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Rayleigh Waves in a Thermoelastic Half-Space Coated by a Maxwell–Cattaneo Thermoelastic Layer

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Abstract: This paper investigates the propagation of in-plane surface waves in a coated thermoelastic half-space. First, it investigates a special case where the surface layer is described by the Maxwell–Cattaneo thermoelastic approach, while the half-space is filled by a thermoelastic material described by the classical Fourier law for the heat flux. The contact between the layer and the half-space is assumed to be welded, i.e., the displacements and the temperature, as well as the stresses and the heat flux are continuous through the interface of the layer and the half-space. The boundary and continuity conditions of the problem are formulated and then the exact dispersion relation of the surface waves is established. An illustrative numerical simulation is presented for the case of an aluminum thermoelastic layer coating a thermoelastic copper half-space, highlighting important aspects regarding the propagation of Rayleigh waves in such structures. The exact effective boundary conditions at the interface are also established replacing the entire effect of the layer on the half-space. The general case of the problem is also investigated when both the surface layer and the half-space are described by the Maxwell–Cattaneo thermoelasticity theory. This study helps to further understand the propagation characteristics of elastic waves in layered structures with thermal effects described by the Maxwell–Cattaneo approach.

Keywords: Rayleigh waves; exact secular equation; isotropic thermoelastic half-space; Maxwell–Cattaneo thermoelastic layer; welded contact; exact effective boundary conditions

MSC: 74J15; 74K35; 80A17; 74M15; 74L05



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1. Introduction

The technique of covering a material substrate with a layer made of a material with superior characteristic properties is widely used in order to improve the protection against chemical corrosion, degradation of any kind, and, in particular, against thermal effects. The measurement of thermomechanical properties of thin elastic layers deposited over the substrate is very important in modern engineering and technology. There was a great interest in studying the influence of the characteristic properties of the layer on the material substrate. An efficient tool in this study is represented by the approach of Rayleigh-type surface waves which is the most versatile mechanism due to its wide range of advantages non-destructive nature in testing materials, laboratory simulations, measurements of the mechanical properties of the covering layer, reduced material costs, as well as less time required for inspections.

Achenbach and Keshava [1] used the plate theory in order to study a layer and a supporting half-space of different elastic properties and so they furnished the dispersion equation for free waves in the form of a sixth-order determinant equation in terms of the non-dimensional-phase velocity. Tiersten [2] investigated the influence of thin film on guided wave propagation in the joined structure of a film over a half-space with comparison

to experiment data. He introduced an approximate procedure in which the entire effect of the plating is treated as a non-zero homogeneous boundary condition at the surface of the substrate.

By considering the overlying material to be very thin, the entire effect of the thin elastic layer on the elastic half-space is approximately replaced by means of so-called effective boundary conditions. This way provides a relationship between the displacement components and the stress components of the half-space at the interface between the elastic layer and the elastic half-space. There are several approaches for obtaining these effective boundary conditions in order to extract the material properties of the thin layers from experimental data. Bøvik [3] made a comparison between the boundary conditions derived by Tiersten [2] and the so-called boundary conditions for elastic surface waves guided by thin films. He also expanded the stress components at the upper surface of a thin layer using the Taylor series in terms of the thickness of the layer. This approach of Taylor series expansion was further employed by several researchers in order to obtain the required boundary conditions. Thus, Vinh and Anh [4–6] and Vinh et al. [7,8] explored the approximate secular equation and the velocity formula for guided wave propagation in an orthotropic elastic half-space, which is coated by a thin orthotropic elastic layer with welded or smooth contact. Other interesting studies were carried out by Haskell [9], Achenbach and Epstein [10], Makarov et al. [11], Kuchler and Richter [12], Cai and Fu [13], Sotiropoulos [14], Every [15], and Chattopadhyay and Raju [16].

Mathematical modeling of the elastic surface waves guided by thin films in layered structures was described in the books by Nayfeh [17] and Rose [18]. A comprehensive review of established and promising technologies under development in the emerging area of structural health monitoring of aerospace composite structures, as well as a study of guided waves in thin-wall composite structures, is developed in the excellent textbooks written by Nowinski [19] and Giurgiutiu [20].

Steigmann and Ogden developed a systematic approximation of the linear equations for small-amplitude surface waves in an elastic half-space, interacting with a residually stressed thin film of a different material bonded to its plane boundary [21]. Recently, interesting results were obtained both in terms of establishing effective boundary conditions and in terms of obtaining the exact secular equation for the propagation of Rayleigh waves in half-spaces on which an elastic layer is laid in the various forms of contact. In this connection, we mention the research developed by Dai et al. [22], Vinh et al. [23,24], Kaplunov and Prikazchikov [25], Erbas et al. [26], Kaplunov et al. [27,28], Phan et al. [29], Mubarak et al. [30], and Kaur et al. [31]. These results were extended in [32,33] by combining the weakly nonlocal elasticity theory with the impedance boundary conditions that are formulated using nonlocal stresses. Moreover, the authors of [34] focused on an amplitude versus offset (AVO) method for analyzing the reflection and transmission of elastic waves at an interface between two fluid-saturated thermo-poroelastic media.

The thermal effects are totally neglected in all research from the specialized literature on the argument of Rayleigh wave in a coated half-space, although they are present in most practical situations of interest. Various researchers have explored the characteristics of Rayleigh/Rayleigh-like surface waves in different thermoelastic half-space models, such as those by Lockett [35], Chadwick [36], Deresiewicz [37], Chiriță [38,39], and the chapter titled Rayleigh's Waves in *The Encyclopedia of Thermal Stresses* [40]. Recently, Zampoli and Chiriță [41] demonstrated that by considering thermal effects, the propagation speed of Rayleigh waves tends to decrease compared to the purely elastic case. Additionally, they highlighted the time-dependent damping of the wave amplitude.

It is well known that the classical Fourier constitutive law for heat flux q_i , namely $q_i = k\theta_{,i}$ where θ denotes the absolute temperature and k denotes heat conductivity, leads to the parabolic heat conduction equation. This implies that heat pulses propagate with infinite speed. To overcome this limitation of classical theory, Cattaneo [42] proposed replacing the Fourier constitutive relation with an evolution equation of the form $\tau\dot{q}_i + q_i = k\theta_{,i}$ where the superposed dot indicates the material time derivative, and τ is an intrinsic relaxation

time. This revised constitutive relation, known as the Maxwell–Cattaneo equation, leads to hyperbolic-type heat conduction and predicts thermal waves, also referred to as second sound. For an overview of thermoelasticity with second sound, we recommend the papers by Chandrasekharaiah [43,44].

In this paper, we study the problem of Rayleigh surface wave propagation in a thermo-mechanical system consisting of a Fourier thermoelastic half-space covered by a different thermoelastic layer modeled by the Maxwell–Cattaneo theory. The contact between the layer and the half-space is assumed to be welded, i.e., the displacements and temperature, as well as the stresses and heat flux, are continuous across the interface of the layer and the half-space. The explicit expressions of free in-plane Rayleigh waves in both the layer and the half-space are first established. Then, by using the boundary conditions on the free surface and the continuity conditions at the interface, the exact secular equation for Rayleigh waves is derived. A numerical simulation of this secular equation is presented for the case of a thermoelastic layer made of aluminum covering a thermoelastic copper half-space, highlighting important aspects of Rayleigh wave propagation in such structures.

Furthermore, the entire effect of the thermoelastic layer on the half-space is replaced by the so-called effective boundary conditions, which relate the displacements and the temperature fields, as well as the stresses and the heat flux of the half-space at its surface. Thus, the wave is then considered a Rayleigh wave propagating in the half-space, without a coating layer, which is subjected to the effective boundary conditions.

We also consider the case where both the (thermoelastic) half-space and coating (thermoelastic) layers are filled by two different Maxwell–Cattaneo thermoelastic materials. When the thickness of the thermoelastic layer tends to zero ($h \rightarrow 0$), i.e., there is no one thermoelastic layer, then our exact secular equation furnishes the secular equation for Rayleigh waves in a Maxwell–Cattaneo thermoelastic half-space.

The objective of this work is as follows. Section 2 presents the basic systems of differential equations describing the evolutionary thermoelastic behavior of the materials in the Maxwell–Cattaneo thermoelastic layer and those of the half-space filled by a Fourier thermoelastic solid, respectively. Section 3 establishes the classes of in-plane wave solutions that allow us to solve the Rayleigh surface wave problem. Section 4 treats the propagating Rayleigh wave problem and establishes the exact secular equation. Section 5 illustrates the Rayleigh wave problem for the structure consisting of an aluminum layer coating a copper half-space with welded contact. Section 6 establishes the exact effective boundary conditions. Section 7 treats the problem of Rayleigh surface wave propagation for the thermomechanical system based on the Maxwell–Cattaneo theory.

2. Formulation of the Rayleigh Wave Propagation Problem

We consider the half-space $0 < x_2 < \infty$ filled by a thermoelastic material modeled according to the classical Fourier's law for heat conduction. The half-space is covered by a layer of uniform thickness, h , occupying the domain $-h < x_2 < 0$, which is filled with a thermoelastic material modeled with the Maxwell–Cattaneo law for heat flux; see Figure 1.

Note that the same quantities related to the half-space and the layer have the same symbol but are systematically distinguished by a tilde if pertaining to the half-space $0 < x_2 < \infty$. In accordance with what was said above and with the classical linear theory of isotropic thermoelasticity, in the half-space $x_2 > 0$, we have the basic motion equations:

$$\begin{aligned}\tilde{S}_{rs,r} &= \tilde{q}\tilde{\ddot{u}}_s, \\ \tilde{q}_{r,r} + T_0\tilde{m}\dot{\tilde{u}}_{r,r} &= T_0\tilde{a}\dot{\tilde{\theta}},\end{aligned}\quad (1)$$

with the following constitutive equations:

$$\begin{aligned}\tilde{S}_{rs} &= \tilde{\lambda}\tilde{u}_{p,p}\delta_{rs} + \tilde{\mu}(\tilde{u}_{r,s} + \tilde{u}_{s,r}) + \tilde{m}\tilde{\theta}\delta_{rs}, \\ \tilde{q}_r &= \tilde{k}\tilde{\theta}_{,r}.\end{aligned}\quad (2)$$

In the above relations, we use the following notations: $\tilde{\lambda}$ and $\tilde{\mu}$ are the elastic Lamé constants, $\tilde{k} > 0$ is the constant thermal conductivity, $\tilde{a} > 0$ is the constant mass specific heat, $\tilde{m} = -\tilde{a}(3\tilde{\lambda} + 2\tilde{\mu})$ is the stress-temperature constant modulus, $\tilde{\alpha}$ is the coefficient of thermal expansion, $\tilde{\rho}$ is the density mass, $\tilde{u}_r$ are the components of the displacement vector, $\tilde{\theta}$ is the temperature variation from the absolute temperature $T_0 > 0$, $\tilde{S}_{rs}$ are the components of the stress tensor, and $\tilde{q}_r$ are the components of the heat flux vector.

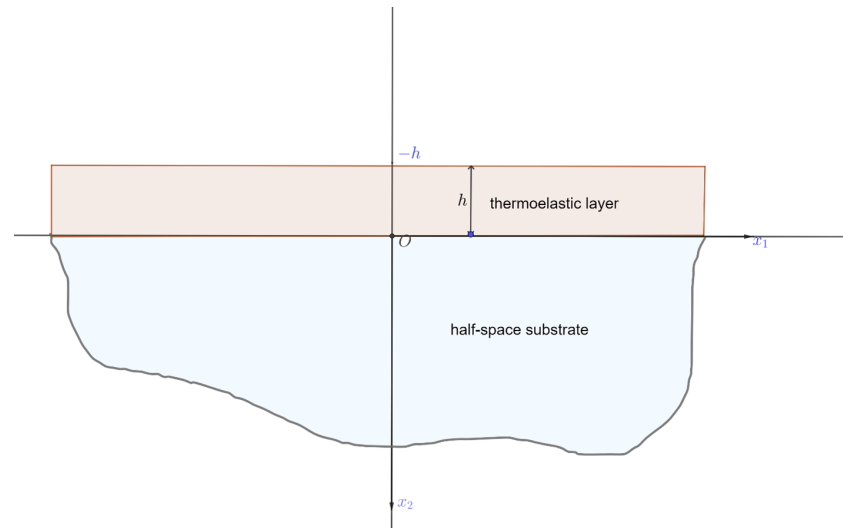


Figure 1. Layered structure.

According to the Maxwell–Cattaneo model for the heat flux vector, in the thermoelastic layer, $-h < x_2 < 0$, we have the following:

$$\begin{aligned} S_{rs,r} &= \rho \ddot{u}_s, \\ q_{r,r} + T_0 m \dot{u}_{r,r} &= T_0 a \dot{\theta}, \end{aligned} \quad (3)$$

and

$$\begin{aligned} S_{rs} &= \lambda u_{p,p} \delta_{rs} + \mu (u_{r,s} + u_{s,r}) + m \theta \delta_{rs}, \\ \tau \dot{q}_r + q_r &= k \theta_{,r}, \end{aligned} \quad (4)$$

where $\tau > 0$ is the relaxation time.

The contact between the thermoelastic layer and the semi-space is supposed to be welded and there are no temperature changes or heat transfer at the contact interface. This means that the displacement and temperature fields, as well as the stresses and the heat flux, are continuous through the interface of the layer and the half-space. Therefore, we must have the following conditions fulfilled:

$$\begin{aligned} \tilde{u}_s(x_1, 0, x_3, t) &= u_s(x_1, 0, x_3, t), \quad \tilde{S}_{2s}(x_1, 0, x_3, t) = S_{2s}(x_1, 0, x_3, t), \\ \tilde{\theta}(x_1, 0, x_3, t) &= \theta(x_1, 0, x_3, t), \quad \tilde{q}_2(x_1, 0, x_3, t) = q_2(x_1, 0, x_3, t), \\ \text{for all } x_1, x_3 &\in \mathbb{R}, t > 0. \end{aligned} \quad (5)$$

Furthermore, we assume that the surface $x_2 = -h$ of the thermoelastic layer is free of traction and that there are no thermal exchanges with the outside; we have the following:

$$\begin{aligned} S_{12}(x_1, -h, x_3, t) &= 0, \quad S_{22}(x_1, -h, x_3, t) = 0, \quad S_{32}(x_1, -h, x_3, t) = 0, \\ q_2(x_1, -h, x_3, t) &= 0, \quad \text{for all } x_1, x_3 \in \mathbb{R}, t > 0. \end{aligned} \quad (6)$$

Alternatively, we consider the free surface of the layer to be thermally insulated when the condition (6)₂ is replaced by the following:

$$\theta(x_1, -h, x_3, t) = 0, \quad \text{for all } x_1, x_3 \in \mathbb{R}, t > 0. \quad (7)$$

Finally, as we intend to talk about surface waves, we will have to impose zero asymptotic conditions on the mechanical and thermal quantities in the depth of the half-space; we assume the following:

$$\begin{aligned} \lim_{x_2 \rightarrow \infty} \tilde{u}_r(x_1, x_2, x_3, t) &= 0, & \lim_{x_2 \rightarrow \infty} \tilde{S}_{rs}(x_1, x_2, x_3, t) &= 0, \\ \lim_{x_2 \rightarrow \infty} \tilde{\theta}(x_1, x_2, x_3, t) &= 0, & \lim_{x_2 \rightarrow \infty} \tilde{q}_r(x_1, x_2, x_3, t) &= 0, \end{aligned} \quad \text{for all } x_1, x_3 \in \mathbb{R}, t > 0. \quad (8)$$

Once the mathematical problem of the thermoelastic layer superimposed on a thermoelastic half-space is formulated, we are interested in the Rayleigh wave propagation in such a material environment.

The following dimensionless coefficients are important in the study of thermoelastic phenomena:

$$\tilde{\varepsilon} = \frac{\tilde{m}^2}{\tilde{a}\tilde{\varrho}\tilde{c}_1^2}, \quad \varepsilon = \frac{m^2}{a\varrho c_1^2}, \quad (9)$$

where

$$\tilde{c}_1^2 = \frac{\tilde{\lambda} + 2\tilde{\mu}}{\tilde{\varrho}}, \quad c_1^2 = \frac{\lambda + 2\mu}{\varrho}. \quad (10)$$

Moreover, we also need the following notations:

$$\tilde{c}_2^2 = \frac{\tilde{\mu}}{\tilde{\varrho}}, \quad c_2^2 = \frac{\mu}{\varrho}. \quad (11)$$

3. Some Classes of In-Plane Wave Solutions of the Basic Equations

Throughout this paper, we study the propagation of a Rayleigh wave, traveling along the interface between the layer and the half-space with velocity c and wave-number $\varkappa$ in the x_1 -direction and decaying in the x_2 -direction. To solve the Rayleigh surface wave problem for the considered mechanical system, we look for x_1Ox_2 -plane wave solutions $\{u_r, \theta; S_{rs}, q_r\}$ of Equations (3) and (4) in the following form:

$$\begin{aligned} u_1(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ \frac{1}{\varkappa} \mathcal{U}_1(x_2) e^{i\varkappa(x_1 - ct)} \right\}, \\ u_2(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ \frac{i}{\varkappa} \mathcal{U}_2(x_2) e^{i\varkappa(x_1 - ct)} \right\}, \\ u_3(x_1, x_2, x_3, t) &= 0, \\ \theta(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ T_0 \mathcal{T}(x_2) e^{i\varkappa(x_1 - ct)} \right\}, \end{aligned} \quad (12)$$

and

$$\begin{aligned} S_{11}(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ \mu \mathcal{S}_{11}(x_2) e^{i\varkappa(x_1 - ct)} \right\}, \\ S_{22}(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ i\mu \mathcal{S}_{22}(x_2) e^{i\varkappa(x_1 - ct)} \right\}, \\ S_{12}(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ \mu \mathcal{S}_{12}(x_2) e^{i\varkappa(x_1 - ct)} \right\}, \\ S_{33}(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ \mu \mathcal{S}_{33}(x_2) e^{i\varkappa(x_1 - ct)} \right\}, \\ q_1(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ \varkappa k T_0 \mathcal{Q}_1(x_2) e^{i\varkappa(x_1 - ct)} \right\}, \\ q_2(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ \varkappa k T_0 \mathcal{Q}_2(x_2) e^{i\varkappa(x_1 - ct)} \right\}, \\ S_{13}(x_1, x_2, x_3, t) &= S_{23}(x_1, x_2, x_3, t) = 0, \quad q_3(x_1, x_2, x_3, t) = 0, \end{aligned} \quad (13)$$

in the layer $-h < x_2 < 0$.

Consequently, in view of the relations (12) and (13), the constitutive Equation (4) implies the following:

$$\begin{aligned}
S_{22}(x_2) &= \frac{c_1^2}{c_2^2} \left(\mathcal{U}_1 + \frac{1}{\varkappa} \mathcal{U}_{2,2} \right) - 2\mathcal{U}_1 - \frac{imT_0}{\varrho c_2^2} \mathcal{T}, \\
S_{12}(x_2) &= \frac{1}{\varkappa} \mathcal{U}_{1,2} - \mathcal{U}_2, \\
Q_2(x_2) &= \frac{1}{\varkappa(1 - i\tau\kappa c)} \mathcal{T}_2,
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
-iS_{11}(x_2) &= \frac{c_1^2}{c_2^2} \left(\mathcal{U}_1 + \frac{1}{\varkappa} \mathcal{U}_{2,2} \right) - \frac{2}{\varkappa} \mathcal{U}_{2,2} - \frac{imT_0}{\varrho c_2^2} \mathcal{T}, \\
-iS_{33}(x_2) &= \left(\frac{c_1^2}{c_2^2} - 2 \right) \left(\mathcal{U}_1 + \frac{1}{\varkappa} \mathcal{U}_{2,2} \right) - \frac{imT_0}{\varrho c_2^2} \mathcal{T}, \\
Q_1(x_2) &= \frac{i}{1 - i\tau\kappa c} \mathcal{T}.
\end{aligned} \tag{15}$$

The basic Equation (3) is as follows:

$$\begin{aligned}
S_{12,2} - \frac{c_1^2}{c_2^2} (\varkappa \mathcal{U}_1 + \mathcal{U}_{2,2}) + 2\mathcal{U}_{2,2} + \frac{imT_0\varkappa}{\varrho c_2^2} \mathcal{T} &= -\frac{\varkappa c^2}{c_2^2} \mathcal{U}_1, \\
\varkappa S_{12} + S_{22,2} &= -\frac{\varkappa c^2}{c_2^2} \mathcal{U}_2, \\
Q_{2,2} - \frac{\varkappa}{1 - i\tau\kappa c} \mathcal{T} + \frac{mc}{k} \left(\mathcal{U}_1 + \frac{1}{\varkappa} \mathcal{U}_{2,2} \right) &= -\frac{iT_0ac}{k} \mathcal{T}.
\end{aligned} \tag{16}$$

Further, we seek solutions of the differential systems (14) and (16) in the following form:

$$\{\mathcal{U}_1, \mathcal{U}_2, \mathcal{T}; S_{12}, S_{22}, Q_2\}(x_2) = \{U_1, U_2, T; s_{12}, s_{22}, Q_2\} e^{i\kappa r x_2}, \tag{17}$$

with $\{U_1, U_2, T; s_{12}, s_{22}, Q_2\}$ being a non-zero constant dimensionless complex vector and r dimensionless constant parameter. Therefore, we have the following:

$$\begin{aligned}
s_{12} &= irU_1 - U_2, \\
s_{22} &= \frac{c_1^2}{c_2^2} (U_1 + irU_2) - 2U_1 - \frac{imT_0}{\varrho c_2^2} T, \\
Q_2 &= \frac{ir}{1 - i\tau\kappa c} T,
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
ir s_{12} - \frac{c_1^2}{c_2^2} (U_1 + irU_2) + 2ir U_2 + \frac{imT_0}{\varrho c_2^2} T + \frac{c^2}{c_2^2} U_1 &= 0, \\
s_{12} + ir s_{22} + \frac{c^2}{c_2^2} U_2 &= 0, \\
irQ_2 - \frac{1}{1 - i\tau\kappa c} T + \frac{mc}{\varkappa k} (U_1 + irU_2) + \frac{iT_0ac}{\varkappa k} T &= 0.
\end{aligned} \tag{19}$$

By replacing the relation (18) in the equations described by (19), we obtain the following:

$$\begin{aligned}
ir(irU_1 + U_2) - \frac{c_1^2}{c_2^2} (U_1 + irU_2) + \frac{imT_0}{\varrho c_2^2} T + \frac{c^2}{c_2^2} U_1 &= 0, \\
-(irU_1 + U_2) + ir \left[\frac{c_1^2}{c_2^2} (U_1 + irU_2) - \frac{imT_0}{\varrho c_2^2} T \right] + \frac{c^2}{c_2^2} U_2 &= 0, \\
\frac{mc}{\varkappa k} (U_1 + irU_2) - \left(\frac{r^2 + 1}{1 - i\tau\kappa c} - \frac{iT_0ac}{\varkappa k} \right) T &= 0,
\end{aligned} \tag{20}$$

and, moreover,

$$\begin{aligned} \left(s^2 - \frac{c^2}{c_2^2}\right)(irU_1 + U_2) &= 0, \\ \left(s^2 - \frac{c^2}{c_1^2}\right)(U_1 + irU_2) - \frac{imT_0s^2}{\varrho c_1^2} T &= 0, \\ -\frac{mc}{k\kappa}(1 - i\tau\kappa c)(U_1 + irU_2) + \left[s^2 - \frac{ic}{c_1K}(1 - i\tau\kappa c)\right]T &= 0, \end{aligned} \quad (21)$$

where

$$\begin{aligned} s^2 &= r^2 + 1, \\ K &= \frac{\kappa k}{T_0 a c_1}. \end{aligned} \quad (22)$$

Since we have $\{U_1, U_2, T; s_{12}, s_{22}, Q_2\} \neq 0$, it follows that the determinant of the algebraic system (21) has to be zero; we have the following:

$$\left(s^2 - \frac{c^2}{c_2^2} C^2\right) \left\{ s^4 - \left[C^2 + \frac{iC}{K}(1 - i\chi C)(\varepsilon + 1) \right] s^2 + \frac{iC^3}{K}(1 - i\chi C) \right\} = 0, \quad (23)$$

with the following roots:

$$\begin{aligned} s_1^2 &= \frac{c_1^2}{c_2^2} C^2, \\ s_2^2 &= \frac{1}{2} \left\{ C^2 + \frac{iC}{K}(1 - i\chi C)(\varepsilon + 1) + \sqrt{\Delta} \right\}, \\ s_3^2 &= \frac{1}{2} \left\{ C^2 + \frac{iC}{K}(1 - i\chi C)(\varepsilon + 1) - \sqrt{\Delta} \right\}, \end{aligned} \quad (24)$$

where

$$\Delta = \left[C^2 + \frac{iC}{K}(1 - i\chi C)(\varepsilon + 1) \right]^2 - \frac{4\varepsilon C^2}{K^2} (1 - i\chi C)^2, \quad (25)$$

and the dimensionless parameters C and χ are defined by the following:

$$C = \frac{c}{c_1}, \quad \chi = \tau\kappa c_1. \quad (26)$$

Consequently, there exist the following admissible values for the dimensionless parameter r :

$$\begin{aligned} r_1 &= i\tilde{\zeta}_1, & r_2 &= i\tilde{\zeta}_2, & r_3 &= i\tilde{\zeta}_3, \\ r_4 &= -i\tilde{\zeta}_1, & r_5 &= -i\tilde{\zeta}_2, & r_6 &= -i\tilde{\zeta}_3, \end{aligned} \quad (27)$$

with

$$\tilde{\zeta}_1 = \sqrt{1 - s_1^2}, \quad \tilde{\zeta}_2 = \sqrt{1 - s_2^2}, \quad \tilde{\zeta}_3 = \sqrt{1 - s_3^2}, \quad (28)$$

and the corresponding vector solutions of the differential system defined by (14) and (16) as follows:

$$\left\{ \mathcal{U}_1^{(n)}, \mathcal{U}_2^{(n)}, \mathcal{T}^{(n)}; \mathcal{S}_{12}^{(n)}, \mathcal{S}_{22}^{(n)}, \mathcal{Q}_2^{(n)} \right\} (x_2) = \left\{ U_1^{(n)}, U_2^{(n)}, T^{(n)}; s_{12}^{(n)}, s_{22}^{(n)}, Q_2^{(n)} \right\} e^{i\kappa r_n x_2}, \quad (29)$$

for $n = 1, 2, \dots, 6$. Moreover, we have the following:

$$U_1^{(1)} = \tilde{\zeta}_1, \quad U_2^{(1)} = 1, \quad T^{(1)} = 0, \quad s_{12}^{(1)} = C^2 - 2, \quad s_{22}^{(1)} = -2\tilde{\zeta}_1, \quad Q_2^{(1)} = 0, \quad (30)$$

$$\begin{aligned} U_1^{(2)} &= s_2^2 - \frac{iC}{K} (1 - i\chi C), \quad U_2^{(2)} = \xi_2 \left[s_2^2 - \frac{iC}{K} (1 - i\chi C) \right], \\ T^{(2)} &= Y C s_2^2 (1 - i\chi C), \quad s_{12}^{(2)} = -2\xi_2 \left[s_2^2 - \frac{iC}{K} (1 - i\chi C) \right], \\ s_{22}^{(2)} &= \left(\frac{c_1^2}{c_2^2} s_2^2 - 2 \right) \left[s_2^2 - \frac{iC}{K} (1 - i\chi C) \right] - \frac{i\epsilon c_1^2}{c_2^2 K} C s_2^2 (1 - i\chi C), \quad Q_2^{(2)} = -Y C \xi_2 s_2^2, \end{aligned} \quad (31)$$

$$\begin{aligned} U_1^{(3)} &= iX, \quad U_2^{(3)} = iX \xi_3, \quad T^{(3)} = s_3^2 - C^2, \\ s_{12}^{(3)} &= -2iX \xi_3, \quad s_{22}^{(3)} = iX \left(\frac{c_1^2}{c_2^2} C^2 - 2 \right), \quad Q_2^{(3)} = \frac{-\xi_3}{1 - i\chi C} (s_3^2 - C^2), \end{aligned} \quad (32)$$

$$U_1^{(4)} = -\xi_1, \quad U_2^{(4)} = 1, \quad T^{(4)} = 0, \quad s_{12}^{(4)} = C^2 - 2, \quad s_{22}^{(4)} = 2\xi_1, \quad Q_2^{(4)} = 0, \quad (33)$$

$$\begin{aligned} U_1^{(5)} &= s_2^2 - \frac{iC}{K} (1 - i\chi C), \quad U_2^{(5)} = -\xi_2 \left[s_2^2 - \frac{iC}{K} (1 - i\chi C) \right], \\ T^{(5)} &= Y C s_2^2 (1 - i\chi C), \quad s_{12}^{(5)} = 2\xi_2 \left[s_2^2 - \frac{iC}{K} (1 - i\chi C) \right], \\ s_{22}^{(5)} &= \left(\frac{c_1^2}{c_2^2} s_2^2 - 2 \right) \left[s_2^2 - \frac{iC}{K} (1 - i\chi C) \right] - \frac{i\epsilon c_1^2}{c_2^2 K} C s_2^2 (1 - i\chi C), \quad Q_2^{(5)} = Y C \xi_2 s_2^2, \end{aligned} \quad (34)$$

$$\begin{aligned} U_1^{(6)} &= iX, \quad U_2^{(6)} = -iX \xi_3, \quad T^{(6)} = s_3^2 - C^2, \\ s_{12}^{(6)} &= 2iX \xi_3, \quad s_{22}^{(6)} = iX \left(\frac{c_1^2}{c_2^2} C^2 - 2 \right), \quad Q_2^{(6)} = \frac{\xi_3}{1 - i\chi C} (s_3^2 - C^2), \end{aligned} \quad (35)$$

where we use the following dimensionless parameters:

$$X = \frac{mT_0}{\rho c_1^2}, \quad Y = \frac{mc_1}{\chi k}. \quad (36)$$

Finally, we note the following:

$$\{\mathcal{U}_1, \mathcal{U}_2, \mathcal{T}; \mathcal{S}_{12}, \mathcal{S}_{22}, \mathcal{Q}_2\}(x_2) = \sum_{n=1}^6 B_n \left\{ U_1^{(n)}, U_2^{(n)}, T^{(n)}; s_{12}^{(n)}, s_{22}^{(n)}, Q_2^{(n)} \right\} e^{i\chi r_n x_2}, \quad (37)$$

with $B_1, B_2, \dots, B_6$ as arbitrary dimensionless constant parameters, representing a solution of the differential system described by relations (14) and (16). Consequently, in view of relations (12), (13), and (37), we note the following:

$$\begin{aligned} u_1(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ \frac{1}{\chi} e^{i\chi(x_1 - ct)} \sum_{n=1}^6 B_n U_1^{(n)} e^{i\chi r_n x_2} \right\}, \\ u_2(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ \frac{i}{\chi} e^{i\chi(x_1 - ct)} \sum_{n=1}^6 B_n U_2^{(n)} e^{i\chi r_n x_2} \right\}, \\ \theta(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ T_0 e^{i\chi(x_1 - ct)} \sum_{n=1}^6 B_n T^{(n)} e^{i\chi r_n x_2} \right\}, \end{aligned} \quad (38)$$

together with

$$\begin{aligned} S_{12}(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ \mu e^{i\chi(x_1 - ct)} \sum_{n=1}^6 B_n s_{12}^{(n)} e^{i\chi r_n x_2} \right\}, \\ S_{22}(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ i\mu e^{i\chi(x_1 - ct)} \sum_{n=1}^6 B_n s_{22}^{(n)} e^{i\chi r_n x_2} \right\}, \\ q_2(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ \chi k T_0 e^{i\chi(x_1 - ct)} \sum_{n=1}^6 B_n Q_2^{(n)} e^{i\chi r_n x_2} \right\}, \end{aligned} \quad (39)$$

and

$$\begin{aligned} u_3(x_1, x_2, x_3, t) &= 0, \quad S_{13}(x_1, x_2, x_3, t) = 0, \quad S_{23}(x_1, x_2, x_3, t) = 0, \quad q_3(x_1, x_2, x_3, t) = 0, \\ S_{11}(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ i\mu e^{i\mathcal{K}(x_1-ct)} \sum_{n=1}^6 B_n s_{11}^{(n)} e^{i\mathcal{K}r_n x_2} \right\}, \\ q_1(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ i\mathcal{K}T_0 e^{i\mathcal{K}(x_1-ct)} \sum_{n=1}^6 B_n T^{(n)} e^{i\mathcal{K}r_n x_2} \right\}, \\ s_{11}^{(n)} &= \frac{c_1^2}{c_2^2} \left(U_1^{(n)} + ir_n U_2^{(n)} \right) - 2ir_n U_2^{(n)} - \frac{imT_0}{\varrho c_2^2} T^{(n)}, \quad n = 1, 2, \dots, 6. \end{aligned} \quad (40)$$

fulfills the basic Equations (3) and (4).

Herewith, we write the corresponding wave solutions for the differential system described by (1) and (2), in the half-space $x_2 > 0$, and we take into account the asymptotic conditions (8) as follows:

$$\begin{aligned} \tilde{u}_1(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ \frac{1}{\mathcal{K}} e^{i\mathcal{K}(x_1-ct)} \sum_{p=1}^3 D_p \tilde{U}_1^{(p)} e^{i\mathcal{K}\tilde{r}_p x_2} \right\} \\ \tilde{u}_2(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ \frac{i}{\mathcal{K}} e^{i\mathcal{K}(x_1-ct)} \sum_{p=1}^3 D_p \tilde{U}_2^{(p)} e^{i\mathcal{K}\tilde{r}_p x_2} \right\}, \\ \tilde{\theta}(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ T_0 e^{i\mathcal{K}(x_1-ct)} \sum_{p=1}^3 D_p \tilde{T}^{(p)} e^{i\mathcal{K}\tilde{r}_p x_2} \right\}, \end{aligned} \quad (41)$$

$$\begin{aligned} \tilde{S}_{12}(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ \tilde{\mu} e^{i\mathcal{K}(x_1-ct)} \sum_{p=1}^3 D_p \tilde{S}_{12}^{(p)} e^{i\mathcal{K}\tilde{r}_p x_2} \right\}, \\ \tilde{S}_{22}(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ i\tilde{\mu} e^{i\mathcal{K}(x_1-ct)} \sum_{p=1}^3 D_p \tilde{S}_{22}^{(p)} e^{i\mathcal{K}\tilde{r}_p x_2} \right\}, \\ \tilde{q}_2(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ \mathcal{K}\tilde{T}_0 e^{i\mathcal{K}(x_1-ct)} \sum_{p=1}^3 D_p \tilde{Q}_2^{(p)} e^{i\mathcal{K}\tilde{r}_p x_2} \right\}, \end{aligned} \quad (42)$$

where D_1, D_2 , and D_3 are complex arbitrary dimensionless parameters, not all zero, and we have the following:

$$\begin{aligned} \tilde{r}_1 &= i\tilde{\xi}_1, \quad \tilde{r}_2 = i\tilde{\xi}_2, \quad \tilde{r}_3 = i\tilde{\xi}_3, \\ \tilde{\xi}_1 &= \sqrt{1 - \tilde{s}_1^2}, \quad \tilde{\xi}_2 = \sqrt{1 - \tilde{s}_2^2}, \quad \tilde{\xi}_3 = \sqrt{1 - \tilde{s}_3^2}, \end{aligned} \quad (43)$$

$$\begin{aligned} \tilde{s}_1^2 &= \frac{c_1^2}{c_2^2} C^2, \\ \tilde{s}_2^2 &= \frac{1}{2} \left\{ \frac{c_1^2}{\tilde{c}_1^2} C^2 + \frac{ic_1}{\tilde{K}\tilde{c}_1} C(\tilde{\varepsilon} + 1) + \sqrt{\left[\frac{c_1^2}{\tilde{c}_1^2} C^2 + \frac{ic_1}{\tilde{K}\tilde{c}_1} C(\tilde{\varepsilon} - 1) \right]^2 - \frac{4\tilde{\varepsilon}c_1^2}{\tilde{K}^2\tilde{c}_1^2} C^2} \right\}, \\ \tilde{s}_3^2 &= \frac{1}{2} \left\{ \frac{c_1^2}{\tilde{c}_1^2} C^2 + \frac{ic_1}{\tilde{K}\tilde{c}_1} C(\tilde{\varepsilon} + 1) - \sqrt{\left[\frac{c_1^2}{\tilde{c}_1^2} C^2 + \frac{ic_1}{\tilde{K}\tilde{c}_1} C(\tilde{\varepsilon} - 1) \right]^2 - \frac{4\tilde{\varepsilon}c_1^2}{\tilde{K}^2\tilde{c}_1^2} C^2} \right\}. \end{aligned} \quad (44)$$

Moreover, we have the following:

$$\begin{aligned}
 \tilde{U}_1^{(1)} &= \tilde{\xi}_1, \quad \tilde{U}_2^{(1)} = 1, \quad \tilde{T}^{(1)} = 0, \quad \tilde{s}_{12}^{(1)} = \frac{c_1^2}{\tilde{c}_2^2} C^2 - 2, \quad \tilde{s}_{22}^{(1)} = -2\tilde{\xi}_1, \quad \tilde{Q}_2^{(1)} = 0, \\
 \tilde{U}_1^{(2)} &= \tilde{s}_2^2 - \frac{ic_1}{\tilde{c}_1 \tilde{K}} C, \quad \tilde{U}_2^{(2)} = \tilde{\xi}_2 \left(\tilde{s}_2^2 - \frac{ic_1}{\tilde{c}_1 \tilde{K}} C \right), \quad \tilde{T}^{(2)} = \tilde{Y} C \tilde{s}_2^2, \\
 \tilde{s}_{12}^{(2)} &= -2\tilde{\xi}_2 \left(\tilde{s}_2^2 - \frac{ic_1}{\tilde{c}_1 \tilde{K}} C \right), \quad \tilde{s}_{22}^{(2)} = \left(\frac{\tilde{c}_1^2}{\tilde{c}_2^2} \tilde{s}_2^2 - 2 \right) \left(\tilde{s}_2^2 - \frac{ic_1}{\tilde{c}_1 \tilde{K}} C \right) - \frac{i\tilde{\xi}_1 \tilde{c}_1}{\tilde{c}_2^2 \tilde{K}} C \tilde{s}_2^2, \\
 \tilde{Q}_2^{(2)} &= -\tilde{Y} C \tilde{\xi}_2 \tilde{s}_2^2, \\
 \tilde{U}_1^{(3)} &= i\tilde{X}, \quad \tilde{U}_2^{(3)} = i\tilde{X} \tilde{\xi}_3, \quad \tilde{T}^{(3)} = \tilde{s}_3^2 - \frac{c_1^2}{\tilde{c}_1^2} C^2, \quad \tilde{s}_{12}^{(3)} = -2i\tilde{X} \tilde{\xi}_3, \\
 \tilde{s}_{22}^{(3)} &= i\tilde{X} \left(\frac{c_1^2}{\tilde{c}_2^2} C^2 - 2 \right), \quad \tilde{Q}_2^{(3)} = -\tilde{\xi}_3 \left(\tilde{s}_3^2 - \frac{c_1^2}{\tilde{c}_1^2} C^2 \right),
 \end{aligned} \tag{45}$$

where

$$\tilde{X} = \frac{\tilde{m}T_0}{\tilde{Q}\tilde{c}_1^2}, \quad \tilde{Y} = \frac{\tilde{m}c_1}{\kappa\tilde{k}}. \tag{46}$$

4. Explicit Secular Equation for Rayleigh Waves

In the previous section, we established the in-plane wave solutions in the thermoelastic layer as well as in the thermoelastic half-space in the form of relations (38), (39), (41), and (42), respectively. Now we will use these wave solutions to establish the secular equation for a Rayleigh wave propagating in the considered structure: the thermoelastic layer of thickness h welded over the thermoelastic half-space. For this purpose, we will replace the relations (38), (39), (41), and (42) in the compatibility relations described by (5), as well as the boundary conditions described by (6). Thus, for arbitrary constants $B_1, B_2, \dots, B_6$ and D_1, D_2 , and D_3 , we obtain the following homogeneous algebraic system:

$$\begin{aligned}
 \sum_{n=1}^6 B_n U_1^{(n)} - \sum_{p=1}^3 D_p \tilde{U}_1^{(p)} &= 0, \\
 \sum_{n=1}^6 B_n U_2^{(n)} - \sum_{p=1}^3 D_p \tilde{U}_2^{(p)} &= 0, \\
 \sum_{n=1}^6 B_n T^{(n)} - \sum_{p=1}^3 D_p \tilde{T}^{(p)} &= 0,
 \end{aligned} \tag{47}$$

$$\begin{aligned}
 \sum_{n=1}^6 B_n s_{12}^{(n)} - \gamma \sum_{p=1}^3 D_p \tilde{s}_{12}^{(p)} &= 0, \\
 \sum_{n=1}^6 B_n s_{22}^{(n)} - \gamma \sum_{p=1}^3 D_p \tilde{s}_{22}^{(p)} &= 0, \\
 \sum_{n=1}^6 B_n Q_2^{(n)} - \delta \sum_{p=1}^3 D_p \tilde{Q}_2^{(p)} &= 0,
 \end{aligned} \tag{48}$$

$$\begin{aligned}
 \sum_{n=1}^6 B_n t_{12}^{(n)} &= 0, \\
 \sum_{n=1}^6 B_n t_{22}^{(n)} &= 0, \\
 \sum_{n=1}^6 B_n P_2^{(n)} &= 0,
 \end{aligned} \tag{49}$$

where

$$\begin{aligned} t_{12}^{(n)} &= s_{12}^{(n)} e^{-i\kappa r_n h}, \\ t_{22}^{(n)} &= s_{22}^{(n)} e^{-i\kappa r_n h}, \\ P_2^{(n)} &= Q_2^{(n)} e^{-i\kappa r_n h}, \quad n = 1, 2, \dots, 6, \end{aligned} \quad (50)$$

and

$$\gamma = \frac{\tilde{\mu}}{\mu}, \quad \delta = \frac{\tilde{k}}{k}. \quad (51)$$

Since not all arbitrary constants $B_1, B_2, \dots, B_6$ and D_1, D_2 and D_3 can be zero, it follows that the determinant of the homogeneous algebraic system described by relations (47) to (49) must vanish; we have the following:

$$\text{Det } \mathbf{A}_{(9 \times 9)}(a_{mn}) = 0, \quad (52)$$

where the elements a_{mn} of the matrix $\mathbf{A}$ are described in Appendix A.

Relation (52) explicitly represents the exact secular equation of the Rayleigh wave in an isotropic thermoelastic half-space overlaid by an isotropic thermoelastic layer, which is used for determining the dimensionless parameter C ; this is a complex parameter so that $\text{Re}(c_1 C) \geq 0$ represents the wave speed and $\text{Im}(c_1 C) \leq 0$ is related to the rate of decay over time of the wave's amplitude. It should be noted that when $\text{Re}(c_1 C) > 0$, we have a genuine wave, while for $\text{Re}(c_1 C) = 0$, we have a standing mode. Moreover, when $\text{Im}(c_1 C) > 0$, we have the phenomenon of damping in time, while $\text{Im}(c_1 C) = 0$ results in undamped waves over time. Therefore, it is necessary to obtain solutions of Equation (52), satisfying the following conditions:

$$\text{Re}(C) \geq 0, \quad \text{Im}(C) \leq 0. \quad (53)$$

Usually Equation (52) is solved numerically to find dispersion phase velocities and rates of damping in time versus wavenumbers.

Remark 1. We can use Laplace's cofactor expansion along the last three rows of the matrix $\mathbf{A}_{(9 \times 9)}$ in order to write the exact secular Equation (52) in the following form:

$$\begin{aligned} &M_{123}N_{123}e^{-(\xi_1+\xi_2+\xi_3)\kappa h} + M_{456}N_{456}e^{(\xi_1+\xi_2+\xi_3)\kappa h} + M_{234}N_{234}e^{(\xi_1-\xi_2-\xi_3)\kappa h} \\ &+ M_{156}N_{156}e^{(\xi_2+\xi_3-\xi_1)\kappa h} + M_{135}N_{135}e^{(\xi_2-\xi_3-\xi_1)\kappa h} + M_{246}N_{246}e^{(\xi_3+\xi_1-\xi_2)\kappa h} \\ &+ M_{126}N_{126}e^{(\xi_3-\xi_1-\xi_2)\kappa h} + M_{345}N_{345}e^{(\xi_1+\xi_2-\xi_3)\kappa h} \\ &+ (M_{125}N_{125} + M_{136}N_{136})e^{-\xi_1\kappa h} + (M_{245}N_{245} + M_{346}N_{346})e^{\xi_1\kappa h} \\ &+ (M_{124}N_{124} + M_{236}N_{236})e^{-\xi_2\kappa h} + (M_{145}N_{145} + M_{356}N_{356})e^{\xi_2\kappa h} \\ &+ (M_{134}N_{134} + M_{235}N_{235})e^{-\xi_3\kappa h} + (M_{146}N_{146} + M_{256}N_{256})e^{\xi_3\kappa h} = 0, \end{aligned} \quad (54)$$

where $M_{123}, M_{124}, M_{125}, M_{126}, M_{134}, M_{135}, M_{136}, M_{145}, M_{146}, M_{156}, M_{234}, M_{235}, M_{236}, M_{245}, M_{246}, M_{256}, M_{345}, M_{346}, M_{356}$, and M_{456} are the minors of the third-order of the matrix, as follows:

$$\begin{pmatrix} s_{12}^{(1)} & s_{12}^{(2)} & s_{12}^{(3)} & s_{12}^{(4)} & s_{12}^{(5)} & s_{12}^{(6)} \\ s_{22}^{(1)} & s_{22}^{(2)} & s_{22}^{(3)} & s_{22}^{(4)} & s_{22}^{(5)} & s_{22}^{(6)} \\ Q_2^{(1)} & Q_2^{(2)} & Q_2^{(3)} & Q_2^{(4)} & Q_2^{(5)} & Q_2^{(6)} \end{pmatrix}, \quad (55)$$

and $N_{123}, N_{124}, N_{125}, N_{126}, N_{134}, N_{135}, N_{136}, N_{145}, N_{146}, N_{156}, N_{234}, N_{235}, N_{236}, N_{245}, N_{246}, N_{256}, N_{345}, N_{346}, N_{356}$, and N_{456} are their corresponding signed complementary co-factors in the matrix $\mathbf{A}_{(9 \times 9)}$, described in Appendix A, respectively.

Remark 2. When the thickness of the thermoelastic layer tends to zero ($h \rightarrow 0$), i.e., there is no one thermoelastic layer, then the secular Equation (52) reduces to the following:

$$\begin{vmatrix} \tilde{s}_{12}^{(1)} & \tilde{s}_{12}^{(2)} & \tilde{s}_{12}^{(3)} \\ \tilde{s}_{22}^{(1)} & \tilde{s}_{22}^{(2)} & \tilde{s}_{22}^{(3)} \\ \tilde{Q}_2^{(1)} & \tilde{Q}_2^{(2)} & \tilde{Q}_2^{(3)} \end{vmatrix} = 0, \quad (56)$$

which, by means of relation (45), can be brought to the following form:

$$\begin{aligned} & \tilde{\xi}_3 \left(\tilde{s}_3^2 - \frac{c^2}{\tilde{c}_1^2} \right) \left(\tilde{s}_2^2 - \frac{ic}{\tilde{c}_1 \tilde{K}} \right) \left[4\tilde{\xi}_1 \tilde{\xi}_2 - \left(\frac{c^2}{\tilde{c}_2^2} - 2 \right) \left(\frac{\tilde{c}_1^2}{\tilde{c}_2^2} \tilde{s}_2^2 - 2 \right) \right] \\ & + \frac{ic\tilde{s}_2^2}{\tilde{c}_1 \tilde{K}} \tilde{\varepsilon} \left\{ \tilde{\xi}_3 \frac{\tilde{c}_1^2}{\tilde{c}_2^2} \left(\tilde{s}_3^2 - \frac{c^2}{\tilde{c}_1^2} \right) \left(\frac{c^2}{\tilde{c}_2^2} - 2 \right) + \tilde{\xi}_2 \left[\left(\frac{c^2}{\tilde{c}_2^2} - 2 \right)^2 - 4\tilde{\xi}_1 \tilde{\xi}_3 \right] \right\} = 0, \end{aligned} \quad (57)$$

which resonates with the secular equation presented by Chiriță and Zampoli [41] for the propagation of Rayleigh waves in isotropic linear thermoelastic half-space.

5. Numerical Simulation

For the numerical illustration of the previous results, we consider that the thermoelastic layer is occupied by an aluminum material with the following characteristics:

$$\begin{aligned} \varrho &= 2700 \text{ kg m}^{-3}, \quad c_1 = 6317.01 \text{ m s}^{-1}, \quad c_2 = 3110.28 \text{ m s}^{-1}, \quad \varepsilon = 2.65, \quad K = 1.272, \\ \alpha &= 23 \times 10^{-6} \text{ Kelvin}^{-1}, \quad T_0 = 293 \text{ Kelvin}, \end{aligned} \quad (58)$$

while the half-space is made of copper with the following characteristics

$$\begin{aligned} \tilde{\varrho} &= 8940 \text{ kg m}^{-3}, \quad \tilde{c}_1 = 4641.67 \text{ m s}^{-1}, \quad \tilde{c}_2 = 2285.40 \text{ m s}^{-1}, \quad \tilde{\varepsilon} = 1.90, \quad \tilde{K} = 2.108, \\ \tilde{\alpha} &= 17 \times 10^{-6} \text{ Kelvin}^{-1}, \quad T_0 = 293 \text{ Kelvin}, \end{aligned} \quad (59)$$

where α is the coefficient of thermal expansion and $m = -\alpha(3\lambda + 2\mu) = -\alpha(3\varrho c_1^2 - 4\varrho c_2^2)$. Moreover, we take the wavenumber to be $\varkappa = 1 \text{ m}^{-1}$.

For the above data, we solve Equation (52) in terms of dimensionless parameter C , and we consider various values of the thickness h of the thermoelastic layer and highlight the dimensionless speed coefficient $\text{Re}(C) > 0$ and the dimensionless decreasing-in-time coefficient $\text{Im}(C) < 0$, as well as their dependence on the relaxation time τ (by means of the dimensionless parameter $\chi = \tau \varkappa c_1$). To solve the dispersion Equation (52), we used the software package Wolfram Mathematica; the results are presented in Tables 1 and 2.

Table 1. For the fixed value of the relaxation time $\tau = 0 \text{ s}$, the values of the two dimensionless characteristics $\text{Re}(C) > 0$ and $\text{Im}(C) < 0$ are presented by solving the secular equation for various values of the thickness h .

h	0	10^{-3} m	$5 \times 10^{-3} \text{ m}$	10^{-2} m
$\text{Re}(C)$	0.33943	0.339262	0.338584	0.337723
$\text{Im}(C)$	−0.00250732	−0.00252427	−0.00259316	−0.00268182

Table 2. For the fixed value of the relaxation time $\tau = 10^{-2} \text{ s}$, the values of the two dimensionless characteristics $\text{Re}(C) > 0$ and $\text{Im}(C) < 0$ are presented by solving the secular equation for various values of the thickness h .

h	0	10^{-3} m	$5 \times 10^{-3} \text{ m}$	10^{-2} m
$\text{Re}(C)$	0.33943	0.339263	0.338589	0.337733
$\text{Im}(C)$	−0.00250732	−0.0025248	−0.00259589	−0.00268751

Table 1 presents the behaviors of the two characteristic coefficients $\text{Re}(C)$ and $\text{Im}(C)$ for the situation when the layer and the half-space are modeled by the classical Fourier's law for the heat flux (when $\tau = 0$). This table expresses the decreasing tendency of the dimensionless speed of propagation of the Rayleigh wave in relation to the increase in the thickness of the protective layer of the half-space. This fact is supported by an increase in the rate of decrease in time of the Rayleigh wave amplitude corresponding to an increase in the thickness of the layer.

Comparatively, Table 2 shows similar effects with the presence of a Maxwell–Cattaneo thermoelastic layer of thickness $h = 10^{-3}$, but the tendency to decrease the dimensionless speed $\text{Re}(C)$ is more moderate, while the decrease in time of the Rayleigh wave amplitude, $\text{Im}(C)$, is more pronounced.

To highlight how the dimensionless speed coefficient $\text{Re}(C) > 0$ and the dimensionless decreasing in time coefficient $\text{Im}(C) < 0$ vary with the relaxation time, we considered a hypothetical scenario where the layer and the half-space share the same material characteristics but are governed by the Maxwell–Cattaneo law with different τ . Thus, Figure 2 illustrates the behavior of the dimensionless propagation speed $\text{Re}(C) > 0$ in relation to the increase of the relaxation time $\tau \in [0, 10^{-2} \text{ s}]$, while Figure 3 shows the behavior of the Rayleigh wave amplitude decreasing over time with respect to the same relaxation time, being the Maxwell–Cattaneo thermoelastic layer of thickness $h = 10^{-3} \text{ m}$. They are obtained through the Mathematica Plot3D tool. It is important to emphasize that the images presented in Figures 2 and 3 have a prospective character, as our simulations across various relaxation time intervals yield quite different results.

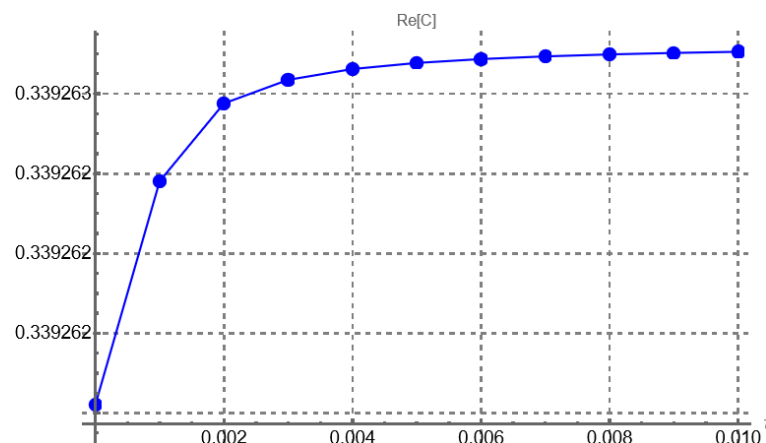


Figure 2. The variation in the dimensionless propagation speed $\text{Re}(C) > 0$ with respect to the relaxation time $\tau \in [0, 10^{-2} \text{ s}]$, (the thickness of the layer is fixed at $h = 10^{-3} \text{ m}$).

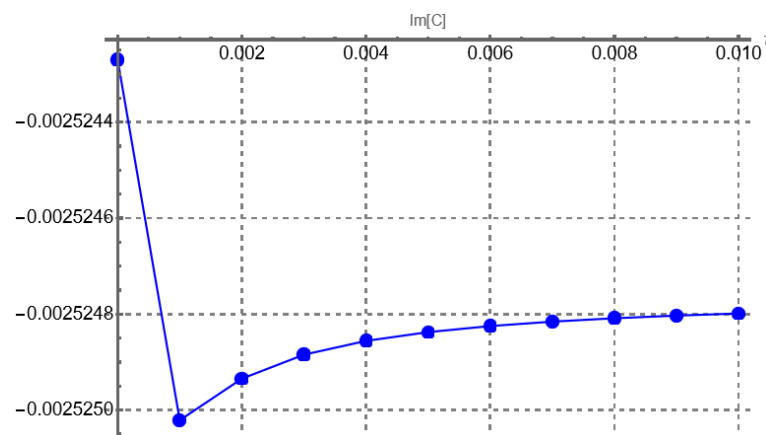


Figure 3. The variation in the dimensionless dumping in time $\text{Im}(C) > 0$ of the amplitude of wave, with respect to the relaxation time $\tau \in [0, 10^{-2} \text{ s}]$, (the thickness of the layer is fixed at $h = 10^{-3} \text{ m}$).

6. Exact Effective Boundary Conditions

We establish effective boundary conditions that could be very useful in deriving approximate explicit secular equations for Rayleigh waves in thermoelastic half-spaces covered with a thin film. It should be noted that the entire effect of the thermoelastic layer on the half-space is replaced by the exact effective boundary conditions at the interface. Consequently, the wave can be envisioned as a Rayleigh wave propagating in the thermoelastic half-space, uncoated but subject to the effective boundary conditions. Following this idea, we should establish boundary surface conditions at $x_2 = 0$ for the displacements and temperature fields, as well as for traction and heat flux.

In this connection, we consider the propagation of a Rayleigh wave, traveling along surface $x_2 = 0$ of the isotropic thermoelastic half-space with velocity c and wavenumber $\varkappa$ in the x_1 -direction, decaying in the x_2 -direction. Consequently, for $x_2 > 0$, we have the following:

$$\begin{aligned}\tilde{u}_1(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ \frac{1}{\varkappa} e^{i\varkappa(x_1 - ct)} \tilde{\mathcal{U}}_1(x_2) \right\}, \\ \tilde{u}_2(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ \frac{i}{\varkappa} e^{i\varkappa(x_1 - ct)} \tilde{\mathcal{U}}_2(x_2) \right\}, \\ \tilde{u}_3(x_1, x_2, x_3, t) &= 0, \\ \tilde{\theta}(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ T_0 e^{i\varkappa(x_1 - ct)} \tilde{\mathcal{T}}(x_2) \right\},\end{aligned}\quad (60)$$

and

$$\begin{aligned}\tilde{S}_{12}(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ \tilde{\mu} e^{i\varkappa(x_1 - ct)} \tilde{\mathcal{S}}_{12}(x_2) \right\}, \\ \tilde{S}_{22}(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ i\tilde{\mu} e^{i\varkappa(x_1 - ct)} \tilde{\mathcal{S}}_{22}(x_2) \right\}, \\ \tilde{S}_{32}(x_1, x_2, x_3, t) &= 0, \\ \tilde{q}_2(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ i\varkappa \tilde{k} T_0 e^{i\varkappa(x_1 - ct)} \tilde{\mathcal{Q}}(x_2) \right\}.\end{aligned}\quad (61)$$

In view of the continuity conditions (5), from relations (38), (39), (60), and (61), we obtain the following matrix equation:

$$\mathcal{A}_{(6 \times 6)} \mathbf{B}_{(6 \times 1)} = \mathbf{D}_{(6 \times 1)}, \quad (62)$$

where

$$\mathcal{A}_{(6 \times 6)} = \begin{pmatrix} U_1^{(1)} & U_1^{(2)} & U_1^{(3)} & U_1^{(4)} & U_1^{(5)} & U_1^{(6)} \\ U_2^{(1)} & U_2^{(2)} & U_2^{(3)} & U_2^{(4)} & U_2^{(5)} & U_2^{(6)} \\ T^{(1)} & T^{(2)} & T^{(3)} & T^{(4)} & T^{(5)} & T^{(6)} \\ s_{12}^{(1)} & s_{12}^{(2)} & s_{12}^{(3)} & s_{12}^{(4)} & s_{12}^{(5)} & s_{12}^{(6)} \\ s_{22}^{(1)} & s_{22}^{(2)} & s_{22}^{(3)} & s_{22}^{(4)} & s_{22}^{(5)} & s_{22}^{(6)} \\ Q_2^{(1)} & Q_2^{(2)} & Q_2^{(3)} & Q_2^{(4)} & Q_2^{(5)} & Q_2^{(6)} \end{pmatrix}, \quad (63)$$

and

$$\begin{aligned}\mathbf{B}_{(6 \times 1)} &= (B_1, B_2, B_3, B_4, B_5, B_6)^T, \\ \mathbf{D}_{(6 \times 1)} &= (\tilde{\mathcal{U}}_1(0), \tilde{\mathcal{U}}_2(0), \tilde{\mathcal{T}}(0), \gamma \tilde{\mathcal{S}}_{12}(0), \gamma \tilde{\mathcal{S}}_{22}(0), \delta \tilde{\mathcal{Q}}_2(0))^T.\end{aligned}\quad (64)$$

On the other hand, the boundary condition on the free surface of the layer in the form (49) can be written in the following matrix form:

$$\Gamma_{(3 \times 6)} \mathbf{B}_{(6 \times 1)} = \mathbf{0}_{(3 \times 1)}, \quad (65)$$

where

$$\Gamma_{(3 \times 6)} = \begin{pmatrix} s_{12}^{(1)} & s_{12}^{(2)} & s_{12}^{(3)} & s_{12}^{(4)} & s_{12}^{(5)} & s_{12}^{(6)} \\ s_{22}^{(1)} & s_{22}^{(2)} & s_{22}^{(3)} & s_{22}^{(4)} & s_{22}^{(5)} & s_{22}^{(6)} \\ p_2^{(1)} & p_2^{(2)} & p_2^{(3)} & p_2^{(4)} & p_2^{(5)} & p_2^{(6)} \end{pmatrix}. \quad (66)$$

Consequently, from relations (62) and (65), we can deduce that on the interface $x_2 = 0$ we have to fulfill the following condition:

$$\Gamma_{(3 \times 6)} \mathcal{A}_{(6 \times 6)}^{-1} \mathbf{D}_{(6 \times 1)} = \mathbf{0}_{(3 \times 1)}. \quad (67)$$

As clearly indicated from relation (64), the matrix $\mathbf{D}_{(6 \times 1)}$ includes the values of the displacement and temperature fields, as well as those of the traction and the heat flux vector on the surface $x_2 = 0$ of the half-space $x_2 > 0$. This implies that in the Rayleigh wave propagation problem for the half-space $x_2 > 0$, the boundary conditions at $x_2 = 0$ must be specified as part of a linear combination. These conditions are described by relation (67), which relates the values of the displacement and temperature fields, along with the values of traction and heat flux.

7. Maxwell–Cattaneo Model for Both Layer and Half-Space

In this section, we assume that both the layer and the half-space are filled by two different thermoelastic materials modeled by the Maxwell–Cattaneo approach. In the half-space $x_2 > 0$, the free wave solutions are described by the same relations (41) to (43); now we have to use the values furnished by the following relations:

$$\begin{aligned} \tilde{s}_1^2 &= \frac{c_1^2}{\tilde{c}_2^2} C^2, \\ \tilde{s}_2^2 &= \frac{1}{2} \left\{ \frac{c_1^2}{\tilde{c}_1^2} C^2 + \frac{ic_1}{\tilde{K}\tilde{c}_1} C(1 - i\tilde{\chi}C)(\tilde{\varepsilon} + 1) + \sqrt{\tilde{\Delta}} \right\}, \\ \tilde{s}_3^2 &= \frac{1}{2} \left\{ \frac{c_1^2}{\tilde{c}_1^2} C^2 + \frac{ic_1}{\tilde{K}\tilde{c}_1} C(1 - i\tilde{\chi}C)(\tilde{\varepsilon} + 1) - \sqrt{\tilde{\Delta}} \right\}, \end{aligned} \quad (68)$$

with $\tilde{\chi} = \tilde{\tau}\kappa c_1$, where $\tilde{\tau}$ represents the thermal relaxation time in the half-space $x_2 > 0$, and we have the following:

$$\tilde{\Delta} = \left[\frac{c_1^2}{\tilde{c}_1^2} C^2 + \frac{ic_1}{\tilde{K}\tilde{c}_1} C(1 - i\tilde{\chi}C)(\tilde{\varepsilon} - 1) \right]^2 - \frac{4\tilde{\varepsilon}c_1^2}{\tilde{K}^2\tilde{c}_1^2} C^2(1 - i\tilde{\chi}C)^2, \quad (69)$$

moreover, we have the following:

$$\begin{aligned} \tilde{U}_1^{(1)} &= \tilde{\xi}_1, \quad \tilde{U}_2^{(1)} = 1, \quad \tilde{T}^{(1)} = 0, \quad \tilde{s}_{12}^{(1)} = \frac{c_1^2}{\tilde{c}_2^2} C^2 - 2, \quad \tilde{s}_{22}^{(1)} = -2\tilde{\xi}_1, \quad \tilde{Q}_2^{(1)} = 0, \\ \tilde{U}_1^{(2)} &= \tilde{s}_2^2 - \frac{ic_1}{\tilde{c}_1\tilde{K}} C(1 - i\tilde{\chi}C), \quad \tilde{U}_2^{(2)} = \tilde{\xi}_2 \left[\tilde{s}_2^2 - \frac{ic_1}{\tilde{c}_1\tilde{K}} C(1 - i\tilde{\chi}C) \right], \\ \tilde{T}^{(2)} &= \tilde{Y} C \tilde{s}_2^2 (1 - i\tilde{\chi}C), \quad \tilde{s}_{12}^{(2)} = -2\tilde{\xi}_2 \left[\tilde{s}_2^2 - \frac{ic_1}{\tilde{c}_1\tilde{K}} C(1 - i\tilde{\chi}C) \right], \\ \tilde{s}_{22}^{(2)} &= \left(\frac{\tilde{c}_1^2}{\tilde{c}_2^2} \tilde{s}_2^2 - 2 \right) \left[\tilde{s}_2^2 - \frac{ic_1}{\tilde{c}_1\tilde{K}} C(1 - i\tilde{\chi}C) \right] - \frac{i\tilde{\varepsilon}c_1\tilde{c}_1}{\tilde{c}_2^2\tilde{K}} C \tilde{s}_2^2 (1 - i\tilde{\chi}C), \quad \tilde{Q}_2^{(2)} = -\tilde{Y} C \tilde{\xi}_2 \tilde{s}_2^2, \\ \tilde{U}_1^{(3)} &= i\tilde{X}, \quad \tilde{U}_2^{(3)} = i\tilde{X} \tilde{\xi}_3, \quad \tilde{T}^{(3)} = \tilde{s}_3^2 - \frac{c_1^2}{\tilde{c}_1^2} C^2, \\ \tilde{s}_{12}^{(3)} &= -2i\tilde{X} \tilde{\xi}_3, \quad \tilde{s}_{22}^{(3)} = i\tilde{X} \left(\frac{c_1^2}{\tilde{c}_2^2} C^2 - 2 \right), \quad \tilde{Q}_2^{(3)} = \frac{-\tilde{\xi}_3}{1 - i\tilde{\chi}C} \left(\tilde{s}_3^2 - \frac{c_1^2}{\tilde{c}_1^2} C^2 \right). \end{aligned} \quad (70)$$

The exact secular equation has the same formal aspect as that described by relation (52); the last three columns of the matrix $\mathbf{A}$ must be written using the values described by the relation (70).

Remark 3. When the thickness of the thermoelastic layer tends to zero ($h \rightarrow 0$), i.e., there is no one thermoelastic layer, then the exact secular equation leads to the secular equation for the propagation of the Rayleigh waves in a half-space filled by a Maxwell–Cattaneo thermoelastic material, as follows:

$$\begin{aligned} & \tilde{\xi}_3 \left(\tilde{s}_3^2 - \frac{c_1^2}{\tilde{c}_1^2} C^2 \right) \left(\tilde{s}_2^2 - \frac{ic_1}{\tilde{c}_1 \tilde{K}} C(1 - i\tilde{\chi}C) \right) \left[4\tilde{\xi}_1 \tilde{\xi}_2 - \left(\frac{c_1^2}{\tilde{c}_2^2} C^2 - 2 \right) \left(\frac{\tilde{c}_1^2}{\tilde{c}_2^2} \tilde{s}_2^2 - 2 \right) \right] \\ & + \frac{ic_1 \tilde{s}_2^2}{\tilde{c}_1 \tilde{K}} C \tilde{\varepsilon} (1 - i\tilde{\chi}C) \left\{ \tilde{\xi}_3 \frac{\tilde{c}_1^2}{\tilde{c}_2^2} \left(\tilde{s}_3^2 - \frac{c_1^2}{\tilde{c}_1^2} C^2 \right) \left(\frac{c_1^2}{\tilde{c}_2^2} C^2 - 2 \right) + \tilde{\xi}_2 \left[\left(\frac{c_1^2}{\tilde{c}_2^2} C^2 - 2 \right)^2 - 4\tilde{\xi}_1 \tilde{\xi}_3 \right] \right\} = 0. \end{aligned} \quad (71)$$

8. Conclusions

This paper examines the propagation of Rayleigh waves in a classical isotropic thermoelastic half-space coated by an isotropic Maxwell–Cattaneo thermoelastic layer, with the contact assumed to be perfectly welded. The explicit and compact expressions for in-plane free Rayleigh waves propagating in the isotropic Maxwell–Cattaneo thermoelastic layer $-h < x_2 < 0$ are established as in relations (38) and (39), while in the half-space, $x_2 > 0$, modeled by classical Fourier thermoelasticity, they are established as in relations (41) and (42). Using these explicit forms allows for a more convenient and straightforward understanding of Rayleigh wave motions in the isotropic thermoelastic layer/half-space system. The exact secular equation for Rayleigh waves in an isotropic layered system is established in determinant form (52) or as expressed by relation (54). The simulations conducted on the structure of an aluminum thermoelastic layer superimposed on a copper thermoelastic half-space highlight interesting aspects regarding the speed of propagation of Rayleigh waves and the rate of decrease in time of their amplitude, both in relation to the increase in the thickness of the layer and the relaxation time.

By establishing the effective boundary conditions (67), the entire influence of the thermoelastic layer on the substrate is captured. These relations, along with the expansion of the stresses and the heat flux at the top surface of the layer into a Taylor series in terms of its thickness, can provide approximate secular equations for the waves supported by thin-film/substrate interactions. These equations can serve as theoretical bases for deriving the thermomechanical properties of thin films from experimental data.

Finally, the relation (70), when coupled with relation (52), provides an explicit exact secular equation for the propagation of Rayleigh waves supported by isotropic Maxwell–Cattaneo thermoelastic layer/half-space interactions. Relation (71) provides the secular equation for the propagation of Rayleigh waves in a half-space composed of Maxwell–Cattaneo thermoelastic material.

Our results highlight the importance of understanding the thermomechanics of layered materials in both natural and engineered systems. Such analysis can be used in experimental research in order to determine the material characteristics of the layer and, thus, improve the material characteristics of the substrate itself. Researchers in the experimental field can use these results to study the thermomechanical properties of composite materials, where different layers may have different thermoelastic properties. This is important for designing materials with specific properties, such as improved strength or thermal resistance, which often involves studying how different layers interact. Engineers consider factors like mechanical strength, thermal conductivity, and chemical resistance to ensure that the layered structures meet the desired high-performance criteria.

On the other hand, the effective boundary condition method helps derive approximate equations (secular equations) that describe the propagation of Rayleigh waves in such layered structures.

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Appendix A

The elements of the matrix **A** are given by the following:

$$\begin{aligned} a_{11} &= \tilde{\zeta}_1, & a_{21} &= 1, & a_{31} &= 0, & a_{41} &= C^2 - 2, & c_{51} &= -2\tilde{\zeta}_1, & a_{61} &= 0, \\ a_{71} &= (C^2 - 2) \times e^{\tilde{\zeta}_1 \times h}, & a_{81} &= -2\tilde{\zeta}_1 \times e^{\tilde{\zeta}_1 \times h}, & a_{91} &= 0, \end{aligned} \quad (\text{A1})$$

$$\begin{aligned} a_{12} &= s_2^2 - \frac{i}{K} C(1 - i\chi C), & a_{22} &= \tilde{\zeta}_2 \left[s_2^2 - \frac{i}{K} C(1 - i\chi C) \right], & a_{32} &= Y C s_2^2 (1 - i\chi C), \\ a_{42} &= -2\tilde{\zeta}_2 \left[s_2^2 - \frac{i}{K} C(1 - i\chi C) \right], \\ a_{52} &= \left(\frac{c_1^2}{c_2^2} s_2^2 - 2 \right) \left[s_2^2 - \frac{i}{K} C(1 - i\chi C) \right] - \frac{i\epsilon}{K} \frac{c_1^2}{c_2^2} C s_2^2 (1 - i\chi C), & a_{62} &= -Y C \tilde{\zeta}_2 s_2^2, \\ a_{72} &= a_{42} \times e^{\tilde{\zeta}_2 \times h}, & a_{82} &= a_{52} \times e^{\tilde{\zeta}_2 \times h}, & a_{92} &= a_{62} \times e^{\tilde{\zeta}_2 \times h}, \end{aligned} \quad (\text{A2})$$

$$\begin{aligned} a_{13} &= iX, & a_{23} &= iX \tilde{\zeta}_3, & a_{33} &= s_3^2 - C^2, & a_{43} &= -2iX \tilde{\zeta}_3, \\ a_{53} &= iX \left(\frac{c_1^2}{c_2^2} C^2 - 2 \right), & a_{63} &= -\frac{1}{1 - i\chi C} \tilde{\zeta}_3 (s_3^2 - C^2), \\ a_{73} &= a_{43} \times e^{\tilde{\zeta}_3 \times h}, & a_{83} &= a_{53} \times e^{\tilde{\zeta}_3 \times h}, & a_{93} &= a_{63} \times e^{\tilde{\zeta}_3 \times h}, \end{aligned} \quad (\text{A3})$$

$$\begin{aligned} a_{14} &= -\tilde{\zeta}_1, & a_{24} &= 1, & a_{34} &= 0, & a_{44} &= C^2 - 2, & c_{54} &= 2\tilde{\zeta}_1, & a_{64} &= 0, \\ a_{74} &= (C^2 - 2) \times e^{-\tilde{\zeta}_1 \times h}, & a_{84} &= 2\tilde{\zeta}_1 \times e^{-\tilde{\zeta}_1 \times h}, & a_{94} &= 0, \end{aligned} \quad (\text{A4})$$

$$\begin{aligned} a_{15} &= s_2^2 - \frac{i}{K} C(1 - i\chi C), & a_{25} &= -\tilde{\zeta}_2 \left[s_2^2 - \frac{i}{K} C(1 - i\chi C) \right], & a_{35} &= Y C s_2^2 (1 - i\chi C), \\ a_{45} &= 2\tilde{\zeta}_2 \left[s_2^2 - \frac{i}{K} C(1 - i\chi C) \right], \\ a_{55} &= \left(\frac{c_1^2}{c_2^2} s_2^2 - 2 \right) \left[s_2^2 - \frac{i}{K} C(1 - i\chi C) \right] - \frac{i\epsilon}{K} \frac{c_1^2}{c_2^2} C s_2^2 (1 - i\chi C), & a_{65} &= Y C \tilde{\zeta}_2 s_2^2, \\ a_{75} &= a_{45} \times e^{-\tilde{\zeta}_2 \times h}, & a_{85} &= a_{55} \times e^{-\tilde{\zeta}_2 \times h}, & a_{95} &= a_{65} \times e^{-\tilde{\zeta}_2 \times h}, \end{aligned} \quad (\text{A5})$$

$$\begin{aligned} a_{16} &= iX, & a_{26} &= -iX \tilde{\zeta}_3, & a_{36} &= s_3^2 - C^2, & a_{46} &= 2iX \tilde{\zeta}_3, \\ a_{56} &= iX \left(\frac{c_1^2}{c_2^2} C^2 - 2 \right), & a_{66} &= \frac{1}{1 - i\chi C} \tilde{\zeta}_3 (s_3^2 - C^2), \\ a_{76} &= a_{46} \times e^{-\tilde{\zeta}_3 \times h}, & a_{86} &= a_{56} \times e^{-\tilde{\zeta}_3 \times h}, & a_{96} &= a_{66} \times e^{-\tilde{\zeta}_3 \times h}, \end{aligned} \quad (\text{A6})$$

$$\begin{aligned} a_{17} &= -\tilde{\zeta}_1, & a_{27} &= -1, & a_{37} &= 0, & a_{47} &= -\gamma \left(\frac{c_1^2}{c_2^2} C^2 - 2 \right), & a_{57} &= 2\gamma \tilde{\zeta}_1, & a_{67} &= 0, \\ a_{77} &= 0, & a_{87} &= 0, & a_{97} &= 0, \end{aligned} \quad (\text{A7})$$

$$\begin{aligned} a_{18} &= -\left(\tilde{s}_2^2 - \frac{ic_1}{K\tilde{c}_1} C \right), & a_{28} &= -\tilde{\zeta}_2 \left(\tilde{s}_2^2 - \frac{ic_1}{K\tilde{c}_1} C \right), & a_{38} &= -\tilde{Y} C \tilde{s}_2^2, \\ a_{48} &= 2\gamma \tilde{\zeta}_2 \left(\tilde{s}_2^2 - \frac{ic_1}{\tilde{c}_1 K} C \right), & a_{58} &= -\gamma \left[\left(\frac{\tilde{c}_1^2}{\tilde{c}_2^2} \tilde{s}_2^2 - 2 \right) \left(\tilde{s}_2^2 - \frac{ic_1}{\tilde{c}_1 K} C \right) - \frac{i\tilde{\epsilon} c_1 \tilde{c}_1}{K\tilde{c}_2^2} C \tilde{s}_2^2 \right], \\ a_{68} &= \delta \tilde{Y} C \tilde{\zeta}_2 \tilde{s}_2^2, & a_{78} &= 0, & a_{88} &= 0, & a_{98} &= 0, \end{aligned} \quad (\text{A8})$$

$$\begin{aligned} a_{19} &= -i\tilde{X}, & a_{29} &= -i\tilde{X} \tilde{\zeta}_3, & a_{39} &= -\left(\tilde{s}_3^2 - \frac{c_1^2}{\tilde{c}_2^2} C^2 \right), & a_{49} &= 2i\tilde{X} \gamma \tilde{\zeta}_3, \\ a_{59} &= -i\tilde{X} \gamma \left(\frac{c_1^2}{\tilde{c}_2^2} C^2 - 2 \right), & a_{69} &= \delta \tilde{\zeta}_3 \left(\tilde{s}_3^2 - \frac{c_1^2}{\tilde{c}_2^2} C^2 \right), & a_{79} &= 0, & a_{89} &= 0, & a_{99} &= 0. \end{aligned} \quad (\text{A9})$$

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