,j}(x, t) - \tau_T k_{ij}(x)\ddot{\alpha}_{,j}(x, t). \tag{6}$$

Equation (6) serves as a generalized constitutive heat conduction law in which the elastic deformation term is ignored.

The three-phase-lags model for a rigid conductor as proposed by Roy Choudhuri [18] is based on the constitutive equation (6) and the well-known heat equation:

$$-q_{i,i}(x, t) + r(x, t) = c(x)\dot{T}(x, t), \tag{7}$$

where $r(x, t)$ represents the heat source acting per unit volume, and $c(x)$ is the specific heat per unit volume.

In terms of the thermal displacement α, the three-phase-lags model for a rigid conductor as proposed by Roy Choudhuri [18] is based upon the following differential equation:

$$c\tau_q \dddot{\alpha} + c\ddot{\alpha} - \left(k_{ij}^*\alpha_{,j}\right)_{,i} - \left[\left(k_{ij} + \tau_\alpha k_{ij}^*\right)\dot{\alpha}_{,j}\right]_{,i} - \tau_T\left(k_{ij}\ddot{\alpha}_{,j}\right)_{,i} = 0, \tag{8}$$

where we assumed the vanishing of the heat source and, moreover, the dependence on the independent variables x and t was suppressed, but implicitly understood.

It can be easily seen that equation (8) generalizes the Moore–Gibson–Thompson equation in the sense that a neglect of the term containing the relaxation time τ_T leads to equation (2). Consequently, all our analysis in this paper will remain valid in the case of the simpler model used in the studies by Kaltenbacher et al. [2], Lasiecka and Wang [3, 4], Pellicer and Said-Houari [5], and Chen and Ikehata [6].

3. Thermodynamic compatibility of the three-phase-lag constitutive equation

In this section, we discover the restrictions imposed by the second law of thermodynamics on the basic constitutive equation of the model described by the relations (6) and (7). In this sense, we note that the constitutive equation (6) can be written in the form:

$$\frac{\partial}{\partial t}\left[e^{t/\tau_q}\, q_i(t)\right] = -\frac{1}{\tau_q}\, e^{t/\tau_q}\left[k_{ij}^*\alpha_{,j}(t) + \left(k_{ij} + \tau_\alpha\, k_{ij}^*\right)\dot{\alpha}_{,j}(t) + \tau_T\, k_{ij}\ddot{\alpha}_{,j}(t)\right], \tag{9}$$

so that, by assuming that $\lim_{t\to-\infty} q_i(t) = 0$, we get:

$$q_i(t) = -\frac{1}{\tau_q}\int_{-\infty}^{t} e^{(z-t)/\tau_q}\left[k_{ij}^*\alpha_{,j}(z) + \left(k_{ij} + \tau_\alpha\, k_{ij}^*\right)\dot{\alpha}_{,j}(z) + \tau_T\, k_{ij}\ddot{\alpha}_{,j}(z)\right]dz, \tag{10}$$

and where we have omitted everywhere the explicit dependence upon the space variable. Furthermore, with the change of variable $t - z = s$, we obtain:

$$q_i(t) = -\frac{1}{\tau_q}\int_{0}^{\infty} e^{-s/\tau_q}\left[k_{ij}^*\alpha^t_{,j}(s) + \left(k_{ij} + \tau_\alpha\, k_{ij}^*\right)\dot{\alpha}^t_{,j}(s) + \tau_T\, k_{ij}\ddot{\alpha}^t_{,j}(s)\right]ds, \tag{11}$$

where:

$$\alpha^t(s) = \alpha(t - s). \tag{12}$$

Some integration by parts in (11) gives:

$$q_i(t) = -\frac{\tau_T}{\tau_q} k_{ij}\dot{\alpha}_{,j}(t) + \frac{1}{\tau_q^2}\left[\tau_T k_{ij} - \tau_q\left(k_{ij} + \tau_\alpha k_{ij}^*\right)\right]\alpha_{,j}(t)$$
$$-\frac{1}{\tau_q^3}\left[\tau_T k_{ij} - \tau_q\left(k_{ij} + \tau_\alpha k_{ij}^*\right) + \tau_q^2 k_{ij}^*\right]\int_0^\infty e^{-s/\tau_q}\alpha_{,j}(t-s)ds. \tag{13}$$

To determine the restrictions imposed by thermodynamics on the constitutive equation in concern, we postulate the second law of thermodynamics in terms of a Clausius–Duhem inequality formulated on cyclic histories of period p, i.e., (see, for example, Amendola et al. [20], Chapter 8, section 8.2: Thermodynamic Constraints for Rigid Heat Conductors, page 216),

$$\oint q_i(t)T_{,i}(t)dt \leq 0, \tag{14}$$

or equivalently,

$$\int_0^p q_i(t)T_{,i}(t)dt \leq 0. \tag{15}$$

Consequently, any cycle for $\alpha_{,i}(t)$, characterized by:

$$\alpha_{,i}(t) = A_i \cos\omega t + B_i \sin\omega t, \tag{16}$$

with $\omega > 0$, and A_i and B_i arbitrary constants in time (but depending on the spatial variable x), with $A_iA_i + B_iB_i \neq 0$, has to fulfill equation (15) as an inequality. Therefore, it has to fulfill the following inequality:

$$\int_0^{2\pi/\omega} q_i(t)\dot{\alpha}_{,i}(t)dt \leq 0, \tag{17}$$

for all $\omega > 0$, and for arbitrary real constants in time A_i and B_i with $A_iA_i + B_iB_i \neq 0$.

If we use relations (13) and (16), we get:

$$\int_0^{2\pi/\omega} q_i(t)\dot{\alpha}_{,i}(t)dt = -\frac{\pi\omega\tau_T}{\tau_q} k_{ij}\left(A_iA_j + B_iB_j\right)$$
$$+\frac{\pi}{\tau_q^3}\left[\tau_T k_{ij} - \tau_q\left(k_{ij} + \tau_\alpha k_{ij}^*\right) + \tau_q^2 k_{ij}^*\right]\left(A_iA_j + B_iB_j\right) I, \tag{18}$$

where:

$$I = \int_0^\infty e^{-s/\tau_q}\sin\omega s ds = \frac{\omega\tau_q^2}{1 + \tau_q^2\omega^2}. \tag{19}$$

Therefore, the second law of thermodynamics implies, for all $\omega > 0$,

$$\tau_T\tau_q k_{ij}\left(A_iA_j + B_iB_j\right)\omega^2 + \varkappa_{ij}\left(A_iA_j + B_iB_j\right) \geq 0, \tag{20}$$

where:

$$\varkappa_{ij} = k_{ij} + \left(\tau_\alpha - \tau_q\right)k_{ij}^*. \tag{21}$$

Consequently, the constitutive equation (6) is compatible with the second law of thermodynamics if:

(R(i)) the tensor k_{ij} is positive semi-definite;

(R(ii)) the tensor $\varkappa_{ij} = k_{ij} + \left(\tau_\alpha - \tau_q\right)k_{ij}^*$ is positive semi-definite.

4. Formulation of the initial boundary value problem—the modified problem

We assume that a regular region B is filled by an inhomogeneous and anisotropic conductor material with three-phase-lag times. Throughout this paper, we consider the initial boundary value problem $\mathcal{P}$ defined by the heat equation (7), the constitutive equation (6), the initial conditions:

$$\alpha(x,0) = 0, \quad \dot{\alpha}(x,0) = \dot{\alpha}^0(x), \quad q_i(x,0) = q_i^0(x) \quad \text{for all} \quad x \in B, \tag{22}$$

and the following boundary conditions:

$$\begin{aligned} &\alpha(x,t) = \Theta(x,t) \quad on \quad \Sigma_1 \times (0,\infty), \\ &q_i(x,t)n_i = Q(x,t) \quad on \quad \Sigma_2 \times (0,\infty). \end{aligned} \tag{23}$$

Here, $\dot{\alpha}^0(x)$ and $q_i^0(x)$, as well as $\Theta(x,t)$ and $Q(x,t)$, are prescribed smooth functions. Moreover, Σ_1 and Σ_2 are the subsets of the boundary ∂B so that $\overline{\Sigma}_1 \cup \Sigma_2 = \partial B$ and $\Sigma_1 \cap \Sigma_2 = \emptyset$.

By a solution of the initial boundary value problem $\mathcal{P}$, corresponding to the given data $\mathcal{D} = \{r; \dot{\alpha}^0, q_i^0; \Theta, Q\}$, we mean the ordered array $\mathcal{S} = \{\alpha, q_i\}$ defined on $B \times (0,\infty)$ with the properties that $\alpha(x,t) \in C^{2,2}(B \times (0,\infty))$, $q_i(x,t) \in C^{1,1}(B \times (0,\infty))$ and which satisfy the field equations (6) and (7), the initial conditions (22), and the boundary conditions (23). In what follows, we denote by $\mathcal{P}_0$ the initial boundary value problem $\mathcal{P}$ corresponding to the zero given data $\mathcal{D} = \{r; \dot{\alpha}^0, q_i^0; \Theta, Q\} = 0$.

In order to conveniently deal with topics regarding the uniqueness, continuous data dependence of solutions, it is necessary to define a modified initial boundary value problem $\tilde{\mathcal{P}}$ associated with the problem in question $\mathcal{P}$. In this sense, we introduce the notations:

$$\tilde{q}_i = q_i + \tau_q \dot{q}_i, \quad \tilde{\alpha} = \alpha + \tau_q \dot{\alpha}, \tag{24}$$

and note that the heat equation (7) implies:

$$-\tilde{q}_{i,i} + \tilde{r} = c\ddot{\tilde{\alpha}} \quad in \quad B \times (0,\infty), \tag{25}$$

while the constitutive equation (6) becomes:

$$\tilde{q}_i = -k_{ij}^* \alpha_{,j} - \left(k_{ij} + \tau_\alpha k_{ij}^*\right) \dot{\alpha}_{,j} - \tau_T k_{ij} \ddot{\alpha}_{,j} \quad in \quad B \times (0,\infty), \tag{26}$$

where:

$$\tilde{r} = r + \tau_q \dot{r}. \tag{27}$$

Furthermore, in view of the initial conditions (22) and the heat equation (7), we have:

$$\tilde{\alpha}(x,0) = \tau_q \dot{\alpha}^0(x), \quad \dot{\tilde{\alpha}}(x,0) = \dot{\alpha}^0(x) + \tau_q \ddot{\alpha}(x,0), \quad c\ddot{\alpha}(x,0) = r(x,0) - q_{i,i}^0(x), \tag{28}$$

while the boundary conditions (23) furnishes:

$$\begin{aligned} &\tilde{\alpha}(x,t) = \tilde{\Theta}(x,t) \quad on \quad \Sigma_1 \times (0,\infty), \\ &\tilde{q}_i(x,t)n_i = \tilde{Q}(x,t) \quad on \quad \Sigma_2 \times (0,\infty), \end{aligned} \tag{29}$$

with $\tilde{\Theta} = \Theta + \tau_q \dot{\Theta}$ and $\tilde{Q} = Q + \tau_q \dot{Q}$.

We now proceed to establish an energetic identity satisfied by any solution of the initial boundary value problem $\tilde{\mathcal{P}}$. Thus, on the basis of the equation (25), we get:

$$\int_B c\dot{\tilde{\alpha}}\ddot{\tilde{\alpha}} dv = \int_B \tilde{r}\dot{\tilde{\alpha}} dv - \int_{\partial B} \tilde{q}_i n_i \dot{\tilde{\alpha}} da + \int_B \dot{\tilde{\alpha}}_{,i} \tilde{q}_i dv. \tag{30}$$

Furthermore, by taking into account the relation (24) and the constitutive equation (26), we obtain:

$$\int_B \dot{\tilde{\alpha}}_{,i}\tilde{q}_i dv = -\int_B \left(\dot{\alpha}_{,i} + \tau_q \ddot{\alpha}_{,i}\right)\left[k^*_{ij}\alpha_{,j} + \left(k_{ij} + \tau_\alpha k^*_{ij}\right)\dot{\alpha}_{,j} + \tau_T k_{ij}\ddot{\alpha}_{,j}\right] dv, \tag{31}$$

and hence, by some appropriate calculations, we deduce that:

$$\begin{aligned}\int_B \dot{\tilde{\alpha}}_{,i}\tilde{q}_i dv = &-\frac{1}{2}\frac{d}{dt}\int_B \left\{k^*_{ij}\left(\alpha_{,i} + \tau_q\dot{\alpha}_{,i}\right)\left(\alpha_{,j} + \tau_q\dot{\alpha}_{,j}\right) + \tau_q\left[k_{ij} + \left(\tau_\alpha - \tau_q\right)k^*_{ij}\right]\dot{\alpha}_{,i}\dot{\alpha}_{,j}\right.\\ &\left.+\tau_T k_{ij}\dot{\alpha}_{,i}\dot{\alpha}_{,j}\right\} dv - \int_B \left\{\left[k_{ij} + \left(\tau_\alpha - \tau_q\right)k^*_{ij}\right]\dot{\alpha}_{,i}\dot{\alpha}_{,j} + \tau_q\tau_T k_{ij}\ddot{\alpha}_{,i}\ddot{\alpha}_{,j}\right\} dv.\end{aligned} \tag{32}$$

Consequently, the above relations (30) and (32) imply:

$$\frac{d\mathcal{E}}{dt}(t) = \int_B \tilde{r}(t)\dot{\tilde{\alpha}}(t)dv - \int_{\partial B} \tilde{q}_i(t)n_i\dot{\tilde{\alpha}}(t)da, \tag{33}$$

where:

$$\begin{aligned}\mathcal{E}(t) = \frac{1}{2}\int_B &\left\{c\dot{\tilde{\alpha}}(t)^2 + k^*_{ij}\tilde{\alpha}_{,i}(t)\tilde{\alpha}_{,j}(t)\right.\\ &\left.+\tau_q\left[k_{ij} + \left(\tau_\alpha - \tau_q\right)k^*_{ij}\right]\dot{\alpha}_{,i}(t)\dot{\alpha}_{,j}(t) + \tau_T k_{ij}\dot{\alpha}_{,i}(t)\dot{\alpha}_{,j}(t)\right\} dv\\ &+\int_0^t\int_B \left\{\left[k_{ij} + \left(\tau_\alpha - \tau_q\right)k^*_{ij}\right]\dot{\alpha}_{,i}(s)\dot{\alpha}_{,j}(s) + \tau_q\tau_T k_{ij}\ddot{\alpha}_{,i}(s)\ddot{\alpha}_{,j}(s)\right\} dvds.\end{aligned} \tag{34}$$

5. Uniqueness and continuous data dependence results

In our further analysis, we will need some of the following hypotheses upon the characteristic material coefficients:

(H1): the specific heat per unit volume is strictly positive in B, i.e.:

$$c(x) > 0 \quad \text{for all} \quad x \in B; \tag{35}$$

(H2): k^*_{ij} is a positive semi-definite tensor as:

$$k^*_{rs}(x)\xi_r\xi_s \geq 0 \quad \text{for all} \quad (\xi_1, \xi_2, \xi_3) \quad \text{and for all} \quad x \in B; \tag{36}$$

(H3): k_{ij} is a positive semi-definite tensor as:

$$k_{rs}(x)\xi_r\xi_s \geq 0 \quad \text{for all} \quad (\xi_1, \xi_2, \xi_3) \quad \text{and for all} \quad x \in B; \tag{37}$$

(H4): $\varkappa_{ij} = k_{ij} + \left(\tau_\alpha - \tau_q\right)k^*_{ij}$ is a positive semi-definite tensor, i.e.:

$$\varkappa_{rs}(x)\xi_r\xi_s \geq 0 \quad \text{for all} \quad (\xi_1, \xi_2, \xi_3) \quad \text{and for all} \quad x \in B. \tag{38}$$

The last two hypotheses prove that the second law of thermodynamics is fulfilled as results from section 3. The first hypothesis proves that a genuine dynamic thermal situation is considered. While the hypothesis (H2) represents an extension of hypothesis (H3) for the conductivity rate tensor.

Theorem 1. (Uniqueness of solution). Suppose the hypotheses (H1) to (H4) to be fulfilled. Then, the initial boundary value problem $\mathcal{P}$ admits at most one solution.

Proof. Having a linear problem, proving the uniqueness of the solution is equivalent to proving that the problem with homogeneous data, $\mathcal{P}_0$, admits only the trivial solution. In this connection, we note that for the initial boundary value problem $\mathcal{P}_0$, from equation (28), we have:

$$\tilde{\alpha}(x,0)=0,\quad \dot{\tilde{\alpha}}(x,0)=0,\quad \dot{\alpha}(x,0)=0,\quad \text{for all}\quad x\in B, \tag{39}$$

and

$$\begin{aligned}&\tilde{\alpha}(x,t)=0 \quad \text{on} \quad \Sigma_1\times(0,\infty),\quad \tilde{q}_i(x,t)n_i=0 \quad \text{on} \quad \Sigma_2\times(0,\infty),\\ &\tilde{r}(x,t)=0 \quad \text{for all} \quad (x,t)\in B\times(0,\infty).\end{aligned} \tag{40}$$

By virtue of relations (39) and (40), the identity (33) provides:

$$\mathcal{E}(t)=0 \quad \text{for all} \quad t\in(0,\infty). \tag{41}$$

But, on the contrary, based on hypotheses (H1) to (H4), $\mathcal{E}(t)$ in equation (34) appears as a sum of positive terms, and therefore, equality (41) implies that each term of the sum has to be zero. In particular, we deduce that:

$$\dot{\tilde{\alpha}}=0, \tag{42}$$

and hence, we have:

$$\dot{\alpha}+\tau_q\ddot{\alpha}=0. \tag{43}$$

which, when integrated under zero initial conditions for α, furnishes:

$$\alpha(x,t)=0 \quad \text{for all} \quad (x,t)\in B\times(0,\infty). \tag{44}$$

Using this result in the constitutive equation (6), we obtain the following differential equation:

$$q_i+\tau_q\dot{q}_i=0, \tag{45}$$

from which and using the initial condition $q_i(x,0)=0$, we get:

$$q_i(x,t)=0 \quad \text{for all} \quad (x,t)\in B\times(0,\infty). \tag{46}$$

Concluding we see that $\mathcal{S}=\{\alpha,q_i\}=0$ and the proof is finished.

Theorem 2. (Continuous dependence of solution with respect to the heat source). Suppose the hypotheses (H1) to (H4) to be fulfilled. Assume that the solution $\mathcal{S}=\{\alpha,q_i\}$ corresponds to the given data $\mathcal{D}=\{r;0,0;0,0\}$ on $B\times(0,\infty)$. Then, the following estimate holds true:

$$\sqrt{\mathcal{E}(t)}\le\frac{1}{\sqrt{2}}\int_0^t\left(\int_B\frac{1}{c}\tilde{r}(s)^2dv\right)^{1/2}ds \quad \textit{for all} \quad t>0. \tag{47}$$

Proof. Under the assumptions of the theorem, the identity (33) furnishes:

$$\mathcal{E}(t)=\int_0^t\int_B\tilde{r}(s)\dot{\tilde{\alpha}}(s)dvds, \tag{48}$$

so that, by means of the Cauchy–Schwarz inequality, we get the estimate:

$$\mathcal{E}(t)\le\int_0^t\left(\int_B\frac{1}{c}\tilde{r}(s)^2dv\right)^{1/2}\left(\int_B c\dot{\tilde{\alpha}}(s)^2dv\right)^{1/2}ds\le\int_0^t\sqrt{2\mathcal{E}(s)}\psi(s)ds, \tag{49}$$

where:

$$\psi(t)=\left(\int_B\frac{1}{c}\tilde{r}(t)^2dv\right)^{1/2}. \tag{50}$$

If we set:

$$\Phi(t)^2 = \int_0^t \sqrt{2\mathcal{E}(s)}\psi(s)ds, \tag{51}$$

then we get:

$$2\Phi(t)\frac{d\Phi}{dt}(t) = \sqrt{2\mathcal{E}(t)}\psi(t) \leq \sqrt{2}\Phi(t)\psi(t), \tag{52}$$

and hence:

$$\frac{d\Phi}{dt}(t) \leq \frac{1}{\sqrt{2}}\psi(t). \tag{53}$$

Consequently, we obtain:

$$\sqrt{\mathcal{E}(t)} \leq \Phi(t) \leq \frac{1}{\sqrt{2}}\int_0^t \psi(s)ds, \tag{54}$$

that is the estimate (47) and the proof is complete.

Theorem 3. Suppose the hypotheses (H1), (H3), and (H4) to be fulfilled. Then, the initial boundary value problem $\mathcal{P}$ admits at most one solution.

Proof 1: Lagrange identity method (see, for example, Brun [21] and Rionero and Chiriţă [22]). We start with the following identity:

$$\frac{\partial}{\partial z}\left[\tilde{\alpha}(t+z)\dot{\tilde{\alpha}}(t-z) + \tilde{\alpha}(t-z)\dot{\tilde{\alpha}}(t+z)\right] = \tilde{\alpha}(t-z)\ddot{\tilde{\alpha}}(t+z) - \tilde{\alpha}(t+z)\ddot{\tilde{\alpha}}(t-z), \tag{55}$$

and then an integration over z on $[0, t]$, followed by the use of the heat equation (25) and the zero given data, gives:

$$2\int_B c\tilde{\alpha}(t)\dot{\tilde{\alpha}}(t)dv = \int_0^t \int_B \left[\tilde{\alpha}(t-z)\tilde{q}_{i,i}(t+z) - \tilde{\alpha}(t+z)\tilde{q}_{i,i}(t-z)\right] dvdz. \tag{56}$$

Then, we use the divergence theorem combined with the null data on the boundary to obtain:

$$2\int_B c\tilde{\alpha}(t)\dot{\tilde{\alpha}}(t)dv = \int_0^t \int_B \left[\tilde{q}_i(t-z)\tilde{\alpha}_{,i}(t+z) - \tilde{q}_i(t+z)\tilde{\alpha}_{,i}(t-z)\right] dvdz. \tag{57}$$

Furthermore, we use the relation (24) and the constitutive equation (26) to get:

$$\begin{aligned}\tilde{q}_i(t-z)\tilde{\alpha}_{,i}(t+z) - \tilde{q}_i(t+z)\tilde{\alpha}_{,i}(t-z) = \frac{\partial}{\partial z}\Big\{\left[k_{ij} + \left(\tau_\alpha - \tau_q\right)k_{ij}^*\right]\alpha_{,i}(t-z)\alpha_{,j}(t+z) \\ +\tau_T k_{ij}\alpha_{,i}(t-z)\dot{\alpha}_{,j}(t+z) + \tau_T k_{ij}\alpha_{,i}(t+z)\dot{\alpha}_{,j}(t-z) + \tau_q\tau_T k_{ij}\dot{\alpha}_{,i}(t-z)\dot{\alpha}_{,j}(t+z)\Big\}.\end{aligned} \tag{58}$$

Now, we use equation (58) into relation (57) and then by an integration combined with null given data, we deduce:

$$\begin{aligned}&\int_B \left[c\tilde{\alpha}(t)^2 + \tau_T k_{ij}\alpha_{,i}(t)\alpha_{,j}(t)\right] dv \\ &+ \int_0^t \int_B \Big\{\left[k_{ij} + \left(\tau_\alpha - \tau_q\right)k_{ij}^*\right]\alpha_{,i}(s)\alpha_{,j}(s) + \tau_q\tau_T\, k_{ij}\dot{\alpha}_{,i}(s)\dot{\alpha}_{,j}(s)\Big\}\, dvds = 0.\end{aligned} \tag{59}$$

As can be seen, by virtue of the constitutive hypotheses (H1), (H3), and (H4), in equation (59) we have a sum of positive terms and this can be equal to zero only if each term is vanishing. In particular, we deduce that:

$$c(x)\tilde{\alpha}^2(x, t) = 0 \quad \text{in} \quad B \times (0, \infty). \tag{60}$$

Since $c(x) > 0$ in B, from equation (60), it follows that the relation (44) holds true and it can be used like in Theorem 1 in order to prove that $\mathcal{S} = \{\alpha, q_i\} = 0$ and the proof is completed.

Proof 2: Logarithmic convexity method (see, for example, Knops and Payne [23]). Guided by the above proof, we introduce now the following function:

$$F(t) = \int_B \left[c\tilde{\alpha}(t)^2 + \tau_T k_{ij}\alpha_{,i}(t)\alpha_{,j}(t)\right] dv + \int_0^t \int_B \left\{\left[k_{ij} + \left(\tau_\alpha - \tau_q\right) k^*_{ij}\right] \alpha_{,i}(s)\alpha_{,j}(s) + \tau_q\tau_T\, k_{ij}\dot{\alpha}_{,i}(s)\dot{\alpha}_{,j}(s)\right\} dvds, \tag{61}$$

and note that:

$$\dot{F}(t) = 2\int_B \left[c\tilde{\alpha}(t)\dot{\tilde{\alpha}}(t) + \tau_T k_{ij}\alpha_{,i}(t)\dot{\alpha}_{,j}(t)\right] dv + 2\int_0^t \int_B \left\{\left[k_{ij} + \left(\tau_\alpha - \tau_q\right) k^*_{ij}\right] \alpha_{,i}(s)\dot{\alpha}_{,j}(s) + \tau_q\tau_T\, k_{ij}\dot{\alpha}_{,i}(s)\ddot{\alpha}_{,j}(s)\right\} dvds, \tag{62}$$

and moreover,

$$\ddot{F}(t) = 2\int_B \left[c\dot{\tilde{\alpha}}(t)^2 + \tau_T k_{ij}\dot{\alpha}_{,i}(t)\dot{\alpha}_{,j}(t) + c\tilde{\alpha}(t)\ddot{\tilde{\alpha}}(t) + \tau_T k_{ij}\alpha_{,i}(t)\ddot{\alpha}_{,j}(t)\right] dv + 2\int_B \left\{\left[k_{ij} + \left(\tau_\alpha - \tau_q\right) k^*_{ij}\right] \alpha_{,i}(t)\dot{\alpha}_{,j}(t) + \tau_q\tau_T\, k_{ij}\dot{\alpha}_{,i}(t)\ddot{\alpha}_{,j}(t)\right\} dv. \tag{63}$$

Furthermore, in view of basic equations (25) and (26) and using the null given data, we have:

$$\begin{aligned}\int_B c\tilde{\alpha}(t)\ddot{\tilde{\alpha}}(t)dv &= -\int_B \tilde{\alpha}(t)\tilde{q}_{i,i}(t)dv = \int_B \tilde{\alpha}_{,i}(t)\tilde{q}_i(t)dv - \int_{\partial B} \tilde{q}_i(t)n_i\tilde{\alpha}(t)da \\ &= -\int_B \Big[k^*_{ij}\alpha_{,i}(t)\alpha_{,j}(t) + \left(k_{ij} + \tau_\alpha k^*_{ij}\right)\alpha_{,i}(t)\dot{\alpha}_{,j}(t) + \tau_T k_{ij}\alpha_{,i}(t)\ddot{\alpha}_{,j}(t) + \tau_q k^*_{ij}\dot{\alpha}_{,i}(t)\alpha_{,j}(t) \\ &\quad + \tau_q\left(k_{ij} + \tau_\alpha k^*_{ij}\right)\dot{\alpha}_{,i}(t)\dot{\alpha}_{,j}(t) + \tau_q\tau_T k_{ij}\dot{\alpha}_{,i}(t)\ddot{\alpha}_{,j}(t)\Big] dv.\end{aligned} \tag{64}$$

Thus, relations (63) and (64) furnish:

$$\ddot{F}(t) = 2\int_B \Big[c\dot{\tilde{\alpha}}(t)^2 + \tau_T k_{ij}\dot{\alpha}_{,i}(t)\dot{\alpha}_{,j}(t) - k^*_{ij}\tilde{\alpha}_{,i}(t)\tilde{\alpha}_{,j}(t) - \tau_q\left[k_{ij} + \left(\tau_\alpha - \tau_q\right) k^*_{ij}\right]\dot{\alpha}_{,i}(t)\dot{\alpha}_{,j}(t)\Big] dv. \tag{65}$$

Then, the conservation law (41) together with relation (34) gives:

$$-\int_B \left\{k^*_{ij}\tilde{\alpha}_{,i}(t)\tilde{\alpha}_{,j}(t) + \tau_q\left[k_{ij} + \left(\tau_\alpha - \tau_q\right) k^*_{ij}\right]\dot{\alpha}_{,i}(t)\dot{\alpha}_{,j}(t)\right\} dv = \int_B \left[\tau_T k_{ij}\dot{\alpha}_{,i}(t)\dot{\alpha}_{,j}(t) + c\dot{\tilde{\alpha}}(t)^2\right] dv + 2\int_0^t \int_B \left\{\left[k_{ij} + \left(\tau_\alpha - \tau_q\right) k^*_{ij}\right]\dot{\alpha}_{,i}(s)\dot{\alpha}_{,j}(s) + \tau_q\tau_T k_{ij}\ddot{\alpha}_{,i}(s)\ddot{\alpha}_{,j}(s)\right\} dvds. \tag{66}$$

Consequently, from relations (65) and (66), we obtain:

$$\ddot{F}(t) = 4\left\{\int_B \left[c\dot{\tilde{\alpha}}(t)^2 + \tau_T k_{ij}\dot{\alpha}_{,i}(t)\dot{\alpha}_{,j}(t)\right] dv + \int_0^t \int_B \left\{\left[k_{ij} + \left(\tau_\alpha - \tau_q\right) k^*_{ij}\right]\dot{\alpha}_{,i}(s)\dot{\alpha}_{,j}(s) + \tau_q\tau_T k_{ij}\ddot{\alpha}_{,i}(s)\ddot{\alpha}_{,j}(s)\right\} dvds\right\}. \tag{67}$$

Based on the Cauchy–Schwarz inequality, from relations (61), (62), and (67), we deduce that:

$$F(t)\ddot{F}(t) - \dot{F}(t)^2 \geq 0, \quad \text{for all} \quad t \geq 0, \tag{68}$$

that proves that $\ln[F(t)]$ is a convex function on $(0, \infty)$. Thus, we conclude that:

$$0 \leq F(t) \leq 0, \tag{69}$$

and the proof follows like the above Proof 1.

Remark 1. A brief look at the two demonstration methods used above shows that, unlike the logarithmic convexity method, the Lagrange identity method does not use the conservation law expressed by the relations (33) and (34), and this gives it a slight advantage in dealing with weak solutions.

Remark 2. It should be noted that hypothesis (H1) can be replaced with one of the following types: (a) $c(x) \geq 0$, but $k_{ij}(x)$ is supposed to be a positive definite tensor and $meas(\Sigma_1) \neq 0$ or (b) $c(x) \geq 0$, but $\varkappa_{ij}(x) = k_{ij}(x) + \left(\tau_\alpha - \tau_q\right) k^*_{ij}(x)$ to be assumed a positive definite tensor and $meas(\Sigma_1) \neq 0$! In such a situation, the relation (59) implies:

$$\alpha_{,i}(x,t) = 0 \quad \text{in} \quad B \times (0, \infty), \tag{70}$$

which leads again to the relation (44) and, moreover, it proves that $\mathcal{S} = \{\alpha, q_i\} = 0$.

6. Instability of solutions

The aim of this section is to prove that when the constitutive equation (6) does not fulfill the second law of thermodynamics, then we can find a solution of the basic equations (6) and (7) which is unstable. To simplify the mathematical calculations, we will consider the case of an isotropic and homogeneous material conductor where $k_{ij} = k\delta_{ij}$ and $k^*_{ij} = k^*\delta_{ij}$ so that the thermodynamic restrictions are written now in the following form:

$$k \geq 0 \quad \text{and} \quad k + (\tau_\alpha - \tau_q)k^* \geq 0. \tag{71}$$

According to the hypotheses (H1) to (H4), we assume now that:

$$c > 0, \quad k^* > 0, \quad k > 0, \quad k + \left(\tau_\alpha - \tau_q\right) k^* > 0. \tag{72}$$

Moreover, the Moore–Gibson–Thomson type equation (8) takes the following form:

$$c\tau_q \dddot{\alpha} + c\,\ddot{\alpha} - k^*\,\Delta\,\alpha - (k + \tau_\alpha\,k^*)\,\Delta\,\dot{\alpha} - \tau_T k\,\Delta\,\ddot{\alpha} = 0, \tag{73}$$

where Δ is the Laplace operator.

Our analysis in this section is related to the possible wave propagation problem for the thermal model in concern. The solutions in this case will be waves of the form:

$$\alpha(x_1, t) = \text{Re}\left\{A\, e^{i\varkappa(x_1 - vt)}\right\}, \tag{74}$$

where $i = \sqrt{-1}$ is the imaginary unit, $\text{Re}\{\cdot\}$ is the real part, $\varkappa > 0$ is the real wave number, and A is a complex nonzero number. Furthermore, x_1 is the spatial coordinate in the propagation direction and v is a complex parameter so that $\text{Re}(v) \geq 0$ will represent the wave speed and $\text{Im}(v) \leq 0$ will be related to the rate of decaying in time. We must note that for $\text{Re}(v) > 0$, there is a genuine wave, while for $\text{Re}(v) = 0$, there is a standing mode. Moreover, when $\text{Im}(v) < 0$, there is the phenomenon of damping in time, while for $\text{Im}(v) = 0$, there is an undamped in time wave.

A substitution of the relation (74) into the equation (73) (or in the basic equations (6) and (7)) leads to the following algebraic equation for determining the parameter v:

$$i\varkappa c\tau_q v^3 - \left(c + \tau_T k\varkappa^2\right) v^2 - i\varkappa v\,(k + \tau_\alpha k^*) + k^* = 0. \tag{75}$$

Moreover, if we set:

$$v = i\chi, \tag{76}$$

the above equation takes the form:

$$f(\chi) \equiv \varkappa c\tau_q \chi^3 + \left(c + \tau_T k\varkappa^2\right)\chi^2 + \varkappa\left(k + \tau_\alpha k^*\right)\chi + k^* = 0. \tag{77}$$

Obviously, this equation admits at least one real and negative root, be it $\chi = \chi_1 = -\zeta$, $\zeta > 0$. Furthermore, we write equation (77) in the form:

$$f(\chi) = \chi^2\left(\varkappa c\tau_q\,\chi + c + \tau_T k\varkappa^2\right) + \left[\varkappa\left(k + \tau_\alpha k^*\right)\chi + k^*\right] = 0, \tag{78}$$

and introduce the values of χ:

$$\tilde{\chi}_1 = -\frac{c + \tau_T k\varkappa^2}{\varkappa c\tau_q}, \quad \tilde{\chi}_2 = -\frac{k^*}{\varkappa\left(k + \tau_\alpha k^*\right)}. \tag{79}$$

Then, we see that:

$$\begin{aligned} f(\tilde{\chi}_1) &= -\frac{1}{\tau_q}\left[k + \left(\tau_\alpha - \tau_q\right)k^* + \frac{\varkappa^2\tau_T k}{c}\left(k + \tau_\alpha k^*\right)\right], \\ f(\tilde{\chi}_2) &= \frac{c\tilde{\chi}_2^2}{k + \tau_\alpha k^*}\left[k + \left(\tau_\alpha - \tau_q\right)k^* + \frac{\varkappa^2\tau_T k}{c}\left(k + \tau_\alpha k^*\right)\right], \end{aligned} \tag{80}$$

and hence, it results that:

$$f(\tilde{\chi}_1)f(\tilde{\chi}_2) = -\frac{c\tilde{\chi}_2^2}{\tau_q\left(k + \tau_\alpha k^*\right)}\left[k + \left(\tau_\alpha - \tau_q\right)k^* + \frac{\varkappa^2\tau_T k}{c}\left(k + \tau_\alpha k^*\right)\right]^2 < 0. \tag{81}$$

We can now conclude that the root $\chi = \chi_1 = -\zeta < 0$ is situated between $\tilde{\chi}_1$ and $\tilde{\chi}_2$. Moreover, we have to note that equation (77) does not admit other real roots outside the interval delimited by the values $\tilde{\chi}_1$ and $\tilde{\chi}_2$. This allows us to conclude that if equation (77) admits all three real roots, then they are all negative and are included in the interval delimited by $\tilde{\chi}_1$ and $\tilde{\chi}_2$. Therefore, in these cases, we are led to thermal standing mode waves that decrease exponentially in time as:

$$\alpha_{thsm}(x_1, t) = \text{Re}\left\{A\, e^{i\varkappa x_1}\right\} e^{\varkappa\xi t}, \tag{82}$$

where ξ is any one of the three negative real roots of the equation (77).

Besides, there is still the situation in which equation (77) also admits a complex conjugate root:

$$\chi_{2,3} = -\gamma \pm i\delta, \quad \gamma \in R, \quad \delta > 0. \tag{83}$$

Let us study these conjugate roots and, in this sense, we recall the first Vièta's formulas for equation (77):

$$\chi_1 + \chi_2 + \chi_3 = -\frac{c + \tau_T k\varkappa^2}{\varkappa c\tau_q}, \tag{84}$$

which furnishes:

$$\gamma = \frac{1}{2}\left(\frac{c + \tau_T k\varkappa^2}{\varkappa c\tau_q} - \zeta\right) = -\frac{1}{2}\left(\tilde{\chi}_1 + \zeta\right). \tag{85}$$

Moreover, there is:

$$\delta^2 = \frac{k^*}{\varkappa c\tau_q\zeta} - \gamma^2. \tag{86}$$

Now, we have two situations to discuss:

(1) the first is when:

$$\tilde{\chi}_1 < \tilde{\chi}_2; \tag{87}$$

(2) the second is when:

$$\tilde{\chi}_1 > \tilde{\chi}_2. \tag{88}$$

The first situation is equivalent to:

$$c\left[k + \left(\tau_\alpha - \tau_q\right) k^*\right] + \tau_T k \varkappa^2 \left(k + \tau_\alpha k^*\right) > 0, \tag{89}$$

which is fulfilled as a consequence of the compatibility of the constitutive equation with the second law (see the consequences (R(i)) and (R(ii))). In this situation, there is:

$$\zeta < |\tilde{\chi}_1| = \frac{c + \tau_T k \varkappa^2}{\varkappa c \tau_q}, \tag{90}$$

and hence, it results from equation (85) that:

$$\gamma > 0. \tag{91}$$

In view of relations (74) and (83), we have the following damped in time thermal wave:

$$\alpha_{thw}(x_1, t) = \mathrm{Re}\left\{A\, e^{i\varkappa(x_1 - \delta t)}\right\}\, e^{-\varkappa\gamma t}. \tag{92}$$

Let us now consider the item (2), i.e., we assume that $\tilde{\chi}_1 > \tilde{\chi}_2$. Then, it follows that:

$$\zeta > |\tilde{\chi}_1| = \frac{c + \tau_T k \varkappa^2}{\varkappa c \tau_q}, \tag{93}$$

and hence, it results from equation (85) that:

$$\gamma < 0. \tag{94}$$

Then, we have the following wave solution:

$$\alpha_{inst}(x_1, t) = \mathrm{Re}\left\{A\, e^{i\varkappa(x_1 - \delta t)}\right\}\, e^{-\varkappa\gamma t}, \tag{95}$$

which becomes infinity when $t \to \infty$, and therefore, the instability of solutions follows.

Remark 3. It can be seen that the discriminant of the algebraic equation (77), which depends on the wave number $\varkappa$, can take positive as well as negative values. This means that the model in concern can have three thermal standing mode waves or one thermal standing mode wave and one damped in time thermal wave, as discussed above.

7. Discussion

Our study in this paper highlights some important aspects in the analysis of mechanical models, which can be summarized as follows:

(a) the restrictions on the thermal coefficients used in the mathematical studies of the Moore–Gibson–Thompson equation acquire here a precise mechanical meaning using the compatibility of the constitutive equation with the second law of thermodynamics;
(b) in this paper, the (unknown) solution is represented by the couple {thermal displacement – heat flux vector} and all the obtained results refer to this couple;
(c) removing one of the variables from the mechanical context of the model (here the heat flux vector) may contribute to the loss of some important characteristics of the problem. The previous studies on the Moore–Gibson–Thompson equation are limited only to the initial and boundary conditions for the remaining variable (here the thermal displacement), leaving aside the initial and boundary conditions on the heat flux vector. Thus, these studies give insufficient information (here only for the thermal displacement), but nothing about the eliminated variable (here the heat flux vector);

(d) in the studies on the Moore–Gibson–Thompson equation, the initial conditions are expressed in terms of the initial values for $\alpha(x, 0)$, $\dot{\alpha}(x, 0)$, and $\ddot{\alpha}(x, 0)$. Giving the initial condition $\ddot{\alpha}(x, 0)$ is not equivalent to giving the initial condition for $q_i(x, 0)$. In fact, if $q_i(x, 0)$ is given, then, from equation (7), $\ddot{\alpha}(x, 0)$ can be determined, but vice versa, knowing $\ddot{\alpha}(x, 0)$ we can only determine $q_{i,i}(x, 0)$ and not $q_i(x, 0)$. However, if we want to see what happens with the heat flux vector, expressed by the constitutive equation (6), we reach the situation when we cannot unique determine it! This can be seen more specific from Theorem 1, where it is clear that we cannot integrate the differential equation (45) to obtain relation (46), i.e., $q_i(x, t) = 0$ for all $(x, t) \in B \times (0, \infty)$, because $q_i(x, 0)$ does not turn out to be zero. In fact, with $\alpha(x, 0) = 0$, $\dot{\alpha}(x, 0) = 0$, and $\ddot{\alpha}(x, 0) = 0$ from the basic equations (6) and (7), we only get $q_{i,i}(x, 0) = 0$ and not $q_i(x, 0) = 0$;

(e) our study considers both boundary data for the thermal displacement, as well as boundary data in terms of the heat flux (these last which have not been considered in the papers on the Moore–Gibson–Thompson equation);

(f) as shown in section 6, the model under discussion admits standing mode waves decreasing exponentially in time, but it can also admit thermal waves damped in time, unlike the classic thermal model (based on the Fourier law) where they are missing;

(g) if the thermodynamic restrictions are not fulfilled, then it is possible to have wave solutions that grow infinitely in time, and thus we are in a situation of instability.

The authors think that the results presented in this paper are not contrary to those existing in the literature, but rather they complement them and give them consistency from a mechanical point of view.

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ORCID iD

Stan Chiriţă https://orcid.org/0000-0001-7560-0367

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