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Structural stability of a three-phase-lag heat conduction model for a rigid conductor

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ABSTRACT

The present work investigates the structural stability of a three-phase-lag thermal conduction model for a rigid conductor, using the integral inequalities techniques as a mathematical basis. A set of mathematical estimates are established to describe the influence of variations in the external given data (boundary and initial conditions, and heat supply), as well as in intrinsic data of the model (material thermal characteristics, as well as relaxation times), on the material behavior response (that is the solutions to the associated initial boundary value problems). The convergence issue is also established by means of an integral inequality that provides an estimate of the difference between the solutions to the initial boundary value problems associated with the initial model and the one obtained by making one of the characteristic coefficients of the equation tend to zero. All presented results concern the structural stability and they allow for the conclusion that the model of the three-phase-lag is well posed.

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1. Introduction

The classical theory of thermoelasticity, which is based on Fourier's law of heat conduction, implies the paradox that thermal signals propagate instantaneously, leading to an unrealistic infinite speed of heat propagation. This paradox is addressed by generalized thermo-elasticity theories that incorporate finite-wave speeds. Generalized thermal models are advanced frameworks used to describe the behavior of materials under thermal loads, extending beyond classical thermoelasticity.

Thus, Cattaneo [1,2] proposed replacing the Fourier constitutive relation with an evolution equation of the form $\tau \dot{q}_i + q_i = k \theta_{,i}$, where the superposed dot indicates the material-time derivative, and τ is an intrinsic relaxation time. Later, Lord-Shulman theory [3] introduces a single relaxation time to account for the finite speed of heat propagation, while Green-Lindsay theory [4] incorporates two relaxation times, providing a more accurate description of heat conduction in materials subjected to rapid thermal changes. In addition, the Green-Naghdi theory [5–8] proposes three types of thermoelastic models (type I, II and III), each with different assumptions about the heat flux and temperature gradient relationship. The dual-phase-lag model proposed by Tzou [9] accounts for both the lag in the heat flux and the temperature gradient, offering a more comprehensive approach to modeling heat conduction in materials with complex thermal responses, while three-phase-lag model extends the dual-phase-lag model by introducing an additional phase lag, further refining the description of heat transfer processes. Roy

Choudhuri [10] developed the three-phase-lag model by incorporating phase lags in the heat flux vector, the temperature gradient, and the thermal displacement gradient.

Several other constitutive relations for the heat flux vector have been discussed by Chandrasekharaiah [11], Hetnarski and Ignaczak [12,13], Straughan [14] and Tzou [15].

On the other side, the structural stability in this context often refers to the continuous dependence of solutions on the constitutive parameters of the material. This means that small changes in the properties of the material or external conditions result in small changes in the response of the material. The stability of thermoelastic structures can be significantly influenced by boundary conditions, such as fixed or free edges, and the type of thermal loading applied. Stability analysis in this context often focuses on ensuring that small perturbations in the system do not lead to large deviations in behavior. Understanding the stability of thermoelastic materials is crucial for designing components that can withstand thermal and mechanical loads without failure. This is particularly important in industries such as aerospace, automotive, and civil engineering.

The importance of studying the structural stability of a mathematical model, as well as related continuous dependence on given data, has been emphasized in several excellent textbooks, as for example those written by Hirsch and Smale [16], Bellomo and Preziosi [17], Straughan [18,19] and Flavin and Rionero [20].

In order to establish continuous dependence and convergence results for the solutions of thermoelastic models, differential inequality techniques are used. These techniques were also used to study the structural stability in various fields of continuum mechanics in many works like: Knops and Payne [21,22], Ames and Hughes [23], Chiriță et al. [24], Chiriță and Ciarletta [25], Castro et al. [26], Harfash [27], Celebi and Kalantarov [28], Hill et al. [29], Markowich et al. [30], Otani and Uchida [31], Celik and Hoang [32], Eltayeb [33], Liu [34], Varsakelis and Papalexandris [35] and de Castro Motta et al. [36].

The present paper considers the three-phase-lag thermal theory developed by Roy Choudhuri [10] and contributes to the study of the well-posedness of the model itself. In fact, we first establish an integral inequality that provides a mathematical estimate describing the continuous dependence on all external given data for the solution of the inhomogeneous Dirichlet problem associated with the model under study. This first result relies on material characteristics and relaxation time assumptions that align with the Second Law of Thermodynamics. On this basis, we approach the problem of structural stability in the sense of the continuous dependence of the solutions with respect to the changes that may occur in the constitutive coefficients or relaxation times involved in the model in concern. Thus, we provide a series of three precise mathematical estimates that describe the continuous dependence of the solution of the initial boundary value problems with respect to small variations in the relaxation times, variations that do not affect the formulated positive definiteness restrictions. Furthermore, we provide three other explicit mathematical estimates that describe the continuous dependence of the solutions with respect to small variations in the other material characteristics of the model: the conductivity tensors and the specific heat. We also show how the solution to the differential problem converges, in an appropriate measure, to the solution of the reduced differential problem, as the respective material coefficients tend to zero.

The present work is motivated by the recent research activity on the Moore-Gibson-Thompson equation, see for example the works [37–43]. It constitutes a continuation of the research presented in the paper by Chiriță and D'Apice [44] on the same three-phase-lag model of a rigid thermal conductor. Both works together provide the structural stability of the model in question.

The plan of the work is the following. [Section 2](#) presents the basic system of differential equations describing the evolutionary behavior of the three-phase-lag model in the line described by Roy Choudhuri [10]. [Section 3](#) formulates the initial boundary value problem associated with the model and it states the main constitutive hypotheses on the material characteristics. [Section 4](#) defines a class of auxiliary elliptic boundary value problems with their appropriate estimates of solutions in terms of the boundary given data. [Section 5](#) treats the continuous dependence of solution with respect to all

external given data (initial and boundary data as well as the external supply loads). [Section 6](#) establishes appropriate estimates describing continuous dependence with respect to the variation of delay times. [Section 7](#) establishes appropriate estimates describing continuous dependence with respect to the other thermal coefficients. [Section 8](#) shows that the solution of the considered initial boundary value problem converges to the solution of the corresponding initial boundary value problem when the conductivity rate tensor in an appropriate measure tends to zero.

2. The three-phase-lag heat conduction model for rigid conductor

We assume that a regular region B is filled by an inhomogeneous and anisotropic conductor material with three-phase-lag times. We suppose that the rigid conductor material is bounded by the piecewise smooth surface ∂B , which is star-shaped with respect to the origin of a Cartesian coordinate system and therefore, we have

$$x_k n_k \geq h_0 > 0, \quad |x_k t_k| \leq \delta_0 \quad \text{on} \quad \partial B, \quad (1)$$

where h_0 and δ_0 are appropriate positive constants, x_k are the components of the vector $\mathbf{x}$, n_k are the components of unit outward normal vector $\mathbf{n}$ to ∂B and t_k are the components of a unit tangent vector to ∂B .

We consider the basic constitutive equation for the generalized heat conduction as proposed by Roy Choudhuri [10]

$$q_i(\mathbf{x}, t) + \tau_q \dot{q}_i(\mathbf{x}, t) = -k_{ij}^*(\mathbf{x}) \alpha_{,j}(\mathbf{x}, t) - \left(k_{ij}(\mathbf{x}) + \tau_\alpha k_{ij}^*(\mathbf{x}) \right) \dot{\alpha}_{,j}(\mathbf{x}, t) - \tau_T k_{ij}(\mathbf{x}) \ddot{\alpha}_{,j}(\mathbf{x}, t), \quad (2)$$

for all $(\mathbf{x}, t) \in B \times (0, \infty)$.

The three-phase-lag model for a rigid conductor as proposed by Roy Choudhuri [10] is based on the constitutive [equation \(2\)](#) and the well-known heat equation:

$$-q_{i,i}(\mathbf{x}, t) + R(\mathbf{x}, t) = c(\mathbf{x}) \dot{T}(\mathbf{x}, t), \quad (\mathbf{x}, t) \in B \times (0, \infty). \quad (3)$$

In the above equations we have used the following notations: q_i are the components of the heat flux vector, T is the temperature variation from the constant absolute reference temperature $T_0 > 0$, α is the thermal displacement defined by (see e.g., [6–8])

$$\alpha(\mathbf{x}, t) = \int_0^t T(\mathbf{x}, z) dz, \quad (4)$$

k_{ij} are the components of the thermal conductivity tensor, k_{ij}^* are the components of the conductivity rate tensor, R represents the heat source acting per unit volume, and c is the specific heat per unit volume. Moreover, the delay time $\tau_q > 0$ is the phase-lag of the heat flux, the delay time $\tau_T > 0$ is the phase-lag of the temperature gradient and $\tau_\alpha > 0$ is the phase-lag of the thermal displacement gradient.

In terms of the thermal displacement α , the three-phase-lag model for a rigid conductor as proposed by Roy Choudhuri [10] is based upon the following differential equation:

$$c \tau_q \ddot{\alpha} + c \ddot{\alpha} - \left(k_{ij}^* \alpha_{,j} \right)_{,i} - \left[\left(k_{ij} + \tau_\alpha k_{ij}^* \right) \dot{\alpha}_{,j} \right]_{,i} - \tau_T (k_{ij} \ddot{\alpha}_{,j})_{,i} = R + \tau_q \dot{R}, \quad (5)$$

where we assumed that the dependence on the independent variables $\mathbf{x}$ and t was suppressed, but implicitly understood.

We have to mention that Chiriță and D'Apice [44] have shown that the constitutive equation (2) is compatible with the second law of thermodynamics if:

(R(i)) the tensor k_{ij} is positive semi-definite;

(R(ii)) the tensor $\kappa_{ij} = k_{ij} + (\tau_\alpha - \tau_q) k_{ij}^*$ is positive semi-definite.

Remark 1. In the particular case of an isotropic and homogeneous rigid conductor, for which $k_{ij}^* = k^* \delta_{ij}$, $k_{ij} = k \delta_{ij}$, $\tau_q = \tau$ and $\tau_T = \tau_\alpha = 0$, the differential equation (5) reduces to the well-known Moore–Gibson–Thompson equation (see e.g., Quintanilla [45])

$$c(\tau \ddot{\alpha} + \ddot{\alpha}) - k^* \Delta \alpha - k \Delta \dot{\alpha} = 0. \quad (6)$$

Results on the continuous dependence and convergence with respect to different parameters in the Moore–Gibson–Thompson heat equation (6) have been established recently by Pellicer and Quintanilla [46].

3. Statement of the initial boundary value problem and some constitutive hypotheses

Throughout this paper, we consider the initial boundary value problem $\mathcal{P}$ defined by the constitutive equation (2) and the energy equation (3), the initial conditions

$$\alpha(\mathbf{x}, 0) = 0, \quad \dot{\alpha}(\mathbf{x}, 0) = \dot{\alpha}^0(\mathbf{x}), \quad q_i(\mathbf{x}, 0) = q_i^0(\mathbf{x}), \quad \text{for all } \mathbf{x} \in \overline{B}, \quad (7)$$

and the following boundary condition

$$\alpha(\mathbf{x}, t) = A(\mathbf{x}, t) \quad \text{for all } (\mathbf{x}, t) \in \partial B \times (0, \infty), \quad (8)$$

where $\dot{\alpha}^0(\mathbf{x})$, $q_i^0(\mathbf{x})$ and $A(\mathbf{x}, t)$ are prescribed smooth functions.

Remark 2. It should be noted that the prescribed initial condition $q_i(\mathbf{x}, 0) = q_i^0(\mathbf{x})$, $\mathbf{x} \in \overline{B}$, allows us to prescribe the initial condition for $\ddot{\alpha}(\mathbf{x}, 0)$ as determined from relation (3) as $c(\mathbf{x})\ddot{\alpha}(\mathbf{x}, 0) = R(\mathbf{x}, 0) - q_{i,i}^0(\mathbf{x})$, and this is useful when referring to the differential equation (5). In particular, in the absence of heat supply ($R(\mathbf{x}, t) = 0$), the initial conditions $q_i(\mathbf{x}, 0) = 0$, $\mathbf{x} \in \overline{B}$, implies that $\ddot{\alpha}(\mathbf{x}, 0) = 0$, $\mathbf{x} \in \overline{B}$.

By a solution of the initial boundary value problem $\mathcal{P}$, corresponding to the external given data $\mathcal{D} = \{R; \dot{\alpha}^0, q_i^0; A\}$ and to the material characteristics $\mathcal{C} = \{c, k_{ij}, k_{rs}^*; \tau_q, \tau_\alpha, \tau_T\}$, we mean the ordered array $\mathcal{S}_C = \{\alpha, q_i\}$ defined on $B \times (0, \infty)$ with the properties that $\alpha(\mathbf{x}, t) \in C^{2 \times 2}(B \times (0, \infty))$, $q_i(\mathbf{x}, t) \in C^{1 \times 1}(B \times (0, \infty))$ and which satisfy the field equations (2) and (3), the initial conditions (7), and the boundary condition (8). Otherwise, when we refer to the three-phase-lag thermal conductor described by equation (5) we will understand by the solution of the problem $\mathcal{P}$ a function $\mathcal{S}_C = \{\alpha(\mathbf{x}, t)\}$ defined on $B \times (0, \infty)$ for which $\alpha(\mathbf{x}, t) \in C^{2 \times 3}(B \times (0, \infty))$ and which satisfies the differential equation (5), the boundary condition (8) and the initial conditions (8)_{1,2} and $\ddot{\alpha}(\mathbf{x}, 0) = \ddot{\alpha}^0(\mathbf{x})$, $\mathbf{x} \in \overline{B}$.

In order to be able to conveniently deal with topics relating the structural stability regarding the continuous dependence of solutions with respect to characteristic material coefficients, it is useful to define a modified initial boundary value problem $\tilde{\mathcal{P}}$ associated with the problem in question $\mathcal{P}$. In this sense we introduce the notations

$$\tilde{q}_i = q_i + \tau_q \dot{q}_i, \quad \tilde{\alpha} = \alpha + \tau_q \dot{\alpha}, \quad (9)$$

and note that the heat equation (3) implies

$$-\tilde{q}_{i,i} + \tilde{R} = c \ddot{\alpha} \quad \text{in } B \times (0, \infty), \quad (10)$$

while the constitutive equation (2) becomes

$$\tilde{q}_i = -k_{ij}^* \alpha_{,j} - (k_{ij} + \tau_\alpha k_{ij}^*) \dot{\alpha}_{,j} - \tau_T k_{ij} \ddot{\alpha}_{,j} \quad \text{in } B \times (0, \infty), \quad (11)$$

where

$$\tilde{R} = R + \tau_q \dot{R}. \quad (12)$$

Furthermore, in view of the initial conditions (7) and the heat [equation \(3\)](#) and the constitutive [equation \(11\)](#), we have

$$\begin{aligned} \tilde{\alpha}(\mathbf{x}, 0) &= \tau_q \dot{\alpha}^0(\mathbf{x}), \quad \dot{\tilde{\alpha}}(\mathbf{x}, 0) = \dot{\alpha}^0(\mathbf{x}) + \tau_q \ddot{\alpha}(\mathbf{x}, 0), \\ \tilde{q}_i(\mathbf{x}, 0) &= -(k_{ij} + \tau_\alpha k_{ij}^*) \dot{\alpha}_{,j}^0(\mathbf{x}) - \tau_T k_{ij} \ddot{\alpha}_{,j}(\mathbf{x}, 0), \\ c(\mathbf{x}) \ddot{\alpha}(\mathbf{x}, 0) &= R(\mathbf{x}, 0) - q_{i,i}^0(\mathbf{x}), \end{aligned} \quad (13)$$

while the boundary condition (8) furnishes

$$\tilde{\alpha}(\mathbf{x}, t) = \tilde{A}(\mathbf{x}, t) \quad \text{on} \quad \partial B \times (0, \infty), \quad (14)$$

with $\tilde{A} = A + \tau_q \dot{A}$.

It is worth noting for later that the constitutive [equation \(11\)](#) can be written in the following form

$$\tilde{q}_i = -k_{ij}^* \tilde{\alpha}_{,j} - \kappa_{ij} \dot{\tilde{\alpha}}_{,j} - \tau_T k_{ij} \ddot{\tilde{\alpha}}_{,j}. \quad (15)$$

Further we note that the relations (9) and (15) imply

$$\begin{aligned} \int_B \dot{\tilde{\alpha}}_{,i} \tilde{q}_i dv &= -\frac{1}{2} \frac{d}{dt} \int_B \left(k_{ij}^* \tilde{\alpha}_{,i} \tilde{\alpha}_{,j} + \tau_q \kappa_{ij} \dot{\tilde{\alpha}}_{,i} \dot{\tilde{\alpha}}_{,j} + \tau_T k_{ij} \dot{\tilde{\alpha}}_{,i} \dot{\tilde{\alpha}}_{,j} \right) dv \\ &\quad - \int_B \left(\kappa_{ij} \dot{\tilde{\alpha}}_{,i} \dot{\tilde{\alpha}}_{,j} + \tau_q \tau_T k_{ij} \ddot{\tilde{\alpha}}_{,i} \ddot{\tilde{\alpha}}_{,j} \right) dv. \end{aligned} \quad (16)$$

Moreover, we introduce

$$\begin{aligned} \mathcal{E}[\mathcal{S}_C](t) &= \frac{1}{2} \int_B \left[c \dot{\tilde{\alpha}}(t)^2 + k_{ij}^* \tilde{\alpha}_{,i}(t) \tilde{\alpha}_{,j}(t) + (\tau_q \kappa_{ij} + \tau_T k_{ij}) \dot{\tilde{\alpha}}_{,i}(t) \dot{\tilde{\alpha}}_{,j}(t) \right] dv \\ &\quad + \int_0^t \int_B \left[\kappa_{ij} \dot{\tilde{\alpha}}_{,i}(s) \dot{\tilde{\alpha}}_{,j}(s) + \tau_q \tau_T k_{ij} \ddot{\tilde{\alpha}}_{,i}(s) \ddot{\tilde{\alpha}}_{,j}(s) \right] dv ds. \end{aligned} \quad (17)$$

To study the structural stability of the three-phase-lag model of the rigid conductor we need a mathematical relationship that expresses the continuous dependence of the solutions with respect to the variations (measurement errors) of the thermal characteristic coefficients of the material and the characteristic relaxation times. Thus, we will establish a quantitative estimate of the solution to the initial boundary value problem $\mathcal{P}$ in terms of the external data of this problem (initial and boundary data, as well as the supply loads). In light of this challenge and in accordance with the thermodynamic constraints (R(i)) and (R(ii)) we have to make the following assumptions on the material characteristics $\mathcal{C} = \{c, k_{ij}, k_{rs}^*, \tau_q, \tau_\alpha, \tau_T\}$:

(H1) the specific heat per unit volume c is strictly positive and

$$0 < c_m \leq c \leq c_M, \quad (c_m, c_M \text{ prescribed constants}), \quad (18)$$

(H2) the conductivity tensor is positive definite and

$$0 < k_m \xi_i \xi_i \leq k_{ij} \xi_i \xi_j \leq k_M \xi_i \xi_i, \quad (\xi_1, \xi_2, \xi_3) \neq 0, \quad (k_m, k_M \text{ prescribed constants}); \quad (19)$$

(H3) the conductivity rate tensor is positive definite and

$$0 < k_m^* \xi_i \xi_i \leq k_{ij}^* \xi_i \xi_j \leq k_M^* \xi_i \xi_i, \quad (\xi_1, \xi_2, \xi_3) \neq 0, \quad (k_m^*, k_M^* \text{ prescribed constants}), \quad (20)$$

(H4) κ_{ij} is a positive definite tensor and

$$0 < \kappa_m \xi_i \xi_i \leq \kappa_{ij} \xi_i \xi_j \leq \kappa_M \xi_i \xi_i, \quad (\xi_1, \xi_2, \xi_3) \neq 0, \quad (\kappa_m, \kappa_M \text{ prescribed constants}). \quad (21)$$

It can be easily seen that these assumptions cause $\mathcal{E}[\mathcal{S}_C](t)$ to be viewed as a measure of the solutions. In fact, we have

$$\begin{aligned} \mathcal{E}[\mathcal{S}_C](t) &\geq \frac{1}{2} \int_B \left[c_m \dot{\tilde{\alpha}}(t)^2 + k_m^* \tilde{\alpha}_i(t) \tilde{\alpha}_{,i}(t) + (\tau_q \kappa_m + \tau_T k_m) \dot{\alpha}_{,i}(t) \dot{\alpha}_{,i}(t) \right] dv \\ &\quad + \int_0^t \int_B \left[\kappa_m \dot{\alpha}_{,i}(s) \dot{\alpha}_{,i}(s) + \tau_q \tau_T k_m \ddot{\alpha}_{,i}(s) \ddot{\alpha}_{,i}(s) \right] dv ds. \end{aligned} \quad (22)$$

Furthermore, from the constitutive [equation \(15\)](#) and by using the Cauchy-Schwarz inequality and the assumptions [\(19\)](#) to [\(21\)](#), we can deduce that

$$[\tilde{q}_i(t) \tilde{q}_i(t)]^{1/2} \leq \left[k_M^* k_{ij}^* \tilde{\alpha}_i(t) \tilde{\alpha}_{,j}(t) \right]^{1/2} + \left[\kappa_M k_{ij} \dot{\alpha}_{,i}(t) \dot{\alpha}_{,j}(t) \right]^{1/2} + \tau_T \left[k_M k_{ij} \ddot{\alpha}_{,i}(t) \ddot{\alpha}_{,j}(t) \right]^{1/2}. \quad (23)$$

Remark 3. *Inequality (23) shows us that any estimate in terms of thermal displacement allows a corresponding estimate for the heat flux. This means that in all our analysis it is sufficient to establish estimates in terms of thermal displacement.*

4. An auxiliary elliptic boundary value problem and related estimates

In this section we will recall an elliptic boundary value problem associated with the region B and the boundary surface ∂B , which is assumed to be star-shaped with respect to the origin of the Cartesian coordinate system, such that the relation [\(1\)](#) is fulfilled. Therefore, we recall [\[36\]](#) and we consider the following Dirichlet boundary value problem $\mathcal{P}_0(\phi^*)$ in terms of the unknown ϕ , defined by

$$\begin{aligned} \Delta \phi &= 0 \quad \text{in } B, \\ \phi &= \phi^* \quad \text{on } \partial B, \end{aligned} \quad (24)$$

where Δ is the Laplace operator. We note that the existence of the solution ϕ of the problem $\mathcal{P}_0(\phi^*)$ is assured by means of the existence theory developed by Fichera [\[47\]](#).

We now multiply the relation $(24)_1$ by $x_k \phi_{,k}$ to obtain the identity

$$0 = \int_B x_k \phi_{,k} \Delta \phi dv = \frac{1}{2} \int_B \phi_{,i} \phi_{,i} dv + \int_{\partial B} x_k \phi_{,k} \phi_{,i} n_i da - \frac{1}{2} \int_{\partial B} x_k n_k \phi_{,i} \phi_{,i} da. \quad (25)$$

Furthermore, we use the decomposition $\nabla \phi = (\partial \phi / \partial n) \mathbf{n} + \nabla_t \phi \mathbf{t}$, on the boundary surface ∂B , where $\mathbf{n}$ is the outward unit normal vector to ∂B , $\mathbf{t}$ is a unit tangent vector to the boundary surface, ∇ is the gradient operator, $(\partial / \partial n)$ is the normal derivative and ∇_t is the tangential derivative. Thus, from relation [\(25\)](#) we get

$$\begin{aligned} &\frac{1}{2} \int_B \phi_{,i} \phi_{,i} dv + \frac{1}{2} \int_{\partial B} x_k n_k \left(\frac{\partial \phi}{\partial n} \right)^2 da \\ &= - \int_{\partial B} x_k t_k \frac{\partial \phi}{\partial n} \nabla_t \phi da + \frac{1}{2} \int_{\partial B} x_k n_k (\nabla_t \phi)^2 da, \end{aligned} \quad (26)$$

and hence, by using of the relation [\(1\)](#) and the Cauchy-Schwarz and arithmetic-geometric mean inequalities, we obtain the following estimate

$$\frac{1}{2} \int_B \phi_{,i} \phi_{,i} dv + \frac{h_0}{4} \int_{\partial B} \left(\frac{\partial \phi}{\partial n} \right)^2 da \leq \frac{1}{2} \int_{\partial B} \left(x_k n_k + \frac{2\delta_0^2}{h_0} \right) (\nabla_t \phi)^2 da. \quad (27)$$

Finally, we recall the Poincaré inequality

$$\lambda_1 \int_B \phi^2 dv \leq \int_B \phi_{,r} \phi_{,r} dv + C_1 \int_{\partial B} \phi^{*2} da, \quad (28)$$

with λ_1 and C_1 positive constants.

Consequently, we can see that the estimates (27) and (28) furnish *a priori* bounds for

$$\int_B \phi^2 dv, \quad \int_B \phi_{,i} \phi_{,i} dv, \quad \int_{\partial B} \left(\frac{\partial \phi}{\partial n} \right)^2 da, \quad (29)$$

in terms of the given data ϕ^* on the boundary surface ∂B .

5. A general estimate for the inhomogeneous Dirichlet problem $\mathcal{P}$

Let us first consider the boundary value problem $\mathcal{P}_0(\dot{\tilde{A}})$ and denote its solution by G . In view of the analysis of the above section, it results that

$$\frac{1}{2} \int_B G_{,r} G_{,r} dv + \frac{h_0}{4} \int_{\partial B} \left(\frac{\partial G}{\partial n} \right)^2 da \leq \frac{1}{2} \int_{\partial B} \left(x_k n_k + \frac{2\delta_0^2}{h_0} \right) (\nabla_t \dot{\tilde{A}})^2 da, \quad (30)$$

and

$$\lambda_1 \int_B G^2 dv \leq \int_B G_{,r} G_{,r} dv + C_1 \int_{\partial B} \dot{\tilde{A}}^2 da. \quad (31)$$

Then we multiply the equation (10) by $G - \dot{\tilde{\alpha}}$ to get

$$\int_B \left(\tilde{q}_{i,i} + c\ddot{\tilde{\alpha}} - \tilde{R} \right) (G - \dot{\tilde{\alpha}}) dv = 0, \quad (32)$$

so that, by means of the divergence theorem and the boundary condition $G - \dot{\tilde{\alpha}} = 0$ on $\partial B \times (0, \infty)$, we get

$$\int_B \left(c\dot{\tilde{\alpha}}\ddot{\tilde{\alpha}} - \tilde{q}_i \dot{\tilde{\alpha}}_{,i} \right) dv - \frac{d}{dt} \int_B c\dot{\tilde{\alpha}} G dv = - \int_B c\dot{\tilde{\alpha}} \dot{G} dv - \int_B \tilde{R} G dv + \int_B \tilde{R} \dot{\tilde{\alpha}} dv - \int_B \tilde{q}_i G_{,i} dv. \quad (33)$$

In view of the relations (16) and (17), from (33) we get

$$\begin{aligned} \mathcal{E}[\mathcal{S}_C](t) &= \int_B c\dot{\tilde{\alpha}}(t) G(t) dv - \int_0^t \int_B c\dot{\tilde{\alpha}}(s) \dot{G}(s) dv ds + \int_0^t \int_B \dot{\tilde{\alpha}}(s) \tilde{R}(s) dv ds \\ &\quad - \int_0^t \int_B \tilde{q}_i(s) G_{,i}(s) dv ds - \int_0^t \int_B \tilde{R}(s) G(s) dv ds + \mathcal{E}[\mathcal{S}_C](0) - \int_B c\dot{\tilde{\alpha}}(0) G(0) dv. \end{aligned} \quad (34)$$

Now we use the Cauchy-Schwarz and arithmetic-geometric mean inequalities in order to get

$$\left| \int_B c\dot{\tilde{\alpha}}(t) G(t) dv \right| \leq \frac{1}{4} \int_B c\dot{\tilde{\alpha}}(t)^2 dv + \int_B cG(t)^2 dv; \quad (35)$$

$$\left| \int_0^t \int_B c\dot{\tilde{\alpha}}(s) \dot{G}(s) dv ds \right| \leq \int_0^t \left(\int_B c\dot{\tilde{\alpha}}(s)^2 dv \right)^{1/2} \left(\int_B c\dot{G}(s)^2 dv \right)^{1/2} ds; \quad (36)$$

$$\left| \int_0^t \int_B \dot{\tilde{\alpha}}(s) \tilde{R}(s) dv ds \right| \leq \int_0^t \left(\int_B c\dot{\tilde{\alpha}}(s)^2 dv \right)^{1/2} \left(\int_B \frac{1}{c} \tilde{R}(s)^2 dv \right)^{1/2} ds. \quad (37)$$

Moreover, by using the estimate (23), we have

$$\begin{aligned}
& \left| \int_0^t \int_B \tilde{q}_i(s) G_{,i}(s) dv ds \right| \leq \int_0^t \left(\int_B \tilde{q}_i(s) \tilde{q}_i(s) dv \right)^{1/2} \left(\int_B G_{,r}(s) G_{,r}(s) dv \right)^{1/2} ds \\
& \leq \int_0^t \sqrt{3} \left[\left(\int_B \kappa_M^* \kappa_{ij}^* \tilde{\alpha}_{,i}(s) \tilde{\alpha}_{,j}(s) dv \right)^{1/2} + \left(\int_B \kappa_M \kappa_{ij} \dot{\alpha}_{,i}(s) \dot{\alpha}_{,j}(s) dv \right)^{1/2} \right] \\
& \quad \times \left(\int_B G_{,r}(s) G_{,r}(s) dv \right)^{1/2} ds + \sqrt{\frac{3\tau_T k_M}{\tau_q}} \int_0^t \left(\int_B \tau_q \tau_T k_{ij} \ddot{\alpha}_{,i}(s) \ddot{\alpha}_{,j}(s) dv \right)^{1/2} \\
& \quad \times \left(\int_B G_{,r}(s) G_{,r}(s) dv \right)^{1/2} ds,
\end{aligned} \tag{38}$$

and

$$\begin{aligned}
& \sqrt{\frac{3\tau_T k_M}{\tau_q}} \int_0^t \left(\int_B \tau_q \tau_T k_{ij} \ddot{\alpha}_{,i}(s) \ddot{\alpha}_{,j}(s) dv \right)^{1/2} \left(\int_B G_{,r}(s) G_{,r}(s) dv \right)^{1/2} ds \\
& \leq \frac{1}{2} \int_0^t \int_B \tau_q \tau_T k_{ij} \ddot{\alpha}_{,i}(s) \ddot{\alpha}_{,j}(s) dv ds + \frac{3\tau_T k_M}{2\tau_q} \int_0^t \int_B G_{,i}(s) G_{,i}(s) dv ds.
\end{aligned} \tag{39}$$

At this stage we introduce the function

$$\begin{aligned}
\mathcal{F}[\mathcal{S}_C](t) &= \frac{1}{2} \int_B \left[\frac{1}{2} c \dot{\alpha}(t)^2 + k_{ij}^* \tilde{\alpha}_{,i}(t) \tilde{\alpha}_{,j}(t) + (\tau_q \kappa_{ij} + \tau_T k_{ij}) \dot{\alpha}_{,i}(t) \dot{\alpha}_{,j}(t) \right] dv \\
& \quad + \int_0^t \int_B \left[\kappa_{ij} \dot{\alpha}_{,i}(s) \dot{\alpha}_{,j}(s) + \frac{1}{2} \tau_q \tau_T k_{ij} \ddot{\alpha}_{,i}(s) \ddot{\alpha}_{,j}(s) \right] dv ds, \quad t \in [0, t_0],
\end{aligned} \tag{40}$$

and note that the use of the estimates (35) to (39) in the identity (34) makes possible to obtain the following integral inequality

$$\mathcal{F}[\mathcal{S}_C](t) \leq \int_0^t \xi(s) \sqrt{\mathcal{F}[\mathcal{S}_C](s)} ds + \mathcal{G}, \quad t \in (0, t_0), \tag{41}$$

where

$$\begin{aligned}
\xi(t) &= 2 \left[\left(\int_B c \dot{G}(t)^2 dv \right)^{1/2} + \left(\int_B \frac{1}{c} \tilde{R}(t)^2 dv \right)^{1/2} \right] \\
& \quad + \left(\sqrt{6\kappa_M^*} + \sqrt{\frac{6\kappa_M}{\tau_q}} \right) \left(\int_B G_{,i}(t) G_{,i}(t) dv \right)^{1/2},
\end{aligned} \tag{42}$$

and

$$\begin{aligned}
\mathcal{G} &= \left| \int_0^{t_0} \int_B \tilde{R}(s) G(s) dv ds \right| + \left| \mathcal{E}[\mathcal{S}_C](0) - \int_B c \dot{\alpha}(0) G(0) dv \right| \\
& \quad + \max_{t \in [0, t_0]} \int_B c G(t)^2 dv + \frac{3\tau_T k_M}{2\tau_q} \int_0^{t_0} \int_B G_{,i}(s) G_{,i}(s) dv ds.
\end{aligned} \tag{43}$$

We now handle the integral inequality (41) to obtain the sought continuous dependence estimate. In this connection we introduce the function

$$\Lambda(t) = \sqrt{\int_0^t \xi(s) \sqrt{\mathcal{F}[\mathcal{S}_C](s)} ds} + \mathcal{G}, \quad t \in (0, t_0), \tag{44}$$

and note that we have

$$\sqrt{\mathcal{F}[\mathcal{S}_c]}(t) \leq \Lambda(t) \quad \text{for all } t \in (0, t_0). \quad (45)$$

Moreover, we get

$$2\Lambda(t)\dot{\Lambda}(t) = \xi(t)\sqrt{\mathcal{F}[\mathcal{S}_c]}(t) \leq \xi(t)\Lambda(t) \quad \text{for all } t \in (0, t_0), \quad (46)$$

so that we obtain

$$\Lambda(t) \leq \frac{1}{2} \int_0^t \xi(s) ds + \Lambda(0) \quad \text{for all } t \in (0, t_0). \quad (47)$$

Consequently, from the relations (45) and (47), we obtain the expected continuous dependence estimate

$$\sqrt{\mathcal{F}[\mathcal{S}_c]}(t) \leq \frac{1}{2} \int_0^t \xi(s) ds + \sqrt{\mathcal{G}} \quad \text{for all } t \in (0, t_0). \quad (48)$$

It is important to note that, based on the assumptions described in relations (18) to (21), it can be seen that [44], $\mathcal{F}[\mathcal{S}_c](t)$ represents a measure of the solution $\mathcal{S}_c = \{\alpha, q_r\}$ of the initial boundary value problem $\mathcal{P}$. On the other hand, through relations (42) and (43), we can see that the right member of the estimate (48) represents a measure of all data of the initial boundary value problem $\mathcal{P}$.

In view of the hypotheses described in (18) to (21) and by calling the relation (40), we can see that the inequality (48) furnishes *a priori* estimates for the following quantities

$$\int_B \dot{\alpha}_{,i} \dot{\alpha}_{,i} d\nu, \quad \int_0^t \int_B \ddot{\alpha}_{,i}(s) \ddot{\alpha}_{,i}(s) dv ds, \quad \int_B \dot{\tilde{\alpha}}^2 d\nu, \quad \int_B \tilde{\alpha}_{,i} \tilde{\alpha}_{,i} d\nu. \quad (49)$$

Furthermore, it is worth to outline that the above procedure allows us to obtain *a priori* bounds for

$$\int_B \dot{\alpha}^2 d\nu, \quad \int_0^t \int_B \ddot{\alpha}(s)^2 dv ds, \quad \int_B \tilde{\alpha}^2 d\nu. \quad (50)$$

6. Some mathematical estimates related to the variation of the delay times

In this section we study the structural stability of the mathematical model of the three-phase-lag heat conduction when the delay times undergo small variations. Generally, the thermal structural stability refers to the ability of a material or structure to maintain its integrity and performance under varying temperature conditions. This is crucial in many engineering applications, such as aerospace, automotive, and construction, where materials are often exposed to extreme temperatures.

In our context, the study of the structural stability of the three-phase-lag model consists in showing that the solutions of the initial boundary value problems depend continuously on the material characteristic coefficients (relaxation times and conductivity coefficients and specific heat), which may be subjected to measurement errors, perturbations in the mathematical modeling process, or even decay and degradation phenomena. This means that small variations in the characteristic coefficients entering into [equation \(5\)](#) and which do not affect the basic constitutive assumptions (H1) to (H4), lead to small variations in the solutions to the initial boundary value problem, that is, these small variations do not destroy the structure of the model.

The estimates of continuous dependence reported in this section can play a central role in obtaining numerical approximations to these kinds of problems.

6.1. Continuous dependence on the delay time of the temperature gradient

Throughout this subsection we investigate the continuous dependence of the solution of the initial boundary value problem $\mathcal{P}$ in the situation when all the characteristic coefficients of the model remain unchanged, only the delay time of the temperature gradient τ_T undergo a variation. That means we consider two material characteristics $\mathcal{C}_1^{(\beta)} = \{c, k_{ij}, k_{rs}^*; \tau_q, \tau_\alpha, \tau_T^{(\beta)}\}$, $\beta = 1, 2$, satisfying the constitutive assumptions (18) to (21) and we consider that $\mathcal{S}_{\mathcal{C}_1}^{(\beta)} = \{\alpha^{(\beta)}, q_r^{(\beta)}\}$, $\beta = 1, 2$, are the corresponding solutions of the initial boundary value problem $\mathcal{P}$ under the same initial, boundary and body loads data, and to the same material characteristics excepting the relaxation time τ_T . We should note that the relaxation time τ_T does not affect the thermodynamic compatibility conditions (R(i)) and (R(ii)).

If we set

$$\begin{aligned} \mathcal{S}_{\mathcal{C}_1} &= \mathcal{S}_{\mathcal{C}_1^{(1)}} - \mathcal{S}_{\mathcal{C}_1^{(2)}} = \{\alpha^{(1)} - \alpha^{(2)}, q_i^{(1)} - q_i^{(2)}\} = \{\alpha, q_i\} \\ \tau_T &= \tau_T^{(1)} - \tau_T^{(2)}, \end{aligned} \quad (51)$$

then the following initial conditions

$$\alpha(\mathbf{x}, 0) = 0, \quad \dot{\alpha}(\mathbf{x}, 0) = 0, \quad q_i(\mathbf{x}, 0) = 0, \quad (\text{or } \ddot{\alpha}(\mathbf{x}, 0) = 0), \quad \text{for all } \mathbf{x} \in \bar{B}, \quad (52)$$

and the boundary condition

$$\alpha(\mathbf{x}, t) = 0, \quad \text{on } \partial B \times (0, \infty), \quad (53)$$

are fulfilled.

In terms of the difference thermal displacement $\alpha = \alpha^{(1)} - \alpha^{(2)}$, the basic differential equation is

$$c\ddot{\alpha} - \left(k_{ij}^* \ddot{\alpha}_{,j}\right)_{,i} - (\kappa_{ij} \dot{\alpha}_{,j})_{,i} - \tau_T^{(1)} (k_{ij} \ddot{\alpha}_{,j})_{,i} = \tau_T \left(k_{ij} \ddot{\alpha}_{,j}\right)_{,i}. \quad (54)$$

We now multiply equation (54) by $\dot{\alpha}$, then integrate result over $B \times (0, t)$ and finally use the initial conditions (52) and the boundary condition (53), to find

$$\mathcal{E}[\mathcal{S}_{\mathcal{C}_1^{(1)}}](t) = -\tau_T \int_0^t \int_B \dot{\alpha}_{,i} k_{ij} \ddot{\alpha}_{,j}^{(2)} dv ds - \tau_q \tau_T \int_0^t \int_B \ddot{\alpha}_{,i} k_{ij} \ddot{\alpha}_{,j}^{(2)} dv ds, \quad (55)$$

where

$$\begin{aligned} \mathcal{E}[\mathcal{S}_{\mathcal{C}_1^{(1)}}](t) &= \frac{1}{2} \int_B \left[c\dot{\alpha}^2(t) + k_{ij}^* \ddot{\alpha}_{,i}(t) \ddot{\alpha}_{,j}(t) + \left(\tau_q \kappa_{ij} + \tau_T^{(1)} k_{ij} \right) \dot{\alpha}_{,i}(t) \dot{\alpha}_{,j}(t) \right] dv \\ &+ \int_0^t \int_B \left[\kappa_{ij} \dot{\alpha}_{,i}(s) \dot{\alpha}_{,j}(s) + \tau_q \tau_T^{(1)} k_{ij} \ddot{\alpha}_{,i}(s) \ddot{\alpha}_{,j}(s) \right] dv ds. \end{aligned} \quad (56)$$

By applying the Cauchy-Schwarz and arithmetic-geometric mean inequalities, we have

$$\begin{aligned} \left| \tau_T \int_0^t \int_B \dot{\alpha}_{,i} k_{ij} \ddot{\alpha}_{,j}^{(2)} dv ds \right| &\leq |\tau_T| \int_0^t \left(\int_B k_{ij} \dot{\alpha}_{,i} \dot{\alpha}_{,j} dv \right)^{1/2} \left(\int_B k_{ij} \ddot{\alpha}_{,i}^{(2)} \ddot{\alpha}_{,j}^{(2)} dv \right)^{1/2} ds \\ &\leq \frac{|\tau_T|}{\sqrt{\tau_T^{(1)}}} \int_0^t \left(\int_B \tau_T^{(1)} k_{ij} \dot{\alpha}_{,i} \dot{\alpha}_{,j} dv \right)^{1/2} \left(\int_B k_{ij} \ddot{\alpha}_{,i}^{(2)} \ddot{\alpha}_{,j}^{(2)} dv \right)^{1/2} ds, \end{aligned} \quad (57)$$

and

$$\begin{aligned} \left| \tau_q \tau_T \int_0^t \int_B \ddot{\alpha}_{,i} k_{ij} \ddot{\alpha}_{,j}^{(2)} dv ds \right| &\leq \tau_q |\tau_T| \left(\int_0^t \int_B k_{ij} \ddot{\alpha}_{,i} \ddot{\alpha}_{,j} dv ds \right)^{1/2} \left(\int_0^t \int_B k_{ij} \ddot{\alpha}_{,i}^{(2)} \ddot{\alpha}_{,j}^{(2)} dv ds \right)^{1/2} \\ &\leq \frac{1}{2} \int_0^t \int_B \tau_q \tau_T^{(1)} k_{ij} \ddot{\alpha}_{,i} \ddot{\alpha}_{,j} dv ds + \frac{\tau_T^2 \tau_q}{2 \tau_T^{(1)}} \int_0^t \int_B k_{ij} \ddot{\alpha}_{,i}^{(2)} \ddot{\alpha}_{,j}^{(2)} dv ds. \end{aligned} \quad (58)$$

Now we use the estimates (57) and (58) into identity (55) to obtain the following integral inequality

$$\mathcal{F}_1[\mathcal{S}_{c_1^{(1)}}](t) \leq |\tau_T| \int_0^t \xi_1(s) \sqrt{\mathcal{F}_1[\mathcal{S}_{c_1^{(1)}}](s)} ds + \tau_T^2 \mathcal{G}_1, \quad t \in (0, t_0), \quad (59)$$

where

$$\mathcal{F}_1[\mathcal{S}_{c_1^{(1)}}](t) = \mathcal{E}[\mathcal{S}_{c_1^{(1)}}](t) - \frac{1}{2} \int_0^t \int_B \tau_q \tau_T^{(1)} k_{ij} \ddot{\alpha}_{,i} \ddot{\alpha}_{,j} dv ds, \quad (60)$$

and

$$\begin{aligned} \xi_1(t) &= \sqrt{\frac{2}{\tau_T^{(1)}}} \left(\int_B k_{ij} \ddot{\alpha}_{,i}^{(2)}(t) \ddot{\alpha}_{,j}^{(2)}(t) dv \right)^{1/2}, \\ \mathcal{G}_1 &= \frac{\tau_q}{2\tau_T^{(1)}} \int_0^{t_0} \int_B k_{ij} \ddot{\alpha}_{,i}^{(2)} \ddot{\alpha}_{,j}^{(2)} dv ds. \end{aligned} \quad (61)$$

Furthermore, we can treat this inequality by the same procedure like that for (41), to obtain

$$\sqrt{\mathcal{F}_1[\mathcal{S}_{c_1^{(1)}}](t)} \leq |\tau_T| \left| \left(\frac{1}{2} \int_0^t \xi_1(s) ds + \sqrt{\mathcal{G}_1} \right) \right|, \quad t \in (0, t_0). \quad (62)$$

By virtue of the results described in (49) and (50), as well as those described by the relation (61), we can say that the term in the parentheses on the right-hand side is bounded, so the inequality (62) shows that small variations τ_T of the relaxation time for the temperature gradient lead to small variations of the thermal displacement as measured by $\mathcal{F}_1[\mathcal{S}_{c_1^{(1)}}](t)$.

6.2. Continuous dependence on the phase-lag of the thermal displacement

We proceed now to investigate the continuous dependence of the solution of the initial boundary value problem $\mathcal{P}$ in the situation when all the material characteristic coefficients of the three-phase-lag model remain unchanged, excepting the delay time τ_α for the thermal displacement, which undergo a variation.

Let $\mathcal{S}_{c_2^{(\beta)}} = \{\alpha^{(\beta)}, q_i^{(\beta)}\}$, $\beta = 1, 2$, be the solution of the initial boundary value problem $\mathcal{P}$ corresponding to the same initial, boundary and body loads data, and to the same material characteristics excepting the delay time for thermal displacement: $\mathcal{C}_2^{(\beta)} = \{c, k_{ij}, k_{rs}^*, \tau_q, \tau_\alpha^{(\beta)}, \tau_T\}$, $\beta = 1, 2$, satisfying the constitutive assumptions (18) to (21). Then the difference $\alpha = \alpha^{(1)} - \alpha^{(2)}$ satisfies the initial conditions described by (52) and the boundary condition (53) and the following differential equation

$$c \ddot{\alpha} - \left(k_{ij}^* \ddot{\alpha}_{,j} \right)_{,i} - \left(\kappa_{ij}^{(1)} \dot{\alpha}_{,j} \right)_{,i} - \tau_T (k_{ij} \ddot{\alpha}_{,j})_{,i} = \tau_\alpha \left(k_{ij}^* \dot{\alpha}_{,j}^{(2)} \right)_{,i}, \quad (63)$$

with

$$\kappa_{ij}^{(1)} = k_{ij} + \left(\tau_\alpha^{(1)} - \tau_q \right) k_{ij}^*, \quad \tau_\alpha = \tau_\alpha^{(1)} - \tau_\alpha^{(2)}. \quad (64)$$

In view of the initial conditions (52) and the boundary condition (53), we have

$$\mathcal{E}[\mathcal{S}_{c_2^{(1)}}](t) = -\tau_\alpha \int_0^t \int_B \dot{\alpha}_{,i} k_{ij}^* \dot{\alpha}_{,j}^{(2)} dv ds - \tau_q \tau_\alpha \int_0^t \int_B \ddot{\alpha}_{,i} k_{ij}^* \dot{\alpha}_{,j}^{(2)} dv ds, \quad (65)$$

where

$$\begin{aligned} \mathcal{E}[\mathcal{S}_{\mathcal{C}_2^{(1)}}](t) &= \frac{1}{2} \int_B \left[c\dot{\alpha}(t)^2 + k_{ij}^* \tilde{\alpha}_i(t) \tilde{\alpha}_j(t) + \left(\tau_q \kappa_{ij}^{(1)} + \tau_T k_{ij} \right) \dot{\alpha}_i(t) \dot{\alpha}_j(t) \right] dv \\ &+ \int_0^t \int_B \left[\kappa_{ij}^{(1)} \dot{\alpha}_i(s) \dot{\alpha}_j(s) + \tau_q \tau_T k_{ij} \ddot{\alpha}_i(s) \ddot{\alpha}_j(s) \right] dv ds. \end{aligned} \quad (66)$$

By applying the Cauchy-Schwarz and arithmetic-geometric mean inequalities, we have

$$\left| \tau_\alpha \int_0^t \int_B \dot{\alpha}_i k_{ij}^* \dot{\alpha}_j^{(2)} dv ds \right| \leq |\tau_\alpha| \sqrt{\frac{k_M^*}{\tau_T k_m}} \int_0^t \left(\int_B \tau_T k_{ij} \dot{\alpha}_i \dot{\alpha}_j dv \right)^{1/2} \left(\int_B k_{ij}^* \dot{\alpha}_i^{(2)} \dot{\alpha}_j^{(2)} dv \right)^{1/2} ds, \quad (67)$$

and

$$\begin{aligned} \left| \tau_q \tau_\alpha \int_0^t \int_B \ddot{\alpha}_i k_{ij}^* \dot{\alpha}_j^{(2)} dv ds \right| &\leq \frac{1}{2} \int_0^t \int_B \tau_q \tau_T k_{ij} \ddot{\alpha}_i \ddot{\alpha}_j dv ds \\ &+ \frac{\tau_\alpha^2 \tau_q k_M^*}{2 \tau_T k_m} \int_0^t \int_B k_{ij}^* \dot{\alpha}_i^{(2)} \dot{\alpha}_j^{(2)} dv ds. \end{aligned} \quad (68)$$

Therefore, we obtain the following integral inequality

$$\mathcal{F}_2[\mathcal{S}_{\mathcal{C}_2^{(1)}}](t) \leq |\tau_\alpha| \int_0^t \xi_2(s) \sqrt{\mathcal{F}_2[\mathcal{S}_{\mathcal{C}_2^{(1)}}](s)} ds + \tau_\alpha^2 \mathcal{G}_2, \quad t \in (0, t_0), \quad (69)$$

where

$$\mathcal{F}_2[\mathcal{S}_{\mathcal{C}_2^{(1)}}](t) = \mathcal{E}[\mathcal{S}_{\mathcal{C}_2^{(1)}}](t) - \frac{1}{2} \int_0^t \int_B \tau_q \tau_T k_{ij} \ddot{\alpha}_i \ddot{\alpha}_j dv ds, \quad (70)$$

and

$$\begin{aligned} \xi_2(t) &= \sqrt{\frac{2k_M^*}{\tau_T k_m}} \left(\int_B k_{ij}^* \dot{\alpha}_i^{(2)}(t) \dot{\alpha}_j^{(2)}(t) dv \right)^{1/2}, \\ \mathcal{G}_2 &= \frac{\tau_q k_M^*}{2 \tau_T k_m} \int_0^{t_0} \int_B k_{ij}^* \dot{\alpha}_i^{(2)} \dot{\alpha}_j^{(2)} dv ds. \end{aligned} \quad (71)$$

Concluding, we obtain the estimate

$$\sqrt{\mathcal{F}_2[\mathcal{S}_{\mathcal{C}_2^{(1)}}](t)} \leq |\tau_\alpha| \left(\frac{1}{2} \int_0^t \xi_2(s) ds + \sqrt{\mathcal{G}_2} \right), \quad t \in (0, t_0), \quad (72)$$

which describes the continuous dependence of solution with respect to the variation τ_α of the delay time for the thermal displacement.

6.3. Continuous dependence on the phase-lag of the heat flux

In this subsection we consider the following two material characteristics $\mathcal{C}_3^{(\beta)} = \{c, k_{ij}, k_{rs}^*; \tau_q^{(\beta)}, \tau_\alpha, \tau_T\}$, $\beta = 1, 2$ satisfying the constitutive assumptions (18) to (21). Then the difference between the solutions $\mathcal{S}_{\mathcal{C}_3^{(\beta)}} = \{\alpha^{(\beta)}, q_i^{(\beta)}\}$, $\beta = 1, 2$, of the initial boundary value problem $\mathcal{P}$ corresponding to the same initial, boundary and body loads data, that is $\alpha = \alpha^{(1)} - \alpha^{(2)}$, satisfies the null initial and boundary conditions and the following differential equation

$$c\ddot{\alpha} - \left(k_{ij}^* \tilde{\alpha}_j \right)_{,i} - \left(\kappa_{ij}^{(1)} \dot{\alpha}_j \right)_{,i} - \tau_T (k_{ij} \ddot{\alpha}_j)_{,i} = -c \tau_q \ddot{\alpha}^{(2)}, \quad (73)$$

where now we have set

$$\tilde{\alpha} = \alpha + \tau_q^{(1)} \dot{\alpha}, \quad \kappa_{ij}^{(1)} = k_{ij} + \left(\tau_\alpha - \tau_q^{(1)} \right) k_{ij}^*, \quad \tau_q = \tau_q^{(1)} - \tau_q^{(2)}. \quad (74)$$

Furthermore, by taking into account null initial and boundary conditions, from (73) we obtain

$$\mathcal{E}[\mathcal{S}_{\mathcal{C}_3^{(1)}}](t) = -\tau_q \int_0^t \int_B c \ddot{\alpha} \ddot{\alpha}^{(2)} dv ds, \quad (75)$$

where

$$\begin{aligned} \mathcal{E}[\mathcal{S}_{\mathcal{C}_3^{(1)}}](t) &= \frac{1}{2} \int_B \left[c \dot{\alpha}(t)^2 + k_{ij}^* \tilde{\alpha}_{,i}(t) \tilde{\alpha}_{,j}(t) + \left(\tau_q^{(1)} \kappa_{ij}^{(1)} + \tau_T k_{ij} \right) \dot{\alpha}_{,i}(t) \dot{\alpha}_{,j}(t) \right] dv \\ &+ \int_0^t \int_B \left[\kappa_{ij}^{(1)} \dot{\alpha}_{,i}(s) \dot{\alpha}_{,j}(s) + \tau_q^{(1)} \tau_T k_{ij} \ddot{\alpha}_{,i}(s) \ddot{\alpha}_{,j}(s) \right] dv ds. \end{aligned} \quad (76)$$

From the identity (75), we deduce the following integral inequality

$$\mathcal{E}[\mathcal{S}_{\mathcal{C}_3^{(1)}}](t) \leq |\tau_q| \int_0^t \xi_3(s) \sqrt{\mathcal{E}[\mathcal{S}_{\mathcal{C}_3^{(1)}}](s)} ds, \quad t \in (0, t_0), \quad (77)$$

where

$$\xi_3(t) = \left(2 \int_B c \ddot{\alpha} \ddot{\alpha}^{(2)}(t)^2 dv \right)^{1/2}, \quad (78)$$

so that we get the following estimate

$$\sqrt{\mathcal{E}[\mathcal{S}_{\mathcal{C}_3^{(1)}}](t)} \leq \frac{|\tau_q|}{2} \int_0^t \xi_3(s) ds, \quad t \in (0, t_0). \quad (79)$$

It is worth to outline that the procedure of Section 5 allows us to obtain a priori bounds for

$$\int_0^t \int_B \ddot{\alpha}^2 dv ds, \quad (80)$$

in terms of the given data. In fact, by direct differentiation with respect to time variable in (5), it follows that $\dot{\alpha}$ satisfies a differential equation like (5) with the body loads $\dot{R} + \tau_q \ddot{R}$, with the boundary data $\dot{A}$ and the initial conditions $\dot{\alpha}(\mathbf{x}, 0) = \dot{\alpha}^0(\mathbf{x})$, $\ddot{\alpha}(\mathbf{x}, 0) = \ddot{\alpha}^0(\mathbf{x})$ and $\ddot{\alpha}(\mathbf{x}, 0) = \ddot{\alpha}^0(\mathbf{x})$ as result calculated from the relation (13)₃ and from the equation (5) when we set $t \rightarrow 0$. Consequently, the inequality (79) provide an estimate describing the continuous dependence of α with respect to the delay time τ_q .

7. Mathematical estimates related to the other material characteristics

In this section we will study the continuous dependence of the solution of the initial boundary value problem $\mathcal{P}$ with respect to the conductivity rate tensor, with respect to the conductivity tensor, and with respect to the specific heat coefficient.

7.1. Continuous dependence on the conductivity rate tensor

Consider two sets of material characteristics $\mathcal{C}_4^{(\beta)} = \{c, k_{ij}, k_{rs}^{*(\beta)}, \tau_q, \tau_\alpha, \tau_T\}$, $\beta = 1, 2$, satisfying the constitutive assumptions (18) to (21). Let $\alpha^{(1)}$ and $\alpha^{(2)}$ be the corresponding solutions of the initial boundary value problem $\mathcal{P}$ with the same external given data. Then the difference $\alpha = \alpha^{(1)} - \alpha^{(2)}$ satisfies the following differential equation

$$c \ddot{\alpha} - \left(k_{ij}^{*(1)} \tilde{\alpha}_{,j} \right)_{,i} - \left(\kappa_{ij}^{(1)} \dot{\alpha}_{,j} \right)_{,i} - \tau_T (k_{ij} \ddot{\alpha}_{,j})_{,i} = \left[k_{ij}^* \left(\alpha_{,j}^{(2)} + \tau_\alpha \dot{\alpha}_{,j}^{(2)} \right) \right]_{,i}, \quad (81)$$

with null external given data and now we have set

$$k_{ij}^* = k_{ij}^{*(1)} - k_{ij}^{*(2)}, \quad \kappa_{ij}^{(1)} = k_{ij} + (\tau_\alpha - \tau_q) k_{ij}^{*(1)}. \quad (82)$$

Therefore, from the differential [equation \(81\)](#) we obtain the following identity

$$\begin{aligned} \mathcal{E}[\mathcal{S}_{\mathcal{C}_4^{(1)}}](t) &= - \int_0^t \int_B k_{ij}^* \dot{\alpha}_{,i} \left(\alpha_{,j}^{(2)} + \tau_\alpha \dot{\alpha}_{,j}^{(2)} \right) dv ds \\ &\quad - \tau_q \int_0^t \int_B k_{ij}^* \ddot{\alpha}_{,i} \left(\alpha_{,j}^{(2)} + \tau_\alpha \dot{\alpha}_{,j}^{(2)} \right) dv ds = I_1 + I_2, \end{aligned} \quad (83)$$

where

$$\begin{aligned} \mathcal{E}[\mathcal{S}_{\mathcal{C}_4^{(1)}}](t) &= \frac{1}{2} \int_B \left[c \dot{\alpha}(t)^2 + k_{ij}^{*(1)} \tilde{\alpha}_{,i}(t) \tilde{\alpha}_{,j}(t) + \left(\tau_q \kappa_{ij}^{(1)} + \tau_T k_{ij} \right) \dot{\alpha}_{,i}(t) \dot{\alpha}_{,j}(t) \right] dv \\ &\quad + \int_0^t \int_B \left[\kappa_{ij}^{(1)} \dot{\alpha}_{,i}(s) \dot{\alpha}_{,j}(s) + \tau_q \tau_T k_{ij} \ddot{\alpha}_{,i}(s) \ddot{\alpha}_{,j}(s) \right] dv ds. \end{aligned} \quad (84)$$

Now we use the inequality

$$\left| k_{ij}^* a_i b_j \right| \leq \left(k_{ij}^* k_{ij}^* \right)^{1/2} (a_r b_s a_r b_s)^{1/2} \leq K^* (a_r a_r)^{1/2} (b_s b_s)^{1/2}, \quad \text{for all } a_r, b_s, \quad (85)$$

with

$$K^* = \max_{\mathbf{x} \in \overline{B}} \left(k_{ij}^* k_{ij}^* \right)^{1/2}, \quad (86)$$

and then we combine it with the Cauchy-Schwarz and arithmetic-geometric mean inequalities in order to get

$$\begin{aligned} |I_1| &\leq K^* \int_0^t \zeta_1(s) \left[\frac{1}{2} \int_B \left(\tau_q \kappa_{ij}^{(1)} + \tau_T k_{ij} \right) \dot{\alpha}_{,i}(s) \dot{\alpha}_{,j}(s) dv \right]^{1/2} ds, \\ |I_2| &\leq \frac{1}{2} \int_0^t \int_B \tau_q \tau_T k_{ij} \ddot{\alpha}_{,i}(s) \ddot{\alpha}_{,j}(s) dv ds + K^{*2} \mathcal{G}_3, \end{aligned} \quad (87)$$

where

$$\zeta_1(t) = \frac{\sqrt{2}}{\sqrt{\tau_q \kappa_m^{(1)} + \tau_T k_m}} \left(\int_B \left(\alpha_{,i}^{(2)}(t) + \tau_\alpha \dot{\alpha}_{,i}^{(2)}(t) \right) \left(\alpha_{,i}^{(2)}(t) + \tau_\alpha \dot{\alpha}_{,i}^{(2)}(t) \right) dv \right)^{1/2}, \quad (88)$$

$$\mathcal{G}_3 = \frac{\tau_q}{2\tau_T k_m} \int_0^{t_0} \int_B \left(\alpha_{,j}^{(2)}(s) + \tau_\alpha \dot{\alpha}_{,j}^{(2)}(s) \right) \left(\alpha_{,j}^{(2)}(s) + \tau_\alpha \dot{\alpha}_{,j}^{(2)}(s) \right) dv ds.$$

On this basis, the identity [\(83\)](#) implies

$$\mathcal{H}_1[\mathcal{S}_{\mathcal{C}_4^{(1)}}](t) \leq K^* \int_0^t \zeta_1(s) \sqrt{\mathcal{H}_1[\mathcal{S}_{\mathcal{C}_4^{(1)}}](s)} ds + K^{*2} \mathcal{G}_3, \quad t \in (0, t_0), \quad (89)$$

where

$$\mathcal{H}_1[\mathcal{S}_{\mathcal{C}_4^{(1)}}](t) = \mathcal{E}[\mathcal{S}_{\mathcal{C}_4^{(1)}}](t) - \frac{1}{2} \int_0^t \int_B \tau_q \tau_T k_{ij} \ddot{\alpha}_{,i}(s) \ddot{\alpha}_{,j}(s) dv ds, \quad (90)$$

so that we get the following estimate

$$\sqrt{\mathcal{H}_1[\mathcal{S}_{C_4^{(1)}}](t)} \leq K^* \left(\frac{1}{2} \int_0^t \zeta_1(s) ds + \sqrt{\mathcal{G}_3} \right), \quad t \in (0, t_0). \quad (91)$$

It now becomes clear how the estimate (91) describes the continuous dependence of the solution α on the variation of the conductivity rate tensor: K^* , as defined in (86), measures the variation of the conductivity rate tensor, $\sqrt{\mathcal{H}_1[\mathcal{S}_{C_4^{(1)}}](t)}$ represents the measure of the solution, while the parenthesis on the right-hand side of the estimate is a bounded quantity based on the results in Section 5.

7.2. Continuous dependence on the conductivity tensor

Let us now consider two sets of material characteristics that satisfy the constitutive hypotheses (18) to (21) and differ only in the conductivity tensor, let them be $\mathcal{C}_5^{(\beta)} = \{c, k_{ij}^{(\beta)}, k_{rs}^*, \tau_q, \tau_\alpha, \tau_T\}$, $\beta = 1, 2$. Then the difference $\alpha = \alpha^{(1)} - \alpha^{(2)}$ satisfies the following differential equation

$$c\ddot{\alpha} - \left(k_{ij}^* \ddot{\alpha}_{,j} \right)_{,i} - \left(\kappa_{ij}^{(1)} \dot{\alpha}_{,j} \right)_{,i} - \tau_T \left(k_{ij}^{(1)} \ddot{\alpha}_{,j} \right)_{,i} = \left[k_{ij} \left(\dot{\alpha}_{,j}^{(2)} + \tau_T \ddot{\alpha}_{,j}^{(2)} \right) \right]_{,i}, \quad (92)$$

where

$$k_{ij} = k_{ij}^{(1)} - k_{ij}^{(2)}, \quad \kappa_{ij}^{(1)} = k_{ij}^{(1)} + (\tau_\alpha - \tau_q) k_{ij}^*. \quad (93)$$

Moreover, we have the following identity

$$\begin{aligned} \mathcal{E}[\mathcal{S}_{C_5^{(1)}}](t) &= - \int_0^t \int_B k_{ij} \dot{\alpha}_{,i} \left(\dot{\alpha}_{,j}^{(2)} + \tau_T \ddot{\alpha}_{,j}^{(2)} \right) dv ds \\ &\quad - \tau_q \int_0^t \int_B k_{ij} \ddot{\alpha}_{,i} \left(\dot{\alpha}_{,j}^{(2)} + \tau_T \ddot{\alpha}_{,j}^{(2)} \right) dv ds = J_1 + J_2, \end{aligned} \quad (94)$$

where

$$\begin{aligned} \mathcal{E}[\mathcal{S}_{C_5^{(1)}}](t) &= \frac{1}{2} \int_B \left[c \dot{\alpha}^2(t) + k_{ij}^* \ddot{\alpha}_{,i}(t) \ddot{\alpha}_{,j}(t) + \left(\tau_q \kappa_{ij}^{(1)} + \tau_T k_{ij}^{(1)} \right) \dot{\alpha}_{,i}(t) \dot{\alpha}_{,j}(t) \right] dv \\ &\quad + \int_0^t \int_B \left[\kappa_{ij}^{(1)} \dot{\alpha}_{,i}(s) \dot{\alpha}_{,j}(s) + \tau_q \tau_T k_{ij}^{(1)} \ddot{\alpha}_{,i}(s) \ddot{\alpha}_{,j}(s) \right] dv ds. \end{aligned} \quad (95)$$

Further, we have

$$\begin{aligned} |J_1| &\leq K \int_0^t \zeta_2(s) \left[\frac{1}{2} \int_B \left(\tau_q \kappa_{ij}^{(1)} + \tau_T k_{ij}^{(1)} \right) \dot{\alpha}_{,i}(s) \dot{\alpha}_{,j}(s) dv \right]^{1/2} ds, \\ |J_2| &\leq \frac{1}{2} \int_0^t \int_B \tau_q \tau_T k_{ij}^{(1)} \ddot{\alpha}_{,i}(s) \ddot{\alpha}_{,j}(s) dv ds + K^2 \mathcal{G}_4, \end{aligned} \quad (96)$$

where

$$\begin{aligned} K &= \max_{\mathbf{x} \in \overline{B}} (k_{ij} k_{ij})^{1/2}, \\ \zeta_2(t) &= \frac{\sqrt{2}}{\sqrt{\tau_q \kappa_m^{(1)} + \tau_T k_m^{(1)}}} \left(\int_B \left(\dot{\alpha}_{,i}^{(2)}(t) + \tau_T \ddot{\alpha}_{,i}^{(2)}(t) \right) \left(\dot{\alpha}_{,i}^{(2)}(t) + \tau_T \ddot{\alpha}_{,i}^{(2)}(t) \right) dv \right)^{1/2}, \\ \mathcal{G}_4 &= \frac{\tau_q}{2\tau_T k_m^{(1)}} \int_0^{t_0} \int_B \left(\dot{\alpha}_{,j}^{(2)}(s) + \tau_T \ddot{\alpha}_{,j}^{(2)}(s) \right) \left(\dot{\alpha}_{,j}^{(2)}(s) + \tau_T \ddot{\alpha}_{,j}^{(2)}(s) \right) dv ds. \end{aligned} \quad (97)$$

On this basis, the identity (94) implies

$$\mathcal{H}_2[\mathcal{S}_{\mathcal{C}_5^{(1)}}](t) \leq K \int_0^t \zeta_2(s) \sqrt{\mathcal{H}_2[\mathcal{S}_{\mathcal{C}_5^{(1)}}](s)} ds + K^2 \mathcal{G}_4, \quad t \in (0, t_0), \quad (98)$$

where

$$\mathcal{H}_2[\mathcal{S}_{\mathcal{C}_5^{(1)}}](t) = \mathcal{E}[\mathcal{S}_{\mathcal{C}_5^{(1)}}](t) - \frac{1}{2} \int_0^t \int_B \tau_q \tau_T k_{ij}^{(1)} \ddot{\alpha}_{,i}(s) \ddot{\alpha}_{,j}(s) dv ds, \quad (99)$$

so that we get the following estimate

$$\sqrt{\mathcal{H}_2[\mathcal{S}_{\mathcal{C}_5^{(1)}}](t)} \leq K \left(\frac{1}{2} \int_0^t \zeta_2(s) ds + \sqrt{\mathcal{G}_4} \right), \quad t \in (0, t_0). \quad (100)$$

Since $K = \max_{\mathbf{x} \in \overline{B}} (k_{ij} k_{ij})^{1/2}$ represents a measure of the conductivity tensor, $\mathcal{H}_2[\mathcal{S}_{\mathcal{C}_5^{(1)}}](t)$ represents a measure for the variation $\alpha = \alpha^{(1)} - \alpha^{(2)}$ and the parenthesis on the right-hand side represents a bounded quantity, it follows that the estimate (100) describes the continuous dependence of the solution of the initial boundary value problem $\mathcal{P}$ on the variation of the conductivity tensor.

7.3. Continuous dependence on the specific heat coefficient

We consider now the following two material characteristics $\mathcal{C}_6^{(\beta)} = \{c^{(\beta)}, k_{ij}, k_{rs}^*, \tau_q, \tau_\alpha, \tau_T\}$, $\beta = 1, 2$, satisfying the constitutive assumptions (18) to (21). Then the difference between the two corresponding solutions, $\alpha = \alpha^{(1)} - \alpha^{(2)}$, satisfies the null initial and boundary conditions and the following differential equation

$$c^{(1)} \ddot{\alpha} - \left(k_{ij}^* \ddot{\alpha}_{,j} \right)_{,i} - (\kappa_{ij} \dot{\alpha}_{,j})_{,i} - \tau_T (k_{ij} \ddot{\alpha}_{,j})_{,i} = -c \ddot{\alpha}^{(2)}, \quad (101)$$

where now we have set

$$c = c^{(1)} - c^{(2)}. \quad (102)$$

Furthermore, we have the following identity

$$\mathcal{E}[\mathcal{S}_{\mathcal{C}_6^{(1)}}](t) = - \int_0^t \int_B c \ddot{\alpha} \ddot{\alpha}^{(2)} dv ds, \quad (103)$$

where

$$\begin{aligned} \mathcal{E}[\mathcal{S}_{\mathcal{C}_6^{(1)}}](t) &= \frac{1}{2} \int_B \left[c^{(1)} \dot{\alpha}(t)^2 + k_{ij}^* \tilde{\alpha}_{,i}(t) \tilde{\alpha}_{,j}(t) + (\tau_q \kappa_{ij} + \tau_T k_{ij}) \dot{\alpha}_{,i}(t) \dot{\alpha}_{,j}(t) \right] dv \\ &+ \int_0^t \int_B \left[\kappa_{ij} \dot{\alpha}_{,i}(s) \dot{\alpha}_{,j}(s) + \tau_q \tau_T k_{ij} \ddot{\alpha}_{,i}(s) \ddot{\alpha}_{,j}(s) \right] dv ds. \end{aligned} \quad (104)$$

From the identity (103), we deduce the following integral inequality

$$\mathcal{E}[\mathcal{S}_{\mathcal{C}_6^{(1)}}](t) \leq C \int_0^t \zeta_3(s) \sqrt{\mathcal{E}[\mathcal{S}_{\mathcal{C}_6^{(1)}}](s)} ds, \quad t \in (0, t_0), \quad (105)$$

where

$$\zeta_3(t) = \sqrt{\frac{2}{c_m^{(1)}}} \left(\int_B \ddot{\alpha}^{(2)}(t)^2 dv \right)^{1/2}, \quad C = \max_{\mathbf{x} \in \overline{B}} |c|, \quad (106)$$

so that we get the following estimate

$$\sqrt{\mathcal{E}[\mathcal{S}_{C_0^{(1)}}](t)} \leq \frac{C}{2} \int_0^t \zeta_3(s) ds, \quad t \in (0, t_0). \quad (107)$$

It is worth to outline that the procedure of [Section 5](#) allows us to obtain a priori bounds for

$$\int_B \ddot{\alpha}^2 dv, \quad (108)$$

in terms of the given data.

8. Convergence as $k_{ij}^* \rightarrow 0$

Convergence result in this section refers to the property that as a thermal characteristic of the differential [equation \(5\)](#) approach zero, its solution approaches the corresponding solution of the reduced differential equation. For the thermal model in discussion to be well-posed, it must be stable and convergent.

In this connection, as an example, we investigate here how the solution of the basic initial boundary value problem $\mathcal{P}$ behaves as the measure of the conductivity rate tensor tends to zero. In fact, if we set $k_{ij}^* = 0$ within the basic [equation \(5\)](#) we obtain the following third order equation

$$c\tau_q \ddot{\alpha} + c\ddot{\alpha} - (k_{ij}\dot{\alpha}_{,j})_{,i} - \tau_T(k_{ij}\ddot{\alpha}_{,j})_{,i} = 0, \quad (109)$$

which, by setting $\psi = \dot{\alpha}$, can be reduced to a second-order partial differential equation of the Jeffreys type (see, for example, Joseph and Preziosi [[48,49](#)])

$$c\tau_q \ddot{\psi} + c\dot{\psi} - (k_{ij}\dot{\psi}_{,j})_{,i} - \tau_T(k_{ij}\dot{\psi}_{,j})_{,i} = 0. \quad (110)$$

Keeping in mind the [Subsection 7.1](#), we take $\alpha^{(1)}$ be the solution of the initial boundary value problem $\mathcal{P}$ corresponding to the material characteristics $\mathcal{C}_4^{(1)} = \{c, k_{ij}, k_{rs}^*, \tau_q, \tau_\alpha, \tau_T\}$, satisfying the constitutive assumptions [\(18\)](#) to [\(21\)](#). We want to compare the solution $\alpha^{(1)}$ with the solution $\alpha^{(2)}$ of the reduced differential equation [\(109\)](#) which correspond to the material characteristics $\mathcal{C}_4^{(2)} = \mathcal{C}_0 = \{c, k_{ij}, 0; \tau_q, \tau_\alpha, \tau_T\}$ that also satisfies the hypotheses (H1) to (H4). It follows then that the difference

$$\alpha = \alpha^{(1)} - \alpha^{(2)}, \quad (111)$$

satisfies the following differential equation

$$c\tau_q \ddot{\alpha} + c\ddot{\alpha} - (k_{ij}\dot{\alpha}_{,j})_{,i} - \tau_T(k_{ij}\ddot{\alpha}_{,j})_{,i} = \left[k_{ij}^* \left(\alpha_{,j}^{(1)} + \tau_\alpha \dot{\alpha}_{,j}^{(1)} \right) \right]_{,i}, \quad (112)$$

and null initial and boundary data. Then we can obtain the following identity

$$\mathcal{E}_0[\mathcal{S}_{C_0}](t) = - \int_0^t \int_B k_{ij}^* \ddot{\alpha}_{,i}(s) \left(\alpha_{,j}^{(1)}(s) + \tau_\alpha \dot{\alpha}_{,j}^{(1)}(s) \right) dv ds, \quad (113)$$

where

$$\mathcal{E}_0[\mathcal{S}_{C_0}](t) = \frac{1}{2} \int_B \left[c\tau_q \ddot{\alpha}(t)^2 + k_{ij}\dot{\alpha}_{,i}(t)\dot{\alpha}_{,j}(t) \right] dv + \int_0^t \int_B \left[c\dot{\alpha}(t)^2 + \tau_T k_{ij}\ddot{\alpha}_{,i}(s)\ddot{\alpha}_{,j}(s) \right] dv ds. \quad (114)$$

Further, we obtain the following integral inequality

$$\mathcal{F}_0[\mathcal{S}_{C_0}](t) \leq K^{*2} \mathcal{G}_0, \quad t \in (0, t_0), \quad (115)$$

where

$$\begin{aligned}
K^* &= \max_{\mathbf{x} \in \bar{B}} \left(k_{ij}^* k_{ij}^* \right)^{1/2}, \\
\mathcal{G}_0 &= \frac{1}{2\tau_T k_m} \int_0^{t_0} \int_B \left(\alpha_{,j}^{(1)}(s) + \tau_\alpha \dot{\alpha}_{,j}^{(1)}(s) \right) \left(\alpha_{,j}^{(1)}(s) + \tau_\alpha \dot{\alpha}_{,j}^{(1)}(s) \right) dv ds, \\
\mathcal{F}_0[\mathcal{S}_{C_0}](t) &= \mathcal{E}_0[\mathcal{S}_{C_0}](t) - \frac{1}{2} \int_0^t \int_B \tau_T k_{ij} \ddot{\alpha}_{,i} \ddot{\alpha}_{,j} dv ds.
\end{aligned} \tag{116}$$

Since $\mathcal{C}_4^{(1)} = \{c, k_{ij}, k_{rs}^*, \tau_q, \tau_\alpha, \tau_T\}$ satisfies the constitutive assumptions (18) to (21), based on the results of Section 5 we can conclude that $\mathcal{G}_0$ is a bounded quantity. Then we can say that $\mathcal{F}_0[\mathcal{S}_{C_0}](t)$ can be viewed as a measure of the difference $\alpha = \alpha^{(1)} - \alpha^{(2)}$. Under these considerations we can conclude that, as k_{ij}^* tends to zero (that is $K^* \rightarrow 0$), the inequality (115) shows that $\alpha = \alpha^{(1)} - \alpha^{(2)}$ tends to zero, that is the solution of the initial boundary value problem $\mathcal{P}$ converges to the solution of the initial boundary value problem associated with the differential equation (109).

9. Concluding remarks

The present study investigated the structural stability of the thermal model of a rigid conductor that is based on the existence of three relaxation times: a delay time for the heat flux, one for the temperature gradient, and one for the thermal displacement gradient. It provides precise mathematical estimates that describe how the solution of the initial boundary value problem behaves at the limit when the prescribed external data or the relaxation times or other thermal coefficients of the model undergo small variations (due to measurements, environmental conditions, etc.). Thus, under the assumption that the material characteristics of the rigid conductor satisfy hypotheses (H1) to (H4), the following were provided:

- i. the estimate (48) provides a continuous dependence result for the solution of the initial boundary value problem $\mathcal{P}$ in terms of all external given data;
- ii. the estimates (62), (72) and (79) furnish continuous dependence of solution with respect to small variations in relaxation times τ_T , τ_α and τ_q , respectively;
- iii. the estimates (91), (100) and (107) show the continuous dependence of the solution with respect the other material characteristics: the conductivity rate tensor k_{ij}^* and the conductivity tensor k_{ij} and the specific heat a ;
- iv. the solution $\alpha^{(1)}$ of the initial boundary value problem $\mathcal{P}$ converges to $\alpha^{(2)}$, the solution of the initial boundary value problem associated with (109), when k_{ij}^* tends to zero.

All these estimates show that suitably small variations of the material characteristics do not destroy the well definiteness of the thermal model, that is the structural stability of the thermal system is established.

These estimates are useful for experimental measurements and numerical treatments, ensuring that small errors in data do not lead to chaos and that numerical approximations remain close to exact values.

The thermal analysis of structures, often performed using finite element analysis, helps engineers understand how these respond to thermal loads. This involves studying temperature distribution, heat flow, and the resulting stresses and deformations.

Finally, we must note that the main objective of this comprehensive report is to ensure the mathematical theoretical fundamentals for justifying the thermal stability of the considered model, as well as its applicability in determining a suitability of materials for specific applications.

Disclosure statement

No potential conflict of interest was reported by the authors.

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