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Cătălin Galeş & Stan Chiriță

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Rayleigh waves in symmetry planes of orthotropic thermoelastic layered structures: Explicit secular equations

Cătălin Galeş  and Stan Chiriță 

Faculty of Mathematics, University Alexandru Ioan Cuza of Iasi, Iasi, Romania

ABSTRACT

The present work studies the propagation of Rayleigh waves in a layered structure consisting of an orthotropic thermoelastic layer of arbitrary thickness covering an orthotropic thermoelastic half-space. The layer and the half-space are in welded contact with each other along one of the orthotropic symmetry planes. The outer surface of the layer is free of traction, and no heat transfer with the environment is allowed. When passing through the contact surface, the plane displacements and temperature fields, as well as the stress and heat flux fields, are assumed to vary continuously, with no jumps being allowed. First of all, the free wave solutions in the orthotropic thermoelastic layer are established in an explicit form for the plane displacements and temperature fields, as well as for the stresses and heat flux fields. Further, these explicit expressions are used to obtain the so-called effective boundary conditions, i.e., linear combinations of values for the displacements and temperature variation fields, as well as for the stresses and heat flux fields on the interface surface between the layer and the half-space. On this basis and by using the free wave solutions in the half-space, the general exact secular equation is established for propagation of Rayleigh waves in the half-space of the considered orthotropic layered structure. Numerical results reveal the impact of anisotropy and thermal effects on the dispersion properties of Rayleigh-type surface waves in orthotropic thermoelastic layered structures, that is, upon the wave propagation speed and on the rate of damping in time of the wave amplitude. In particular, the present paper provides secular equations for the propagation of Rayleigh waves in orthotropic elastic layered structures as well as in isotropic thermoelastic layered structures.

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1. Introduction

The wave propagation problem in layered structures is a fundamental problem in dynamics of continua. The propagation of free guided waves in layered structures is theorized in many important textbooks, as, for example, those written by Ewing et al. [1], Achenbach [2], Nayfeh [3], Kundu [4] and Rose [5]. The purpose of using layered structures is to improve the engineering properties of the material substrate to resist external damage that may come from weather conditions, uncontrolled friction, chemical corrosion or high temperature barriers. This interesting topic is investigated by means of guided waves in numerous research articles. In this sense, we mention here the studies carried out by Sezawa [6], Sezawa and Kanai [7], Haskell [8], Achenbach and Epstein [9], Tiersten [10], Makarov et al. [11], Kuchler and Richter [12], Cai and

CONTACT Cătălin Galeş  cgales@uaic.ro  Faculty of Mathematics, University Alexandru Ioan Cuza of Iasi, Blvd. Carol I, no. 11, 700506 Iasi, Romania.

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Fu [13], Sotiropoulos [14], Every [15] and Chattopadhyay and Raju [16]. Furthermore, an interesting treatment of aerospace composite structures as well as a study of guided waves in thin-walled composite structures can be found in the excellent textbooks written by Nowinski [17] and Giurgiutiu [18].

Other recent and interesting results have been obtained both in terms of establishing effective boundary conditions and in terms of obtaining the exact secular equation for the propagation of Rayleigh waves in elastic half-spaces on which an elastic layer is laid in the various forms of contact: Steigmann and Ogden [19], Dai et al. [20], Vinh and Anh [21], Vinh et al. [22], Kaplunov and Prikazchikov [23], Erbaş et al. [24], Kaplunov et al. [25,26], Mubarak et al. [27] and Kaur et al. [28].

Orthotropic materials are used in designs of various real-world applications, particularly in civil and aerospace engineering. Thus, many modern aircraft wings are designed using orthotropic composite materials. These materials provide the necessary strength and stiffness while minimizing weight, which is crucial for flight efficiency and performance. Orthotropic materials are also used in various components of spacecraft, where directional strength and lightweight properties are essential for withstanding the harsh conditions of space. In this connection, we note that Destrade [29] established the explicit secular equation for surface waves propagating in any direction of a plane of symmetry for crystals possessing at least one plane of symmetry.

On the other hand, it is well known that the problem of Rayleigh wave propagation in a thermoelastic medium has been a topic of major interest investigated by numerous researchers starting with Lockett [30], Chadwick [31], Deresiewicz [32] and continuing with Ieşan [33], Ieşan and Scalia [34], Brock [35], Lykotrafitis and Georgiadis [36,37], Ieşan [38], Chiriţă [39,40], Zampoli and Chiriţă [41] and the Chapter *Rayleigh's Waves* of the Encyclopedia of Thermal Stresses [42]. In-plane surface wave propagation in a thermoelastic isotropic half-space covered with an isotropic thermoelastic layer was recently studied by Chiriţă [43]. The problem of a layered structure consisting of an isotropic thermoelastic half-space and a welded isotropic Maxwell-Cattaneo thermoelastic layer was treated by Chiriţă and D'Apice [44].

We have to note that a thermoelastic orthotropic layered structure combines the principles of thermoelasticity with the directional properties of orthotropic materials, making it highly useful in various engineering applications from aerospace, civil engineering and electronics. Understanding and managing these effects is essential for the successful design and application of thermoelastic orthotropic layered structures.

In this paper, we investigate the motion of Rayleigh surface waves in a layered structure modeled as an orthotropic thermoelastic layer overlying an orthotropic thermoelastic half-space. The layer and the half-space are in welded contact with each other along one of the planes of orthotropic symmetry. The explicit expressions of free Rayleigh waves in the orthotropic thermoelastic layer are provided, and then they are invoked to derive the effective boundary conditions in terms of values of the displacements and temperature fields, as well as the stresses and the heat flux of the half-space at its surface. Using these effective boundary conditions on the interface and the free Rayleigh wave solutions in the half-space, the exact secular equation of Rayleigh waves is provided. In this way, the effects of the thermoelastic layer on the material substrate are taken into account, and thus the problem of Rayleigh wave propagation in a layered structure reduces in determining the half-space-free waves that satisfy the effective boundary conditions, now ignoring the existence of the coated thermoelastic layer. As a normal consequence of our analysis, we establish the exact secular equation for an elastic orthotropic layered structure in the absence of any thermal effects, as well as for a layered structure in which the layer and the half-space are made of different isotropic thermoelastic materials. Finally, an illustrative numerical analysis is carried out on a model of a layered structure consisting of a layer and a half-space made of different thermoelastic orthotropic materials, highlighting the behavior of the two

characteristic parameters of Rayleigh wave propagation: the dimensionless speed of wave propagation and the rate of damping in time of the wave amplitude.

The plan of the work is the following. Section 2 presents the basic differential equation systems that describe the evolutionary thermoelastic behavior of orthotropic materials in the layer and the half-space, respectively. Section 3 introduces the classes of wave solutions in the thermoelastic layer, and this allows us to establish the exact effective boundary conditions in the Section 4. Section 5 treats the propagating Rayleigh wave problem in the thermoelastic half-space and then establishes the corresponding exact secular equation. Subsection 6.1 treats the limiting case when the layer and the half-space are both elastic, that is, thermal effects are absent, while Subsection 6.2 provides an exact secular equation for the propagation of the Rayleigh wave in an isotropic thermoelastic layered structure. Section 7 provides a numerical simulation that exemplifies the analysis developed in the previous sections.

2. The orthotropic thermoelastic layered structure

Throughout this paper, we will consider a thermoelastic layered structure consisting of an orthotropic thermoelastic layer, of constant thickness h , which is welded on an orthotropic thermoelastic half-space, the contact surface coinciding with one of the orthotropic symmetry planes. Furthermore, we will choose a Cartesian system of coordinate axes $Ox_1x_2x_3$ whose planes coincide with the planes of orthotropic symmetry and so that the plane $x_2 = 0$ coincides with the flat surface of contact between the layer and the half-space, see Figure 1.

Therefore, the layered thermoelastic structure consists of the orthotropic thermoelastic half-space $0 < x_2 < \infty$ overlaid by an orthotropic thermoelastic layer with uniform thickness h occupying the domain $-h < x_2 < 0$. The layer is assumed to be perfectly bound to the half-space in the symmetry plane $x_2 = 0$. We are interested in the plane motion in the (x_1, x_2) plane with displacement components (u_1, u_2, u_3) and the temperature variation θ such that

$$\begin{aligned} u_1 &= u_1(x_1, x_2, t), & u_2 &= u_2(x_1, x_2, t), & u_3 &= 0, \\ \theta &= \theta(x_1, x_2, t), & x_1 &\in \mathbb{R}, x_2 \in (-h, \infty), & t &> 0, \end{aligned} \quad (1)$$

where t is the time variable.

Note that the same quantities related to the half-space and the layer have the same symbol but are systematically distinguished by a star if pertaining to the half-space $0 < x_2 < \infty$. In accordance with what was said above and with the linear theory of orthotropic thermoelasticity, in the half-space $x_2 > 0$ we have the basic motion equations

$$\begin{aligned} t_{11,1}^* + t_{21,2}^* &= \varrho^* \ddot{u}_1^*, \\ t_{12,1}^* + t_{22,2}^* &= \varrho^* \ddot{u}_2^*, \\ q_{1,1}^* + q_{2,2}^* + T_0 \beta_1^* \dot{u}_{1,1}^* + T_0 \beta_2^* \dot{u}_{2,2}^* &= T_0 a^* \dot{\theta}^*, \end{aligned} \quad (2)$$

with the constitutive equations

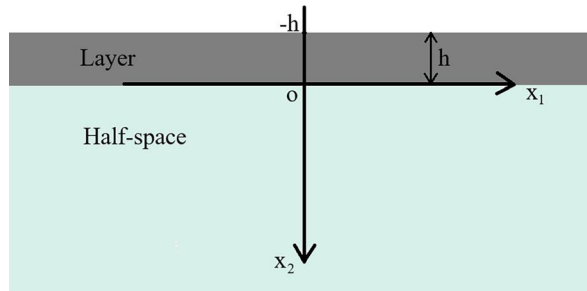


Figure 1. Thermoelastic layered structure.

$$\begin{aligned}
t_{11}^* &= c_{11}^* u_{1,1}^* + c_{12}^* u_{2,2}^* + \beta_1^* \theta^*, \\
t_{22}^* &= c_{12}^* u_{1,1}^* + c_{22}^* u_{2,2}^* + \beta_2^* \theta^*, \\
t_{12}^* &= c_{66}^* (u_{1,2}^* + u_{2,1}^*), \\
q_1^* &= k_1^* \theta_{,1}^*, \\
q_2^* &= k_2^* \theta_{,2}^*;
\end{aligned} \tag{3}$$

while in the layer $-h < x_2 < 0$ we have

$$\begin{aligned}
t_{11,1} + t_{21,2} &= \varrho \ddot{u}_1, \\
t_{12,1} + t_{22,2} &= \varrho \ddot{u}_2, \\
q_{1,1} + q_{2,2} + T_0 \beta_1 \dot{u}_{1,1} + T_0 \beta_2 \dot{u}_{2,2} &= T_0 a \dot{\theta},
\end{aligned} \tag{4}$$

with the constitutive equations

$$\begin{aligned}
t_{11} &= c_{11} u_{1,1} + c_{12} u_{2,2} + \beta_1 \theta, \\
t_{22} &= c_{12} u_{1,1} + c_{22} u_{2,2} + \beta_2 \theta, \\
t_{12} &= c_{66} (u_{1,2} + u_{2,1}), \\
q_1 &= k_1 \theta_{,1}, \\
q_2 &= k_2 \theta_{,2}.
\end{aligned} \tag{5}$$

In the above relations we have used the following notations: c_{11} , c_{22} , c_{66} and c_{12} are the elastic constant characteristics satisfying the necessary and sufficient conditions for the strain energy of the material to be positive definite

$$c_{11} > 0, \quad c_{22} > 0, \quad c_{66} > 0, \quad c_{12}^2 < c_{11} c_{22}. \tag{6}$$

Note that because of the restriction to plane strain and a plane of symmetry, the usual nine constants of orthotropy reduce to the four considered here. Furthermore, k_1 and k_2 are the components of the constant conductivity tensor, satisfying

$$k_1 > 0, \quad k_2 > 0; \tag{7}$$

moreover, β_1 and β_2 are the components of the stress-temperature constant tensor, $a > 0$ is the constant mass specific heat, ϱ is the density mass, while u_1 and u_2 are the components of the displacement vector and θ is the temperature variation from the absolute temperature $T_0 > 0$, t_{11} , t_{22} and t_{12} are the components of the stress tensor, q_1 and q_2 are the components of the heat flux vector.

In terms of the displacements and temperature variation fields, the above equations can be written as follows

$$\begin{aligned}
(c_{11}^* u_{1,11}^* + c_{66}^* u_{1,22}^*) + (c_{12}^* + c_{66}^*) u_{2,21}^* + \beta_1^* \theta_{,1}^* &= \varrho^* \ddot{u}_1^*, \\
(c_{12}^* + c_{66}^*) u_{1,12}^* + (c_{66}^* u_{2,11}^* + c_{22}^* u_{2,22}^*) + \beta_2^* \theta_{,2}^* &= \varrho^* \ddot{u}_2^*, \\
k_1^* \theta_{,11}^* + k_2^* \theta_{,22}^* + T_0 (\beta_1^* \dot{u}_{1,1}^* + \beta_2^* \dot{u}_{2,2}^*) &= T_0 a^* \dot{\theta}^*,
\end{aligned} \tag{8}$$

within the half-space $x_2 > 0$ and

$$\begin{aligned}
(c_{11} u_{1,11} + c_{66} u_{1,22}) + (c_{12} + c_{66}) u_{2,21} + \beta_1 \theta_{,1} &= \varrho \ddot{u}_1, \\
(c_{12} + c_{66}) u_{1,12} + (c_{66} u_{2,11} + c_{22} u_{2,22}) + \beta_2 \theta_{,2} &= \varrho \ddot{u}_2, \\
k_1 \theta_{,11} + k_2 \theta_{,22} + T_0 (\beta_1 \dot{u}_{1,1} + \beta_2 \dot{u}_{2,2}) &= T_0 a \dot{\theta},
\end{aligned} \tag{9}$$

in the layer $-h < x_2 < 0$.

The contact between the thermoelastic layer and the semi-space is supposed to be welded, and there are no temperature changes or heat transfer at the contact interface. This means that the

displacement and temperature fields, as well as the stresses and heat flux, are continuous through the interface of the layer and the half-space. Therefore, we must have the following conditions fulfilled

$$\begin{aligned} u_1^*(x_1, 0, t) &= u_1(x_1, 0, t), & u_2^*(x_1, 0, t) &= u_2(x_1, 0, t), \\ t_{12}^*(x_1, 0, t) &= t_{12}(x_1, 0, t), & t_{22}^*(x_1, 0, t) &= t_{22}(x_1, 0, t), \\ \theta^*(x_1, 0, t) &= \theta(x_1, 0, t), & q_2^*(x_1, 0, t) &= q_2(x_1, 0, t), \end{aligned} \quad (10)$$

for all $x_1 \in \mathbb{R}, t > 0$.

Further, we assume that the surface $x_2 = -h$ of the thermoelastic layer is free of traction and that there are no thermal exchanges with the outside, that is we have

$$\begin{aligned} t_{12}(x_1, -h, t) &= 0, & t_{22}(x_1, -h, t) &= 0, \\ q_2(x_1, -h, t) &= 0, & \text{for all } x_1 \in \mathbb{R}, t > 0. \end{aligned} \quad (11)$$

Alternatively, we can consider that the free surface of the layer is thermally insulated, when the condition $(11)_2$ is replaced by the following one

$$\theta(x_1, -h, t) = 0, \quad \text{for all } x_1 \in \mathbb{R}, t > 0. \quad (12)$$

Finally, because we intend to talk about surface waves, we will have to impose zero asymptotic conditions on the mechanical and thermal quantities in the depth of the half-space, that is we will assume that

$$\begin{aligned} \lim_{x_2 \rightarrow \infty} u_\alpha^*(x_1, x_2, t) &= 0, & \lim_{x_2 \rightarrow \infty} t_{\alpha\beta}^*(x_1, x_2, t) &= 0, \\ \lim_{x_2 \rightarrow \infty} \theta^*(x_1, x_2, t) &= 0, & \lim_{x_2 \rightarrow \infty} q_\alpha^*(x_1, x_2, t) &= 0, \end{aligned} \quad (13)$$

for all $x_1 \in \mathbb{R}, t > 0, \alpha, \beta = 1, 2$.

For our further analysis, we introduce the speed of the longitudinal wave in the direction of the axis Ox_1 , c_1 , the speed of the longitudinal wave in the direction of the axis Ox_2 , c_2 , and the speed of the transverse wave, c_6 , defined by the relation

$$c_1^2 = \frac{c_{11}}{\varrho}, \quad c_2^2 = \frac{c_{22}}{\varrho}, \quad c_6^2 = \frac{c_{66}}{\varrho}, \quad (14)$$

and the corresponding

$$c_1^{*2} = \frac{c_{11}^*}{\varrho^*}, \quad c_2^{*2} = \frac{c_{22}^*}{\varrho^*}, \quad c_6^{*2} = \frac{c_{66}^*}{\varrho^*}. \quad (15)$$

3. Free wave solutions of the basic equations in the thermoelastic layer

Throughout this paper we will study the propagation of a Rayleigh wave, traveling along the interface between the layer and the half-space with velocity c and wave-number $\varkappa > 0$ in the x_1 -direction and decaying in the x_2 -direction. To solve the Rayleigh surface wave problem for the considered layered structure, we are looking for wave solutions of basic equations (9) in the form of waves with assigned wavelength, that align with those presented in [41]:

$$\begin{aligned} u_1(x_1, x_2, t) &= \operatorname{Re} \left\{ \frac{1}{\varkappa} \mathcal{U}_1(x_2) e^{i\varkappa(x_1 - ct)} \right\}, \\ u_2(x_1, x_2, t) &= \operatorname{Re} \left\{ \frac{1}{\varkappa} \mathcal{U}_2(x_2) e^{i\varkappa(x_1 - ct)} \right\}, \\ \theta(x_1, x_2, t) &= \operatorname{Re} \left\{ T_0 T(x_2) e^{i\varkappa(x_1 - ct)} \right\}, \quad x_1 \in \mathbb{R}, -h < x_2 < 0, t > 0, \end{aligned} \quad (16)$$

leading to the following differential system for the unknown dimensionless vector $(\mathcal{U}_1(x_2), \mathcal{U}_2(x_2), T(x_2))$

$$\begin{aligned}
c_{66} \mathcal{U}_{1,22} + i\kappa(c_{12} + c_{66})\mathcal{U}_{2,2} - \kappa^2(c_{11} - qc^2)\mathcal{U}_1 + i\kappa^2 T_0 \beta_1 T &= 0, \\
c_{22} \mathcal{U}_{2,22} - \kappa^2(c_{66} - qc^2)\mathcal{U}_2 + i\kappa(c_{12} + c_{66})\mathcal{U}_{1,2} + \kappa T_0 \beta_2 T_{,2} &= 0, \\
k_2 T_{,22} - \kappa^2 \left(k_1 - \frac{iT_0 ac}{\kappa} \right) T - i\beta_2 c \mathcal{U}_{2,2} + \kappa \beta_1 c \mathcal{U}_1 &= 0.
\end{aligned} \tag{17}$$

Further, we seek for solutions of the differential system (17) in the form

$$(\mathcal{U}_1(x_2), \mathcal{U}_2(x_2), T(x_2)) = (U_1, U_2, T) e^{i\kappa\sigma x_2}, \tag{18}$$

with (U_1, U_2, T) a non-zero constant complex vector and σ a dimensionless constant parameter. Consequently, the differential system (17) leads to the homogeneous algebraic system

$$\begin{aligned}
\left(\sigma^2 + \frac{c_1^2 - c_1^{*2} \psi^2}{c_6^2} \right) U_1 + \frac{c_{12} + c_{66}}{c_{66}} \sigma U_2 - iM_1 T &= 0, \\
\frac{c_{12} + c_{66}}{c_{22}} \sigma U_1 + \left(\sigma^2 + \frac{c_6^2 - c_1^{*2} \psi^2}{c_2^2} \right) U_2 - iM_2 \sigma T &= 0, \\
-\frac{N_1 c_1^*}{c_1} \psi U_1 - \frac{N_2 c_1^*}{c_1} \psi \sigma U_2 + \left(\sigma^2 + \mathcal{K} - \frac{ic_1^* \psi}{c_1 \mathcal{K}_2} \right) T &= 0,
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
M_1 &= \frac{T_0 \beta_1}{qc_6^2}, \quad N_1 = \frac{\beta_1 c_1}{\kappa k_2}, \quad M_2 = \frac{T_0 \beta_2}{qc_2^2}, \quad N_2 = \frac{\beta_2 c_1}{\kappa k_2}, \\
\mathcal{K} &= \frac{k_1}{k_2}, \quad \mathcal{K}_2 = \frac{\kappa k_2}{T_0 a c_1}, \quad \psi = \frac{c}{c_1^*}.
\end{aligned} \tag{20}$$

Since we have $(U_1, U_2, T) \neq 0$, it is necessary that the determinant of the algebraic system (19) to be zero, that is

$$\text{Det} \Lambda_{(3 \times 3)}(\sigma, \psi) = 0, \tag{21}$$

where

$$\Lambda_{(3 \times 3)}(\sigma, \psi) \equiv (\lambda_{pr}(\sigma, \psi)) \equiv \begin{pmatrix} \sigma^2 + \frac{c_1^2 - c_1^{*2} \psi^2}{c_6^2} & \frac{c_{12} + c_{66}}{c_{66}} \sigma & -iM_1 \\ \frac{c_{12} + c_{66}}{c_{22}} \sigma & \sigma^2 + \frac{c_6^2 - c_1^{*2} \psi^2}{c_2^2} & -iM_2 \sigma \\ -\frac{c_1^* N_1}{c_1} \psi & -\frac{c_1^* N_2}{c_1} \psi \sigma & \sigma^2 + \mathcal{K} - \frac{ic_1^* \psi}{c_1 \mathcal{K}_2} \end{pmatrix}. \tag{22}$$

Thus, we obtain the following cubic equation in terms of unknown $\pi = \sigma^2$,

$$\pi^3 + A_1 \pi^2 + A_2 \pi + A_3 = 0, \tag{23}$$

where

$$\begin{aligned}
A_1 &= \frac{c_1^2 - c_1^{*2} \psi^2}{c_6^2} + \frac{c_6^2 - c_1^{*2} \psi^2}{c_2^2} - \frac{(c_{12} + c_{66})^2}{c_{22} c_{66}} + \mathcal{K} - \frac{ic_1^* \psi}{c_1 \mathcal{K}_2} - iM_2 N_2 \psi \frac{c_1^*}{c_1}, \\
A_2 &= \left(\frac{c_1^2 - c_1^{*2} \psi^2}{c_6^2} + \frac{c_6^2 - c_1^{*2} \psi^2}{c_2^2} - \frac{(c_{12} + c_{66})^2}{c_{22} c_{66}} \right) \left(\mathcal{K} - \frac{ic_1^* \psi}{c_1 \mathcal{K}_2} \right) + \frac{c_1^2 - c_1^{*2} \psi^2}{c_6^2} \times \frac{c_6^2 - c_1^{*2} \psi^2}{c_2^2} \\
&\quad + \frac{ic_1^* \psi}{c_1} \left[\frac{c_{12} + c_{66}}{c_{22} c_{66}} (c_{22} M_2 N_1 + c_{66} M_1 N_2) - M_1 N_1 - \frac{c_1^2 - c_1^{*2} \psi^2}{c_6^2} M_2 N_2 \right], \\
A_3 &= \frac{c_1^2 - c_1^{*2} \psi^2}{c_6^2} \times \frac{c_6^2 - c_1^{*2} \psi^2}{c_2^2} \left(\mathcal{K} - \frac{ic_1^* \psi}{c_1 \mathcal{K}_2} \right) - iM_1 N_1 \frac{c_1^* \psi}{c_1} \frac{c_6^2 - c_1^{*2} \psi^2}{c_2^2}.
\end{aligned} \tag{24}$$

Let us denote by π_1 , π_2 and π_3 the three roots of the cubic Equation (23). Consequently, there exist the following admissible values for the dimensionless parameter σ :

$$\begin{aligned}\sigma_1 &= \xi_1, & \sigma_2 &= \xi_2, & \sigma_3 &= \xi_3, \\ \sigma_4 &= -\xi_1, & \sigma_5 &= -\xi_2, & \sigma_6 &= -\xi_3,\end{aligned}\quad (25)$$

with

$$\begin{aligned}\xi_1 &= \sqrt{\pi_1}, & \xi_2 &= \sqrt{\pi_2}, & \xi_3 &= \sqrt{\pi_3}, \\ \text{Im}(\xi_1) &> 0, & \text{Im}(\xi_2) &> 0, & \text{Im}(\xi_3) &> 0.\end{aligned}\quad (26)$$

For writing the wave solutions, corresponding to the six roots of Equation (23), it is useful to introduce the adjunct matrix $\text{adj}(\Lambda) = (\Lambda_{rs}(\sigma, \psi))_{1 \leq r, s \leq 3}$ associated with the matrix Λ , whose elements are given in Appendix 1.

Then the corresponding vector solutions of the differential system (17) are

$$(\mathcal{U}_1^{(n)}, \mathcal{U}_2^{(n)}, T^{(n)})(x_2) = (U_1^{(n)}, U_2^{(n)}, T^{(n)})e^{i\omega\sigma_n x_2}, \quad n = 1, 2, \dots, 6, \quad (27)$$

where

$$\begin{aligned}(\mathcal{U}_1^{(1)}, \mathcal{U}_2^{(1)}, T^{(1)}) &= (\Lambda_{11}(\sigma_1, \psi), \Lambda_{21}(\sigma_1, \psi), \Lambda_{31}(\sigma_1, \psi)), \\ (\mathcal{U}_1^{(2)}, \mathcal{U}_2^{(2)}, T^{(2)}) &= (\Lambda_{12}(\sigma_2, \psi), \Lambda_{22}(\sigma_2, \psi), \Lambda_{32}(\sigma_2, \psi)), \\ (\mathcal{U}_1^{(3)}, \mathcal{U}_2^{(3)}, T^{(3)}) &= (\Lambda_{13}(\sigma_3, \psi), \Lambda_{23}(\sigma_3, \psi), \Lambda_{33}(\sigma_3, \psi)),\end{aligned}\quad (28)$$

$$\begin{aligned}(\mathcal{U}_1^{(4)}, \mathcal{U}_2^{(4)}, T^{(4)}) &= (\Lambda_{11}(\sigma_4, \psi), \Lambda_{21}(\sigma_4, \psi), \Lambda_{31}(\sigma_4, \psi)), \\ (\mathcal{U}_1^{(5)}, \mathcal{U}_2^{(5)}, T^{(5)}) &= (\Lambda_{12}(\sigma_5, \psi), \Lambda_{22}(\sigma_5, \psi), \Lambda_{32}(\sigma_5, \psi)), \\ (\mathcal{U}_1^{(6)}, \mathcal{U}_2^{(6)}, T^{(6)}) &= (\Lambda_{13}(\sigma_6, \psi), \Lambda_{23}(\sigma_6, \psi), \Lambda_{33}(\sigma_6, \psi)).\end{aligned}\quad (29)$$

Finally, we note that

$$(\mathcal{U}_1, \mathcal{U}_2, T)(x_2) = \sum_{n=1}^6 B_n (U_1^{(n)}, U_2^{(n)}, T^{(n)})e^{i\omega\sigma_n x_2}, \quad (30)$$

with $B_1, B_2, \dots, B_6$ arbitrary dimensionless constant parameters, represents a solution of the differential system (17). Consequently, in view of relations (16) and (30), we have that

$$\begin{aligned}u_1(x_1, x_2, t) &= \text{Re} \left\{ \frac{1}{\omega} e^{i\omega(x_1 - c_1^* \psi t)} \sum_{n=1}^6 B_n U_1^{(n)} e^{i\omega\sigma_n x_2} \right\}, \\ u_2(x_1, x_2, t) &= \text{Re} \left\{ \frac{1}{\omega} e^{i\omega(x_1 - c_1^* \psi t)} \sum_{n=1}^6 B_n U_2^{(n)} e^{i\omega\sigma_n x_2} \right\}, \\ \theta(x_1, x_2, t) &= \text{Re} \left\{ T_0 e^{i\omega(x_1 - c_1^* \psi t)} \sum_{n=1}^6 B_n T^{(n)} e^{i\omega\sigma_n x_2} \right\},\end{aligned}\quad (31)$$

is a solution of the differential system (9) in the layer $-h < x_2 < 0$. In view of the constitutive Equations (5), from (31) we get

$$\begin{aligned}
t_{12}(x_1, x_2, t) &= \text{Re} \left\{ i c_{66} e^{i\omega(x_1 - c_1^* \psi t)} \sum_{n=1}^6 B_n t_{12}^{(n)} e^{i\omega \sigma_n x_2} \right\}, \\
t_{22}(x_1, x_2, t) &= \text{Re} \left\{ i c_{22} e^{i\omega(x_1 - c_1^* \psi t)} \sum_{n=1}^6 B_n t_{22}^{(n)} e^{i\omega \sigma_n x_2} \right\}, \\
q_2(x_1, x_2, t) &= \text{Re} \left\{ i \omega k_2 T_0 e^{i\omega(x_1 - c_1^* \psi t)} \sum_{n=1}^6 B_n q_2^{(n)} e^{i\omega \sigma_n x_2} \right\},
\end{aligned} \tag{32}$$

where

$$\begin{aligned}
t_{12}^{(n)} &= \sigma_n U_1^{(n)} + U_2^{(n)}, \\
t_{22}^{(n)} &= \frac{c_{12}}{c_{22}} U_1^{(n)} + \sigma_n U_2^{(n)} - \frac{i\beta_2 T_0}{c_{22}} T^{(n)}, \\
q_2^{(n)} &= \sigma_n T^{(n)}, \quad n = 1, 2, \dots, 6.
\end{aligned} \tag{33}$$

Further, on the basis of the relations (28), (29) and (33), we have

$$\begin{aligned}
t_{12}^{(1)} &= \sigma_1 \Lambda_{11}(\sigma_1, \psi) + \Lambda_{21}(\sigma_1, \psi), \quad q_2^{(1)} = \sigma_1 \Lambda_{31}(\sigma_1, \psi), \\
t_{22}^{(1)} &= \frac{c_{12}}{c_{22}} \Lambda_{11}(\sigma_1, \psi) + \sigma_1 \Lambda_{21}(\sigma_1, \psi) - \frac{i\beta_2 T_0}{c_{22}} \Lambda_{31}(\sigma_1, \psi); \\
t_{12}^{(2)} &= \sigma_2 \Lambda_{12}(\sigma_2, \psi) + \Lambda_{22}(\sigma_2, \psi), \quad q_2^{(2)} = \sigma_2 \Lambda_{32}(\sigma_2, \psi), \\
t_{22}^{(2)} &= \frac{c_{12}}{c_{22}} \Lambda_{12}(\sigma_2, \psi) + \sigma_2 \Lambda_{22}(\sigma_2, \psi) - \frac{i\beta_2 T_0}{c_{22}} \Lambda_{32}(\sigma_2, \psi); \\
t_{12}^{(3)} &= \sigma_3 \Lambda_{13}(\sigma_3, \psi) + \Lambda_{23}(\sigma_3, \psi), \quad q_2^{(3)} = \sigma_3 \Lambda_{33}(\sigma_3, \psi), \\
t_{22}^{(3)} &= \frac{c_{12}}{c_{22}} \Lambda_{13}(\sigma_3, \psi) + \sigma_3 \Lambda_{23}(\sigma_3, \psi) - \frac{i\beta_2 T_0}{c_{22}} \Lambda_{33}(\sigma_3, \psi),
\end{aligned} \tag{34}$$

and

$$\begin{aligned}
t_{12}^{(4)} &= \sigma_4 \Lambda_{11}(\sigma_4, \psi) + \Lambda_{21}(\sigma_4, \psi), \quad q_2^{(4)} = \sigma_4 \Lambda_{31}(\sigma_4, \psi), \\
t_{22}^{(4)} &= \frac{c_{12}}{c_{22}} \Lambda_{11}(\sigma_4, \psi) + \sigma_4 \Lambda_{21}(\sigma_4, \psi) - \frac{i\beta_2 T_0}{c_{22}} \Lambda_{31}(\sigma_4, \psi); \\
t_{12}^{(5)} &= \sigma_5 \Lambda_{12}(\sigma_5, \psi) + \Lambda_{22}(\sigma_5, \psi), \quad q_2^{(5)} = \sigma_5 \Lambda_{32}(\sigma_5, \psi), \\
t_{22}^{(5)} &= \frac{c_{12}}{c_{22}} \Lambda_{12}(\sigma_5, \psi) + \sigma_5 \Lambda_{22}(\sigma_5, \psi) - \frac{i\beta_2 T_0}{c_{22}} \Lambda_{32}(\sigma_5, \psi); \\
t_{12}^{(6)} &= \sigma_6 \Lambda_{13}(\sigma_6, \psi) + \Lambda_{23}(\sigma_6, \psi), \quad q_2^{(6)} = \sigma_6 \Lambda_{33}(\sigma_6, \psi), \\
t_{22}^{(6)} &= \frac{c_{12}}{c_{22}} \Lambda_{13}(\sigma_6, \psi) + \sigma_6 \Lambda_{23}(\sigma_6, \psi) - \frac{i\beta_2 T_0}{c_{22}} \Lambda_{33}(\sigma_6, \psi).
\end{aligned} \tag{35}$$

4. Exact effective boundary conditions

The entire effect of the thermoelastic layer on the half-space should be replaced by the exact effective boundary conditions at the interface. Thus the wave should be then considered as a Rayleigh wave propagating in the half-space, without the coating layer, that is subjected to the effective boundary conditions. In fact, the wave solutions in the half-space $x_2 > 0$ are searched in the following form

$$\begin{aligned}
u_1^*(x_1, x_2, t) &= \operatorname{Re} \left\{ \frac{1}{\varkappa} \mathcal{U}_1^*(x_2) e^{i\varkappa(x_1 - c_1^* \psi t)} \right\}, \\
u_2^*(x_1, x_2, t) &= \operatorname{Re} \left\{ \frac{1}{\varkappa} \mathcal{U}_2^*(x_2) e^{i\varkappa(x_1 - c_1^* \psi t)} \right\}, \\
\theta^*(x_1, x_2, t) &= \operatorname{Re} \{ T_0 T^*(x_2) e^{i\varkappa(x_1 - c_1^* \psi t)} \},
\end{aligned} \tag{36}$$

with the stresses and the heat flux given by

$$\begin{aligned}
t_{12}^*(x_1, x_2, t) &= \operatorname{Re} \left\{ i c_{66}^* T_{12}^*(x_2) e^{i\varkappa(x_1 - c_1^* \psi t)} \right\}, \\
t_{22}^*(x_1, x_2, t) &= \operatorname{Re} \left\{ i c_{22}^* T_{22}^*(x_2) e^{i\varkappa(x_1 - c_1^* \psi t)} \right\}, \\
q_2^*(x_1, x_2, t) &= \operatorname{Re} \{ i \varkappa k_2^* T_0 \mathcal{Q}_2^*(x_2) e^{i\varkappa(x_1 - c_1^* \psi t)} \},
\end{aligned} \tag{37}$$

and satisfying the asymptotic conditions

$$\lim_{x_2 \rightarrow \infty} (\mathcal{U}_1^*(x_2), \mathcal{U}_2^*(x_2), T^*(x_2), T_{12}^*(x_2), T_{22}^*(x_2), \mathcal{Q}_2^*(x_2)) = 0. \tag{38}$$

In this idea we note that the relations (31), (32) and (36) and (37), when replaced in the conditions on the interface (10) lead to

$$\begin{aligned}
\sum_{n=1}^6 B_n U_1^{(n)} &= \mathcal{U}_1^*(0), \quad \sum_{n=1}^6 B_n U_2^{(n)} = \mathcal{U}_2^*(0), \quad \sum_{n=1}^6 B_n T^{(n)} = T^*(0), \\
\sum_{n=1}^6 B_n t_{12}^{(n)} &= \gamma T_{12}^*(0), \quad \sum_{n=1}^6 B_n t_{22}^{(n)} = \delta T_{22}^*(0), \quad \sum_{n=1}^6 B_n q_2^{(n)} = \zeta \mathcal{Q}_2^*(0),
\end{aligned} \tag{39}$$

where

$$\gamma = \frac{c_{66}^*}{c_{66}}, \quad \delta = \frac{c_{22}^*}{c_{22}}, \quad \zeta = \frac{k_2^*}{k_2}. \tag{40}$$

Therefore, we can write

$$\Gamma_{(6 \times 6)} \times \mathbf{B}_{(6 \times 1)} = \mathbf{D}_{(6 \times 1)}, \tag{41}$$

where

$$\Gamma_{(6 \times 6)} = \begin{pmatrix} U_1^{(1)} & U_1^{(2)} & U_1^{(3)} & U_1^{(4)} & U_1^{(5)} & U_1^{(6)} \\ U_2^{(1)} & U_2^{(2)} & U_2^{(3)} & U_2^{(4)} & U_2^{(5)} & U_2^{(6)} \\ T^{(1)} & T^{(2)} & T^{(3)} & T^{(4)} & T^{(5)} & T^{(6)} \\ t_{12}^{(1)} & t_{12}^{(2)} & t_{12}^{(3)} & t_{12}^{(4)} & t_{12}^{(5)} & t_{12}^{(6)} \\ t_{22}^{(1)} & t_{22}^{(2)} & t_{22}^{(3)} & t_{22}^{(4)} & t_{22}^{(5)} & t_{22}^{(6)} \\ q_2^{(1)} & q_2^{(2)} & q_2^{(3)} & q_2^{(4)} & q_2^{(5)} & q_2^{(6)} \end{pmatrix}, \tag{42}$$

and

$$\begin{aligned}
\mathbf{B}_{(6 \times 1)} &= (B_1, B_2, B_3, B_4, B_5, B_6)^T, \\
\mathbf{D}_{(6 \times 1)} &= (\mathcal{U}_1^*(0), \mathcal{U}_2^*(0), T^*(0), \gamma T_{12}^*(0), \delta T_{22}^*(0), \zeta \mathcal{Q}_2^*(0))^T.
\end{aligned} \tag{43}$$

On the other hand, based on relation (32), the boundary condition on the free surface of the layer, that is relation (11), can be written in the following matrix form

$$\Pi_{(3 \times 6)} \mathbf{B}_{(6 \times 1)} = \mathbf{0}_{(3 \times 1)}, \tag{44}$$

where

$$\Pi_{(3 \times 6)} = \begin{pmatrix} s_{12}^{(1)} & s_{12}^{(2)} & s_{12}^{(3)} & s_{12}^{(4)} & s_{12}^{(5)} & s_{12}^{(6)} \\ s_{22}^{(1)} & s_{22}^{(2)} & s_{22}^{(3)} & s_{22}^{(4)} & s_{22}^{(5)} & s_{22}^{(6)} \\ p_2^{(1)} & p_2^{(2)} & p_2^{(3)} & p_2^{(4)} & p_2^{(5)} & p_2^{(6)} \end{pmatrix}, \quad (45)$$

with

$$s_{12}^{(n)} = t_{12}^{(n)} e^{-i\kappa\sigma_n h}, \quad s_{22}^{(n)} = t_{22}^{(n)} e^{-i\kappa\sigma_n h}, \quad p_2^{(n)} = q_2^{(n)} e^{-i\kappa\sigma_n h}, \quad n = 1, 2, \dots, 6. \quad (46)$$

By eliminating the dimensionless parameters $\mathbf{B}_{(6 \times 1)} = (B_1, B_2, B_3, B_4, B_5, B_6)^T$ between the relations (41) and (44), we obtain the so-called exact effective boundary condition

$$\Pi_{(3 \times 6)} \times \Gamma_{(6 \times 6)}^{-1} \times \mathbf{D}_{(6 \times 1)} = \mathbf{0}_{(3 \times 1)}, \quad (47)$$

a relation involving $\mathbf{D}_{(6, 1)}$, that is the values of the displacements and the temperature variation, as well as the values of the stress and the heat flux on the interface $x_2 = 0$.

Consequently, we can now proceed to determine the exact secular equation for the Rayleigh wave propagating in the direction of the Ox_1 axis, with a decrease in the depth of the half-space $x_2 > 0$, under the effective boundary condition (47) on the surface $x_2 = 0$, without taking into consideration the existence of the thermoelastic layer $-h < x_2 < 0$, but only its influence described by relation (47) through the matrices Π and Γ .

5. Explicit secular equation of Rayleigh waves

In Section 3, we establish the free wave solutions in the thermoelastic layer in the form of relations (31) and (32). Herewith, we use our analysis in Section 3 in order to establish the corresponding wave solutions for the differential system (8), in the thermoelastic half-space $x_2 > 0$. Thus, we take into account the asymptotic conditions (13), to write

$$\begin{aligned} u_1^*(x_1, x_2, t) &= \text{Re} \left\{ \frac{1}{\kappa} e^{i\kappa(x_1 - c_1^* \psi t)} \sum_{p=1}^3 E_p U_1^{*(p)} e^{i\kappa\sigma_p^* x_2} \right\}, \\ u_2^*(x_1, x_2, t) &= \text{Re} \left\{ \frac{1}{\kappa} e^{i\kappa(x_1 - c_1^* \psi t)} \sum_{p=1}^3 E_p U_2^{*(p)} e^{i\kappa\sigma_p^* x_2} \right\}, \\ \theta^*(x_1, x_2, t) &= \text{Re} \left\{ T_0 e^{i\kappa(x_1 - c_1^* \psi t)} \sum_{p=1}^3 E_p T^{*(p)} e^{i\kappa\sigma_p^* x_2} \right\}, \end{aligned} \quad (48)$$

where E_1 , E_2 and E_3 are complex arbitrary dimensionless parameters, not all zero. Further, in view of the constitutive Equations (3) and by using relation (48), we have

$$\begin{aligned} t_{12}^*(x_1, x_2, t) &= \text{Re} \left\{ i c_{66}^* e^{i\kappa(x_1 - c_1^* \psi t)} \sum_{p=1}^3 E_p t_{12}^{*(p)} e^{i\kappa\sigma_p^* x_2} \right\}, \\ t_{22}^*(x_1, x_2, t) &= \text{Re} \left\{ i c_{22}^* e^{i\kappa(x_1 - c_1^* \psi t)} \sum_{p=1}^3 E_p t_{22}^{*(p)} e^{i\kappa\sigma_p^* x_2} \right\}, \\ q_{12}^*(x_1, x_2, t) &= \text{Re} \left\{ i \kappa k_2^* T_0 e^{i\kappa(x_1 - c_1^* \psi t)} \sum_{p=1}^3 E_p q_{12}^{*(p)} e^{i\kappa\sigma_p^* x_2} \right\}, \end{aligned} \quad (49)$$

where

$$\begin{aligned}
t_{12}^{*(p)} &= \sigma_p^* U_1^{*(p)} + U_2^{*(p)}, \\
t_{22}^{*(p)} &= \frac{c_{12}^*}{c_{22}^*} U_1^{*(p)} + \sigma_p^* U_2^{*(p)} - \frac{i\beta_2^* T_0}{c_{22}^*} T^{*(p)}, \\
q_2^{*(p)} &= \sigma_p^* T^{*(p)}, \quad p = 1, 2, 3.
\end{aligned} \tag{50}$$

At this instant, we must note that the quantities π_1^* , π_2^* and π_3^* are the roots of [equation \(23\)](#), where the material characteristics of [relation \(24\)](#) must be taken as those corresponding to the material of the half-space $x_2 > 0$. Furthermore, the quantities σ_1^* , σ_2^* and σ_3^* are calculated using the [relations \(25\)](#) and [\(26\)](#), with these values thus calculated, we obtain the values $U_1^{*(p)}$, $U_2^{*(p)}$ and $T^{*(p)}$, $p = 1, 2, 3$, based on the [relation \(28\)](#), which are involved in the [relations \(48\)](#) and [\(49\)](#).

Corroborating now with what was said in [Section 4](#), it follows that

$$\begin{aligned}
\mathcal{U}_1^*(0) &= \sum_{p=1}^3 E_p U_1^{*(p)}, \quad \mathcal{U}_2^*(0) = \sum_{p=1}^3 E_p U_2^{*(p)}, \quad \mathcal{T}^*(0) = \sum_{p=1}^3 E_p T^{*(p)}, \\
\mathcal{T}_{12}^*(0) &= \sum_{p=1}^3 E_p t_{12}^{*(p)}, \quad \mathcal{T}_{22}^*(0) = \sum_{p=1}^3 E_p t_{22}^{*(p)}, \quad \mathcal{Q}_2^*(0) = \sum_{p=1}^3 E_p q_2^{*(p)},
\end{aligned} \tag{51}$$

and hence, in view of [relation \(43\)₂](#), we have

$$\mathbf{D}_{(6 \times 1)} = \mathbf{\Omega}_{(6 \times 3)} \times \mathbf{E}_{(3 \times 1)}, \tag{52}$$

where

$$\mathbf{\Omega}_{(6 \times 3)} = \begin{pmatrix} U_1^{*(1)} & U_1^{*(2)} & U_1^{*(3)} \\ U_2^{*(1)} & U_2^{*(2)} & U_2^{*(3)} \\ T^{*(1)} & T^{*(2)} & T^{*(3)} \\ \gamma t_{12}^{*(1)} & \gamma t_{12}^{*(2)} & \gamma t_{12}^{*(3)} \\ \delta t_{22}^{*(1)} & \delta t_{22}^{*(2)} & \delta t_{22}^{*(3)} \\ \zeta q_2^{*(1)} & \zeta q_2^{*(2)} & \zeta q_2^{*(3)} \end{pmatrix}, \quad \mathbf{E}_{(3 \times 1)} = (E_1, E_2, E_3)^T. \tag{53}$$

If we now use the effective boundary conditions [\(47\)](#) for the free wave solution [\(48\)](#) and [\(49\)](#), we obtain the following homogeneous algebraic system

$$\mathbf{\Pi}_{(3 \times 6)} \times \mathbf{\Gamma}_{(6 \times 6)}^{-1} \times \mathbf{\Omega}_{(6 \times 3)} \times \mathbf{E}_{(3 \times 1)} = \mathbf{0}_{(3 \times 1)}, \tag{54}$$

which must admit a non-zero solution $\mathbf{E}_{(3 \times 1)} \neq \mathbf{0}$. Therefore, we obtain the exact secular equation in the following form

$$\text{Det} \left(\mathbf{\Pi}_{(3 \times 6)} \times \mathbf{\Gamma}_{(6 \times 6)}^{-1} \times \mathbf{\Omega}_{(6 \times 3)} \right) = 0. \tag{55}$$

Consequently, the wave solution represented by the [relations \(48\)](#) and [\(49\)](#) represents the Rayleigh wave propagating in the half-space $x_2 > 0$, assuming that the effective boundary conditions [\(47\)](#) are fulfilled on its surface $x_2 = 0$, as if the thermoelastic layer was absent!

[Relation \(55\)](#) represents explicitly the exact secular equation of the Rayleigh wave in an orthotropic thermoelastic half-space overlaid by an orthotropic thermoelastic layer for determining the dimensionless parameter ψ or for determining $c = c_1^* \psi$. In general, the parameter ψ is a complex number, so $\text{Re}(\psi) \geq 0$ will represent the dimensionless wave speed and $\text{Im}(\psi) \leq 0$ will be related to the decay rate in time of the wave amplitude. We must note that for $\text{Re}(\psi) > 0$, we will have a genuine wave, while for $\text{Re}(\psi) = 0$, we will have a standing mode. Moreover, when $\text{Im}(\psi) > 0$,

we will have the phenomenon of damping in time of the amplitude, while for $\text{Im}(\psi) = 0$, we will have undamped in time waves. Consequently, there have to get solutions of the exact secular Equation (55) satisfying conditions

$$\text{Re}(\psi) \geq 0, \quad \text{Im}(\psi) \leq 0. \quad (56)$$

Remark 1. We can avoid calculating the inverse of the $\Gamma_{(6 \times 6)}$ matrix, obtaining an equivalent form of the exact secular equation (55). In fact, we will replace the relations (31), (32), (48) and (49) in the compatibility relations described by (10), as well as in the boundary conditions described by (11). Thus, we obtain for the arbitrary constants $B_1, B_2, \dots, B_6$ and E_1, E_2 and E_3 , the following homogeneous algebraic system

$$\begin{aligned} \sum_{n=1}^6 B_n U_1^{(n)} - \sum_{p=1}^3 E_p U_1^{*(p)} &= 0, \quad \sum_{n=1}^6 B_n U_2^{(n)} - \sum_{p=1}^3 E_p U_2^{*(p)} = 0, \\ \sum_{n=1}^6 B_n T^{(n)} - \sum_{p=1}^3 E_p T^{*(p)} &= 0, \end{aligned} \quad (57)$$

$$\begin{aligned} \sum_{n=1}^6 B_n t_{12}^{(n)} - \gamma \sum_{p=1}^3 E_p t_{12}^{*(p)} &= 0, \quad \sum_{n=1}^6 B_n t_{22}^{(n)} - \delta \sum_{p=1}^3 E_p t_{22}^{*(p)} = 0, \\ \sum_{n=1}^6 B_n q_2^{(n)} - \zeta \sum_{p=1}^3 E_p q_2^{*(p)} &= 0, \end{aligned} \quad (58)$$

$$\sum_{n=1}^6 B_n s_{12}^{(n)} = 0, \quad \sum_{n=1}^6 B_n s_{22}^{(n)} = 0, \quad \sum_{n=1}^6 B_n p_2^{(n)} = 0. \quad (59)$$

Since not all arbitrary constants $B_1, B_2, \dots, B_6$ and E_1, E_2 and E_3 can be zero, it follows that the determinant of the homogeneous algebraic system described by relations (57) to (59) must be vanishing, that is

$$\text{Det} \left(\Psi_{(9 \times 9)}(\psi_{mn}) \right) = 0, \quad (60)$$

where ψ_{mn} , $m, n = 1, 2, \dots, 9$, are defined in the Appendix 2.

6. Some interesting limiting cases

In this section we will use our analysis from the previous sections to establish the exact secular equation for the situation where the thermal effects are canceled, as well as the one where both the thermoelastic layer and the thermoelastic half-space are isotropic.

6.1. Elastic orthotropic layered structure

In this subsection, we will eliminate the thermal effects both in the thermoelastic layer and in the half-space, that is, we will consider that they are filled with orthotropic elastic materials so that the matrix Λ in relation (22) becomes

$$\Lambda_{(2 \times 2)}^e(\sigma, \psi) \equiv \begin{pmatrix} \sigma^2 + \frac{c_1^2 - c_1^{*2} \psi^2}{c_6^2} & \frac{c_{12} + c_{66}}{c_{22}} \sigma \\ \frac{c_{12} + c_{66}}{c_{22}} \sigma & \sigma^2 + \frac{c_2^2 - c_1^{*2} \psi^2}{c_2^2} \end{pmatrix}. \quad (61)$$

Thus, we have

$$\pi_{1,2}^e = \frac{1}{2c_{22}c_{66}} \left[(c_{12} + c_{66})^2 - c_{22}(c_{11} - \varrho c_1^{*2} \psi^2) - c_{66}(c_{66} - \varrho c_1^{*2} \psi^2) \pm \sqrt{\Delta} \right], \quad (62)$$

where

$$\Delta = \left[(c_{12} + c_{66})^2 - c_{22}(c_{11} - \varrho c_1^{*2} \psi^2) + c_{66}(c_{66} - \varrho c_1^{*2} \psi^2) \right]^2 - 4c_{66}(c_{12} + c_{66})^2(c_{66} - \varrho c_1^{*2} \psi^2). \quad (63)$$

Therefore, now we have

$$\begin{aligned} \sigma_1^e &= \zeta_1, & \sigma_2^e &= \zeta_2, \\ \sigma_3^e &= -\zeta_1, & \sigma_4^e &= -\zeta_2, \\ \zeta_1 &= \sqrt{\pi_1^e}, & \zeta_2 &= \sqrt{\pi_2^e}, \quad \text{with } \text{Im}(\zeta_1) > 0, \quad \text{Im}(\zeta_2) > 0, \end{aligned} \quad (64)$$

and the wave solution (31), in the layer $-h < x_2 < 0$, is now as follows

$$\begin{aligned} u_1(x_1, x_2, t) &= \text{Re} \left\{ \frac{1}{\varkappa} e^{i\varkappa(x_1 - c_1^* \psi t)} \sum_{m=1}^4 F_m V_1^{(m)} e^{i\varkappa \sigma_m^e x_2} \right\}, \\ u_2(x_1, x_2, t) &= \text{Re} \left\{ \frac{1}{\varkappa} e^{i\varkappa(x_1 - c_1^* \psi t)} \sum_{m=1}^4 F_m V_2^{(m)} e^{i\varkappa \sigma_m^e x_2} \right\}, \end{aligned} \quad (65)$$

where

$$\begin{aligned} V_1^{(1)} &= \zeta_1^2 + \frac{c_6^2 - c_1^{*2} \psi^2}{c_2^2}, & V_2^{(1)} &= -\frac{c_{12} + c_{66}}{c_{22}} \zeta_1, \\ V_1^{(2)} &= -\frac{c_{12} + c_{66}}{c_{66}} \zeta_2, & V_2^{(2)} &= \zeta_2^2 + \frac{c_1^2 - c_1^{*2} \psi^2}{c_6^2}, \\ V_1^{(3)} &= \zeta_1^2 + \frac{c_6^2 - c_1^{*2} \psi^2}{c_2^2}, & V_2^{(3)} &= \frac{c_{12} + c_{66}}{c_{22}} \zeta_1, \\ V_1^{(4)} &= \frac{c_{12} + c_{66}}{c_{66}} \zeta_2, & V_2^{(4)} &= \zeta_2^2 + \frac{c_1^2 - c_1^{*2} \psi^2}{c_6^2}, \end{aligned} \quad (66)$$

and F_m , $m = 1, 2, 3, 4$, are arbitrary dimensionless constant parameters, not all zero. Moreover, we have

$$\begin{aligned} t_{12}^{e(1)} &= \zeta_1 \left(\zeta_1^2 - \frac{c_{12} + \varrho c_1^{*2} \psi^2}{c_{22}} \right), & t_{22}^{e(1)} &= -\frac{c_{66}}{c_{22}} \zeta_1^2 + \frac{c_{12}}{c_{22}} \frac{c_{66} - \varrho c_1^{*2} \psi^2}{c_{22}}, \\ t_{12}^{e(2)} &= -\frac{c_{12}}{c_{66}} \zeta_2^2 + \frac{c_{11} - \varrho c_1^{*2} \psi^2}{c_{66}}, & t_{22}^{e(2)} &= \zeta_2 \left(\zeta_2^2 + \frac{c_{11} - \varrho c_1^{*2} \psi^2}{c_{66}} - \frac{c_{12}(c_{12} + c_{66})}{c_{22}c_{66}} \right), \\ t_{12}^{e(3)} &= -\zeta_1 \left(\zeta_1^2 - \frac{c_{12} + \varrho c_1^{*2} \psi^2}{c_{22}} \right), & t_{22}^{e(3)} &= -\frac{c_{66}}{c_{22}} \zeta_1^2 + \frac{c_{12}}{c_{22}} \frac{c_{66} - \varrho c_1^{*2} \psi^2}{c_{22}}, \\ t_{12}^{e(4)} &= -\frac{c_{12}}{c_{66}} \zeta_2^2 + \frac{c_{11} - \varrho c_1^{*2} \psi^2}{c_{66}}, & t_{22}^{e(4)} &= -\zeta_2 \left(\zeta_2^2 + \frac{c_{11} - \varrho c_1^{*2} \psi^2}{c_{66}} - \frac{c_{12}(c_{12} + c_{66})}{c_{22}c_{66}} \right). \end{aligned} \quad (67)$$

Furthermore, in the half-space $x_2 > 0$ and by taking into account the asymptotic conditions (13), we have

$$\begin{aligned} u_1^*(x_1, x_2, t) &= \operatorname{Re} \left\{ \frac{1}{\varkappa} e^{i\varkappa(x_1 - c_1^* \psi t)} \left[G_1 V_1^{*(1)} e^{i\varkappa \sigma_1^* x_2} + G_2 V_1^{*(2)} e^{i\varkappa \sigma_2^* x_2} \right] \right\}, \\ u_2^*(x_1, x_2, t) &= \operatorname{Re} \left\{ \frac{1}{\varkappa} e^{i\varkappa(x_1 - c_1^* \psi t)} \left[G_1 V_2^{*(1)} e^{i\varkappa \sigma_1^* x_2} + G_2 V_2^{*(2)} e^{i\varkappa \sigma_2^* x_2} \right] \right\}, \end{aligned} \quad (68)$$

where G_1 and G_2 are complex arbitrary dimensionless parameters, not both zero. Moreover, there exists

$$\begin{aligned} t_{12}^{*e(1)} &= \zeta_1^* \left(\zeta_1^{*2} - \frac{c_{12}^* + \varrho^* c_1^{*2} \psi^2}{c_{22}^*} \right), \quad t_{22}^{*e(1)} = -\frac{c_{66}^*}{c_{22}^*} \zeta_1^{*2} + \frac{c_{12}^*}{c_{22}^*} \frac{c_{66}^* - \varrho^* c_1^{*2} \psi^2}{c_{22}^*}, \\ t_{12}^{*e(2)} &= -\frac{c_{12}^*}{c_{66}^*} \zeta_2^{*2} + \frac{c_{11}^* - \varrho^* c_1^{*2} \psi^2}{c_{66}^*}, \quad t_{22}^{*e(2)} = \zeta_2^* \left(\zeta_2^{*2} + \frac{c_{11}^* - \varrho^* c_1^{*2} \psi^2}{c_{66}^*} - \frac{c_{12}^* (c_{12}^* + c_{66}^*)}{c_{22}^* c_{66}^*} \right). \end{aligned} \quad (69)$$

Consequently, the secular equation is

$$\begin{vmatrix} V_1^{(1)} & V_1^{(2)} & V_1^{(3)} & V_1^{(4)} & -V_1^{*(1)} & -V_1^{*(2)} \\ V_2^{(1)} & V_2^{(2)} & V_2^{(3)} & V_2^{(4)} & -V_2^{*(1)} & -V_2^{*(2)} \\ t_{12}^{e(1)} & t_{12}^{e(2)} & t_{12}^{e(3)} & t_{12}^{e(4)} & -\gamma t_{12}^{*e(1)} & -\gamma t_{12}^{*e(2)} \\ t_{22}^{e(1)} & t_{22}^{e(2)} & t_{22}^{e(3)} & t_{22}^{e(4)} & -\delta t_{22}^{*e(1)} & -\delta t_{22}^{*e(2)} \\ s_{12}^{e(1)} & s_{12}^{e(2)} & s_{12}^{e(3)} & s_{12}^{e(4)} & 0 & 0 \\ s_{22}^{e(1)} & s_{22}^{e(2)} & s_{22}^{e(3)} & s_{22}^{e(4)} & 0 & 0 \end{vmatrix} = 0, \quad (70)$$

where

$$s_{12}^{e(m)} = t_{12}^{e(m)} e^{-i\varkappa \sigma_m^* h}, \quad s_{22}^{e(m)} = t_{22}^{e(m)} e^{-i\varkappa \sigma_m^* h}, \quad m = 1, 2, 3, 4. \quad (71)$$

The relationship (70) provides the exact secular equation for the propagation of a Rayleigh wave in an orthotropic elastic layered structure and represents an alternative to the exact secular equation established by Vinh et al. [22].

6.2. Isotropic thermoelastic layered structure

Throughout this subsection, we will consider that both the layer and the half-space are filled with isotropic thermoelastic materials so that we have

$$\begin{aligned} c_{11} &= c_{22} = \lambda + 2\mu, \quad c_{66} = \mu, \quad c_{12} = \lambda, \\ k_1 &= k_2 = k > 0, \quad \beta_1 = \beta_2 = \beta, \end{aligned} \quad (72)$$

where λ and μ are Lamé constants satisfying the restrictions

$$\mu > 0, \quad \lambda + 2\mu > 0. \quad (73)$$

We shall use the following wave speeds

$$C_1^2 = \frac{\lambda + 2\mu}{\varrho}, \quad C_2^2 = \frac{\mu}{\varrho}, \quad C_1^{*2} = \frac{\lambda^* + 2\mu^*}{\varrho^*}, \quad C_2^{*2} = \frac{\mu^*}{\varrho^*}, \quad (74)$$

and note that

$$c_1 = c_2 = C_1, \quad c_6 = C_2, \quad c_1^* = c_2^* = C_1^*, \quad c_6^* = C_2^*. \quad (75)$$

We further introduce the following non-dimensional parameters

$$\begin{aligned} \varepsilon &= \frac{\beta^2}{a\varrho C_1^2}, \quad K = \frac{\varkappa k}{T_0 a C_1}, \quad M = \frac{\beta T_0}{\varrho C_1^2}, \quad N = \frac{\beta C_1}{\varkappa k}, \\ \varepsilon^* &= \frac{\beta^{*2}}{a^* \varrho^* C_1^{*2}}, \quad K^* = \frac{\varkappa k^*}{T_0 a^* C_1^*}, \quad M^* = \frac{\beta^* T_0}{\varrho^* C_1^{*2}}, \quad N^* = \frac{\beta^* C_1^*}{\varkappa k^*}. \end{aligned} \quad (76)$$

Then the Equation (21) becomes

$$\left(s^2 - \frac{C_1^{*2}}{C_2^2} \psi^2 \right) \left\{ s^4 - \frac{C_1^*}{C_1} \psi \left[\frac{C_1^*}{C_1} \psi + \frac{i(1+\varepsilon)}{K} \right] s^2 + \frac{iC_1^{*3}}{C_1^3 K} \psi^3 \right\} = 0, \quad (77)$$

with

$$s^2 = \pi + 1 = \sigma^2 + 1. \quad (78)$$

Therefore, the roots of the Equation (76) are

$$\begin{aligned} s_1^2 &= \frac{C_1^{*2}}{C_2^2} \psi^2, \\ s_2^2 &= \frac{C_1^*}{2C_1} \psi \left\{ \frac{C_1^*}{C_1} \psi + \frac{i(\varepsilon+1)}{K} + \sqrt{\left(\frac{C_1^*}{C_1} \psi + \frac{i(\varepsilon-1)}{K} \right)^2 - \frac{4\varepsilon}{K^2}} \right\}, \\ s_3^2 &= \frac{C_1^*}{2C_1} \psi \left\{ \frac{C_1^*}{C_1} \psi + \frac{i(\varepsilon+1)}{K} - \sqrt{\left(\frac{C_1^*}{C_1} \psi + \frac{i(\varepsilon-1)}{K} \right)^2 - \frac{4\varepsilon}{K^2}} \right\}, \end{aligned} \quad (79)$$

and, moreover, there corresponding values for σ are

$$\begin{aligned} \tilde{\sigma}_1 &= i\tilde{\xi}_1, \quad \tilde{\sigma}_2 = i\tilde{\xi}_2, \quad \tilde{\sigma}_3 = i\tilde{\xi}_3, \\ \tilde{\sigma}_4 &= -i\tilde{\xi}_1, \quad \tilde{\sigma}_5 = -i\tilde{\xi}_2, \quad \tilde{\sigma}_6 = -i\tilde{\xi}_3, \\ \tilde{\xi}_1 &= \sqrt{1-s_1^2}, \quad \tilde{\xi}_2 = \sqrt{1-s_2^2}, \quad \tilde{\xi}_3 = \sqrt{1-s_3^2}. \end{aligned} \quad (80)$$

Then the wave solution in the thermoelastic layer $-h < x_2 < 0$ is

$$\begin{aligned} u_1(x_1, x_2, t) &= \text{Re} \left\{ \frac{1}{\varkappa} e^{i\varkappa(x_1 - C_1^* \psi t)} \sum_{n=1}^6 G_n W_1^{(n)} e^{i\varkappa \tilde{\sigma}_n x_2} \right\}, \\ u_2(x_1, x_2, t) &= \text{Re} \left\{ \frac{1}{\varkappa} e^{i\varkappa(x_1 - C_1^* \psi t)} \sum_{n=1}^6 G_n W_2^{(n)} e^{i\varkappa \tilde{\sigma}_n x_2} \right\}, \\ \theta(x_1, x_2, t) &= \text{Re} \left\{ T_0 e^{i\varkappa(x_1 - C_1^* \psi t)} \sum_{n=1}^6 G_n \Theta^{(n)} e^{i\varkappa \tilde{\sigma}_n x_2} \right\}, \end{aligned} \quad (81)$$

where

$$\begin{aligned} (W_1^{(1)}, W_2^{(1)}, \Theta^{(1)}) &= (-i\tilde{\xi}_1, 1, 0), \\ (W_1^{(2)}, W_2^{(2)}, \Theta^{(2)}) &= \left(s_2^2 - \frac{iC_1^* \psi}{C_1 K}, i\tilde{\xi}_2 \left(s_2^2 - \frac{iC_1^* \psi}{C_1 K} \right), \frac{NC_1^* \psi}{C_1} s_2^2 \right), \\ (W_1^{(3)}, W_2^{(3)}, \Theta^{(3)}) &= \left(iM, -M\tilde{\xi}_3, s_3^2 - \frac{C_1^{*2}}{C_1^2} \psi^2 \right), \end{aligned} \quad (82)$$

$$\begin{aligned}
(W_1^{(4)}, W_2^{(4)}, \Theta^{(4)}) &= (i\tilde{\xi}_1, 1, 0), \\
(W_1^{(5)}, W_2^{(5)}, \Theta^{(5)}) &= \left(s_2^2 - \frac{iC_1^* \psi}{C_1 K}, -i\tilde{\xi}_2 \left(s_2^2 - \frac{iC_1^* \psi}{C_1 K} \right), \frac{C_1^* N \psi}{C_1} s_2^2 \right), \\
(W_1^{(6)}, W_2^{(6)}, \Theta^{(6)}) &= \left(iM, M\tilde{\xi}_3, s_3^2 - \frac{C_1^*}{C_1^2} \psi^2 \right).
\end{aligned} \tag{83}$$

Furthermore, we have

$$\begin{aligned}
t_{12}(x_1, x_2, t) &= \operatorname{Re} \left\{ i\mu e^{i\lambda(x_1 - C_1^* \psi t)} \sum_{n=1}^6 G_n \tilde{t}_{12}^{(n)} e^{i\lambda \tilde{\sigma}_n x_2} \right\}, \\
t_{22}(x_1, x_2, t) &= \operatorname{Re} \left\{ i(\lambda + 2\mu) e^{i\lambda(x_1 - C_1^* \psi t)} \sum_{n=1}^6 G_n \tilde{t}_{22}^{(n)} e^{i\lambda \tilde{\sigma}_n x_2} \right\}, \\
q_2(x_1, x_2, t) &= \operatorname{Re} \left\{ i\lambda k T_0 e^{i\lambda(x_1 - C_1^* \psi t)} \sum_{n=1}^6 G_n \tilde{q}_2^{(n)} e^{i\lambda \tilde{\sigma}_n x_2} \right\},
\end{aligned} \tag{84}$$

where

$$\begin{aligned}
\tilde{t}_{12}^{(n)} &= \tilde{\sigma}_n W_1^{(n)} + W_2^{(n)}, \quad \tilde{q}_2^{(n)} = \tilde{\sigma}_n \Theta^{(n)}, \\
\tilde{t}_{22}^{(n)} &= \left(1 - \frac{2C_2^2}{C_1^2} \right) W_1^{(n)} + \tilde{\sigma}_n W_2^{(n)} - iMT^{(n)}.
\end{aligned} \tag{85}$$

In view of the relations (82), (83) and (85), we get

$$\begin{aligned}
\tilde{t}_{12}^{(1)} &= 2 - \frac{C_1^{*2}}{C_2^2} \psi^2, \quad \tilde{t}_{22}^{(1)} = 2i\tilde{\xi}_1 \frac{C_2^2}{C_1^2}, \quad \tilde{q}_2^{(1)} = 0, \\
\tilde{t}_{12}^{(2)} &= 2i\tilde{\xi}_2 \left(s_2^2 - \frac{iC_1^* \psi}{C_1 K} \right), \quad \tilde{t}_{22}^{(2)} = \left(s_2^2 - \frac{2C_2^2}{C_1^2} \right) \left(s_2^2 - \frac{iC_1^* \psi}{C_1 K} \right) - \frac{i\epsilon C_1^* \psi}{C_1 K} s_2^2, \\
\tilde{q}_2^{(2)} &= i \frac{NC_1^* \psi}{C_1} \tilde{\xi}_2 s_2^2, \\
\tilde{t}_{12}^{(3)} &= -2M\tilde{\xi}_3, \quad \tilde{t}_{22}^{(3)} = \frac{iM}{C_1^2} (C_1^{*2} \psi^2 - 2C_2^2), \quad \tilde{q}_2^{(3)} = i\tilde{\xi}_3 \left(s_3^2 - \frac{C_1^*}{C_1^2} \psi^2 \right),
\end{aligned} \tag{86}$$

and

$$\begin{aligned}
\tilde{t}_{12}^{(4)} &= 2 - \frac{C_1^{*2}}{C_2^2} \psi^2, \quad \tilde{t}_{22}^{(4)} = -2i\tilde{\xi}_1 \frac{C_2^2}{C_1^2}, \quad \tilde{q}_2^{(4)} = 0, \\
\tilde{t}_{12}^{(5)} &= -2i\tilde{\xi}_2 \left(s_2^2 - \frac{iC_1^* \psi}{C_1 K} \right), \quad \tilde{t}_{22}^{(5)} = \left(s_2^2 - \frac{2C_2^2}{C_1^2} \right) \left(s_2^2 - \frac{iC_1^* \psi}{C_1 K} \right) - \frac{i\epsilon C_1^* \psi}{C_1 K} s_2^2, \\
\tilde{q}_2^{(5)} &= -i \frac{NC_1^* \psi}{C_1} \tilde{\xi}_2 s_2^2, \\
\tilde{t}_{12}^{(6)} &= 2M\tilde{\xi}_3, \quad \tilde{t}_{22}^{(6)} = \frac{iM}{C_1^2} (C_1^{*2} \psi^2 - 2C_2^2), \quad \tilde{q}_2^{(6)} = -i\tilde{\xi}_3 \left(s_3^2 - \frac{C_1^*}{C_1^2} \psi^2 \right).
\end{aligned} \tag{87}$$

Finally, the exact secular equation for the isotropic thermoelastic layered structure takes the following form

$$\begin{vmatrix}
W_1^{(1)} & W_1^{(2)} & W_1^{(3)} & W_1^{(4)} & W_1^{(5)} & W_1^{(6)} & -W_1^{*(1)} & -W_1^{*(2)} & -W_1^{*(3)} \\
W_2^{(1)} & W_2^{(2)} & W_2^{(3)} & W_2^{(4)} & W_2^{(5)} & W_2^{(6)} & -W_2^{*(1)} & -W_2^{*(2)} & -W_2^{*(3)} \\
\Theta^{(1)} & \Theta^{(2)} & \Theta^{(3)} & \Theta^{(4)} & \Theta^{(5)} & \Theta^{(6)} & -\Theta^{*(1)} & -\Theta^{*(2)} & -\Theta^{*(3)} \\
\tilde{t}_{12}^{(1)} & \tilde{t}_{12}^{(2)} & \tilde{t}_{12}^{(3)} & \tilde{t}_{12}^{(4)} & \tilde{t}_{12}^{(5)} & \tilde{t}_{12}^{(6)} & -\tilde{\gamma}\tilde{t}_{12}^{*(1)} & -\tilde{\gamma}\tilde{t}_{12}^{*(2)} & -\tilde{\gamma}\tilde{t}_{12}^{*(3)} \\
\tilde{t}_{22}^{(1)} & \tilde{t}_{22}^{(2)} & \tilde{t}_{22}^{(3)} & \tilde{t}_{22}^{(4)} & \tilde{t}_{22}^{(5)} & \tilde{t}_{22}^{(6)} & -\tilde{\delta}\tilde{t}_{22}^{*(1)} & -\tilde{\delta}\tilde{t}_{22}^{*(2)} & -\tilde{\delta}\tilde{t}_{22}^{*(3)} \\
\tilde{q}_2^{(1)} & \tilde{q}_2^{(2)} & \tilde{q}_2^{(3)} & \tilde{q}_2^{(4)} & \tilde{q}_2^{(5)} & \tilde{q}_2^{(6)} & -\tilde{\zeta}\tilde{q}_2^{*(1)} & -\tilde{\zeta}\tilde{q}_2^{*(2)} & -\tilde{\zeta}\tilde{q}_2^{*(3)} \\
\tilde{s}_{12}^{(1)} & \tilde{s}_{12}^{(2)} & \tilde{s}_{12}^{(3)} & \tilde{s}_{12}^{(4)} & \tilde{s}_{12}^{(5)} & \tilde{s}_{12}^{(6)} & 0 & 0 & 0 \\
\tilde{s}_{22}^{(1)} & \tilde{s}_{22}^{(2)} & \tilde{s}_{22}^{(3)} & \tilde{s}_{22}^{(4)} & \tilde{s}_{22}^{(5)} & \tilde{s}_{22}^{(6)} & 0 & 0 & 0 \\
\tilde{p}_2^{(1)} & \tilde{p}_2^{(2)} & \tilde{p}_2^{(3)} & \tilde{p}_2^{(4)} & \tilde{p}_2^{(5)} & \tilde{p}_2^{(6)} & 0 & 0 & 0
\end{vmatrix} = 0, \quad (88)$$

where

$$\begin{aligned}
\tilde{\gamma} &= \frac{\mu^*}{\mu}, \quad \tilde{\delta} = \frac{\lambda^* + 2\mu^*}{\lambda + 2\mu}, \quad \tilde{\zeta} = \frac{k^*}{k}, \\
\tilde{s}_{12}^{(n)} &= \tilde{t}_{12}^{(n)} e^{-i\kappa\tilde{\sigma}_n h}, \quad \tilde{s}_{22}^{(n)} = \tilde{t}_{22}^{(n)} e^{-i\kappa\tilde{\sigma}_n h}, \quad \tilde{p}_2^{(n)} = \tilde{q}_2^{(n)} e^{-i\kappa\tilde{\sigma}_n h}, \quad n = 1, 2, \dots, 6.
\end{aligned} \quad (89)$$

The secular Equation (88) congruent with that established by Chiriță [43].

7. Numerical simulations

In order to illustrate our theoretical analysis from the preceding sections, we now present some numerical simulations. In this connection, we will consider that the thermoelastic half-space is made of Dolomite rock, and the thermoelastic layer consists of cobalt material. We are interested in how the two characteristics of the Rayleigh wave, the dimensionless wave speed $\text{Re}(\psi)$ and the dimensionless decay rate of the wave amplitude $\text{Im}(\psi)$, behave in relation to the thickness of the covering layer. Throughout this section, we take the wave number to be $\kappa = 1 \text{ m}^{-1}$ and the initial temperature to be $T_0 = 300 \text{ K}$.

We take for the thermoelastic half-space a Dolomite rock material as considered an anisotropic thermoelastic medium in Rasolofosaon and Zinszner [45] and Sharma [46] (see, also, [47]), with the following material characteristics

$$\begin{aligned}
\rho^* &= 2727 \text{ kg/m}^3, \quad c_{11}^* = 106.8 \times 10^9 \text{ kg/(ms}^2\text{)}, \quad c_{22}^* = 99.0 \times 10^9 \text{ kg/(ms}^2\text{)}, \\
c_{66}^* &= 37.82 \times 10^9 \text{ kg/(ms}^2\text{)}, \quad c_{12}^* = 27.1 \times 10^9 \text{ kg/(ms}^2\text{)}, \\
k_1^* &= 180 \text{ (kg m)/(s}^3\text{K)}, \quad k_2^* = 250 \text{ (kg m)/(s}^3\text{K)}, \quad a^* = 8181 \text{ kg/(ms}^2\text{K}^2\text{)}, \\
\beta_1^* &= 32 \times 10^4 \text{ kg/(m s}^2\text{K)}, \quad \beta_2^* = 14 \times 10^5 \text{ kg/(m s}^2\text{K)}.
\end{aligned} \quad (90)$$

Further, we assume that the half-space is welded covered by an orthotropic layer of cobalt material with the following characteristic coefficients (Biswas et al. [48])

$$\begin{aligned}
\rho &= 8836 \text{ kg/m}^3, \quad c_{11} = 3071 \times 10^8 \text{ kg/(m s}^2\text{)}, \quad c_{22} = 3581 \times 10^8 \text{ kg/(m s}^2\text{)}, \\
c_{66} &= 1510 \times 10^8 \text{ kg/(m s}^2\text{)}, \quad c_{12} = 1650 \times 10^8 \text{ kg/(m s}^2\text{)}, \\
k_1 &= 69.21 \text{ (kg m)/(s}^3\text{K)}, \quad k_2 = 70.1 \text{ (kg m)/(s}^3\text{K)}, \quad a = 8274.12 \text{ kg/(ms}^2\text{K}^2\text{)}, \\
\beta_1 &= 0.70 \times 10^5 \text{ kg/(ms}^2\text{K)}, \quad \beta_2 = 0.69 \times 10^5 \text{ kg/(ms}^2\text{K)}.
\end{aligned} \quad (91)$$

Therefore, we have

$$\begin{aligned}
c_1^* &= 6258.11 \text{ m/s}, \quad c_2^* = 6025.25 \text{ m/s}, \quad c_6^* = 3724.07 \text{ m/s}, \\
c_1 &= 5895.38 \text{ m/s}, \quad c_2 = 6366.11 \text{ m/s}, \quad c_6 = 4133.91 \text{ m/s},
\end{aligned} \quad (92)$$

and, moreover, we get the following values for the dimensionless parameters

Table 1. The behavior of solution of the secular equation (70) with respect to the layer's thickness.

$h[m]$	$Re(\psi)$	$Im(\psi)$
0	0.543811386817941007328	0
25×10^{-4}	0.543501798886273427784	0
5×10^{-3}	0.543188423530168354445	0
75×10^{-4}	0.542871325027236872759	0
10^{-2}	0.542550567178041533010	0
5×10^{-2}	0.536971298707644382977	0
10^{-1}	0.529075223536377032411	0

Table 2. Evolution of the solution for the secular equation (60) with respect to the layer's thickness.

$h[m]$	$Re(\psi)$	$Im(\psi)$
0	0.5438926286423674104	$-2.488975735073 \cdot 10^{-12}$
$25 \cdot 10^{-4}$	0.5435017998687430877	$-2.2335524280374 \cdot 10^{-12}$
$5 \cdot 10^{-3}$	0.5431884235076495129	$-4.9503906922908 \cdot 10^{-12}$
$75 \cdot 10^{-4}$	0.5428713250663208765	$-4.7513298109200 \cdot 10^{-12}$
10^{-2}	0.5425505671252486780	$-4.7473743662923 \cdot 10^{-12}$
$5 \cdot 10^{-2}$	0.5369712987189717452	$-4.69387383566661 \cdot 10^{-12}$
10^{-1}	0.5290752235376430346	$-4.61912939992415 \cdot 10^{-12}$

$$\begin{aligned}
M_1^* &= 0.002538, \quad N_1^* = 8.01038 \times 10^6, \quad M_2^* = 0.004242, \quad N_2^* = 3.50454 \times 10^7, \\
\mathcal{K}^* &= 0.72, \quad \mathcal{K}_2^* = 1.62768 \times 10^{-8}, \\
M_1 &= 0.000742, \quad N_1 = 6.2927 \times 10^6, \quad M_2 = 0.0000674, \quad N_2 = 620281, \\
\mathcal{K} &= 0.987304, \quad \mathcal{K}_2 = 4.48144 \times 10^{-9}, \\
\gamma &= 0.250464, \quad \delta = 0.276459, \quad \zeta = 0.0757347.
\end{aligned} \tag{93}$$

Table 1 reports the solution of the secular Equation (70), for various thickness values. For the elastic case $Im(\Psi) = 0$, while $Re(\Psi)$ decreases with increasing thickness h .

In contrast, Table 2 describes the behavior of the dimensionless propagation speed and the rate of decreasing of the Rayleigh wave amplitude with respect to the thickness of the thermoelastic layer. As can be seen, the dimensionless propagation speed of the Rayleigh wave decreases with increasing thickness of the thermoelastic layer. Compared to Table 1 which describes the elastic case, Table 2 highlights the fact that the presence of thermal effects leads to the appearance of time-dependent damping of the wave amplitude.

8. Conclusions

Choosing the right materials that exhibit both the desired thermal and orthotropic properties of a layered structure is crucial for the success of the design. In the context of layered structures, orthotropic materials are often used to enhance mechanical performance, such as stiffness and strength, in specific directions. For example, in aerospace and civil engineering, orthotropic layered structures can be designed to withstand various loads and stresses more efficiently than isotropic materials. Understanding and managing the thermal and orthotropic properties of an elastic material is essential for the successful design and application of thermoelastic orthotropic layered structures.

The present analysis provides the theoretical bases necessary for experimental research regarding the determination of the thermal and orthotropic characteristics of both the layer and the half-space by means of information regarding propagation of guided waves. In fact, this paper provides the explicit exact secular Equation (55) for the propagation of the Rayleigh wave in the orthotropic thermoelastic half-space $x_2 > 0$ under the exact effective boundary conditions (47). We establish the secular equation in terms of a characteristic dimensionless parameter relating to its speed of propagation and to its rate of decreasing in time. To the authors' knowledge, there is

no secular equation or effective boundary conditions similar to those in the present paper, nor is there any evidence of thermal and orthotropic effects on the wave propagation speed or on the time-dependent damping rate of the wave amplitude.

The present results, by themselves, allow for the determination of the material characteristics of the layered structure on the basis of those of the thermoelastic layer. Furthermore, for thin thermoelastic layers, both the exact effective boundary conditions and the exact secular equation can be developed in Taylor series in κh , obtaining simplified Rayleigh wave propagation problems that can lead to approximations of the thermal and orthotropic characteristics of the layered structure.

The numerical simulation of our analysis regarding the exact secular Equation (60) reveals the behavior in relation to the thickness of the covering layer, h , for the two characteristics of the Rayleigh wave propagating in the orthotropic thermoelastic layered structure. This suggests that the dimensionless speed propagation of the wave, $\text{Re}(\psi) > 0$, tends to decrease in relation to the increase of h , corroborated with the appearance of the damping rate in time of the wave amplitude term, $\text{Im}(\psi)$.

By establishing the effective boundary conditions (47), the entire influence of the thermoelastic layer on the substrate is known. These relations, together with the expansion of the stresses and of the heat flux at the top surface of the layer into Taylor series in its thickness furnish approximate secular equations of the waves supported by thin-film/substrate interactions that can be employed as theoretical bases for extracting the thermomechanical properties of the thin films from experimental data.

Finally, the relation (70) provides an explicit alternative to the exact secular equation for the propagation of Rayleigh waves supported by orthotropic elastic layer/elastic half-space interactions, previously presented in [22]. While the relation (84) represents the exact secular equation for the propagation of Rayleigh waves in an isotropic thermoelastic layer/isotropic thermoelastic substrate interaction as in [43].

Disclosure statement

No potential conflict of interest was reported by the authors.

ORCID

Cătălin Galeş  <http://orcid.org/0000-0003-3470-3276>

Stan Chiriță  <http://orcid.org/0000-0001-7560-0367>

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Appendix 1

The elements of the matrix $\text{adj}(\Lambda) = (\Lambda_{rs}(\sigma, \psi))_{1 \leq r, s \leq 3}$ are given by

$$\begin{aligned}\Lambda_{11}(\sigma, \psi) &= \left(\sigma^2 + \frac{c_6^2 - c_1^{*2} \psi^2}{c_2^2} \right) \left(\sigma^2 + \mathcal{K} - \frac{ic_1^* \psi}{c_1 \mathcal{K}_2} \right) - iM_2 N_2 \frac{c_1^* \psi}{c_1} \sigma^2, \\ \Lambda_{12}(\sigma, \psi) &= \sigma \left[iM_1 N_2 \frac{c_1^* \psi}{c_1} - \frac{c_{12} + c_{66}}{c_{66}} \left(\sigma^2 + \mathcal{K} - \frac{ic_1^* \psi}{c_1 \mathcal{K}_2} \right) \right],\end{aligned}\tag{94}$$

$$\begin{aligned}\Lambda_{13}(\sigma, \psi) &= iM_1 \left(\sigma^2 + \frac{c_6^2 - c_1^{*2} \psi^2}{c_2^2} \right) - iM_2 \frac{c_{12} + c_{66}}{c_{66}} \sigma^2, \\ \Lambda_{21}(\sigma, \psi) &= \sigma \left[iM_2 N_1 \frac{c_1^* \psi}{c_1} - \frac{c_{12} + c_{66}}{c_{22}} \left(\sigma^2 + \mathcal{K} - \frac{ic_1^* \psi}{c_1 \mathcal{K}_2} \right) \right], \\ \Lambda_{22}(\sigma, \psi) &= \left(\sigma^2 + \frac{c_1^2 - c_1^{*2} \psi^2}{c_6^2} \right) \left(\sigma^2 + \mathcal{K} - \frac{ic_1^* \psi}{c_1 \mathcal{K}_2} \right) - iM_1 N_1 \frac{c_1^* \psi}{c_1}, \\ \Lambda_{23}(\sigma, \psi) &= i\sigma \left[M_2 \left(\sigma^2 + \frac{c_1^2 - c_1^{*2} \psi^2}{c_6^2} \right) - \frac{c_{12} + c_{66}}{c_{22}} M_1 \right],\end{aligned}\tag{95}$$

$$\begin{aligned}
\Lambda_{31}(\sigma, \psi) &= \frac{c_1^* \psi}{c_1} \left[-\frac{c_{12} + c_{66}}{c_{22}} N_2 \sigma^2 + N_1 \left(\sigma^2 + \frac{c_6^2 - c_1^{*2} \psi^2}{c_2^2} \right) \right], \\
\Lambda_{32}(\sigma, \psi) &= \frac{c_1^* \psi \sigma}{c_1} \left[N_2 \left(\sigma^2 + \frac{c_6^2 - c_1^{*2} \psi^2}{c_2^2} \right) - \frac{c_{12} + c_{66}}{c_{66}} N_1 \right], \\
\Lambda_{33}(\sigma, \psi) &= \left(\sigma^2 + \frac{c_6^2 - c_1^{*2} \psi^2}{c_2^2} \right) \left(\sigma^2 + \frac{c_6^2 - c_1^{*2} \psi^2}{c_2^2} \right) - \frac{(c_{12} + c_{66})^2}{c_{22} c_{66}} \sigma^2.
\end{aligned} \tag{96}$$

Appendix 2

The elements ψ_{mn} , $m, n = 1, 2, \dots, 9$, of the matrix $\Psi_{(9 \times 9)}(\psi_{mn})$ are given by

$$\begin{aligned}
\psi_{11} &= \Lambda_{11}(\sigma_1, \psi), & \psi_{12} &= \Lambda_{12}(\sigma_2, \psi), & \psi_{13} &= \Lambda_{13}(\sigma_3, \psi), \\
\psi_{14} &= \Lambda_{11}(\sigma_4, \psi), & \psi_{15} &= \Lambda_{12}(\sigma_5, \psi), & \psi_{16} &= \Lambda_{13}(\sigma_6, \psi), \\
\psi_{17} &= -\Lambda_{11}^*(\sigma_1^*, \psi), & \psi_{18} &= -\Lambda_{12}^*(\sigma_2^*, \psi), & \psi_{19} &= -\Lambda_{13}^*(\sigma_3^*, \psi),
\end{aligned} \tag{97}$$

$$\begin{aligned}
\psi_{21} &= \Lambda_{21}(\sigma_1, \psi), & \psi_{22} &= \Lambda_{22}(\sigma_2, \psi), & \psi_{23} &= \Lambda_{23}(\sigma_3, \psi), \\
\psi_{24} &= \Lambda_{21}(\sigma_4, \psi), & \psi_{25} &= \Lambda_{22}(\sigma_5, \psi), & \psi_{26} &= \Lambda_{23}(\sigma_6, \psi), \\
\psi_{27} &= -\Lambda_{21}^*(\sigma_1^*, \psi), & \psi_{28} &= -\Lambda_{22}^*(\sigma_2^*, \psi), & \psi_{29} &= -\Lambda_{23}^*(\sigma_3^*, \psi),
\end{aligned} \tag{98}$$

$$\begin{aligned}
\psi_{31} &= \Lambda_{31}(\sigma_1, \psi), & \psi_{32} &= \Lambda_{32}(\sigma_2, \psi), & \psi_{33} &= \Lambda_{33}(\sigma_3, \psi), \\
\psi_{34} &= \Lambda_{31}(\sigma_4, \psi), & \psi_{35} &= \Lambda_{32}(\sigma_5, \psi), & \psi_{36} &= \Lambda_{33}(\sigma_6, \psi), \\
\psi_{37} &= -\Lambda_{31}^*(\sigma_1^*, \psi), & \psi_{38} &= -\Lambda_{32}^*(\sigma_2^*, \psi), & \psi_{39} &= -\Lambda_{33}^*(\sigma_3^*, \psi),
\end{aligned} \tag{99}$$

$$\begin{aligned}
\psi_{41} &= \sigma_1 \Lambda_{11}(\sigma_1, \psi) + \Lambda_{21}(\sigma_1, \psi), & \psi_{42} &= \sigma_2 \Lambda_{12}(\sigma_2, \psi) + \Lambda_{22}(\sigma_2, \psi), \\
\psi_{43} &= \sigma_3 \Lambda_{13}(\sigma_3, \psi) + \Lambda_{23}(\sigma_3, \psi), & \psi_{44} &= \sigma_4 \Lambda_{11}(\sigma_4, \psi) + \Lambda_{21}(\sigma_4, \psi), \\
\psi_{45} &= \sigma_5 \Lambda_{12}(\sigma_5, \psi) + \Lambda_{22}(\sigma_5, \psi), & \psi_{46} &= \sigma_6 \Lambda_{13}(\sigma_6, \psi) + \Lambda_{23}(\sigma_6, \psi), \\
\psi_{47} &= -\gamma [\sigma_1^* \Lambda_{11}^*(\sigma_1^*, \psi) + \Lambda_{21}^*(\sigma_1^*, \psi)], & \psi_{48} &= -\gamma [\sigma_2^* \Lambda_{12}^*(\sigma_2^*, \psi) + \Lambda_{22}^*(\sigma_2^*, \psi)], \\
\psi_{49} &= -\gamma [\sigma_3^* \Lambda_{13}^*(\sigma_3^*, \psi) + \Lambda_{23}^*(\sigma_3^*, \psi)],
\end{aligned} \tag{100}$$

$$\begin{aligned}
\psi_{51} &= \frac{c_{12}}{c_{22}} \Lambda_{11}(\sigma_1, \psi) + \sigma_1 \Lambda_{21}(\sigma_1, \psi) - \frac{i\beta_2 T_0}{c_{22}} \Lambda_{31}(\sigma_1, \psi), \\
\psi_{52} &= \frac{c_{12}}{c_{22}} \Lambda_{12}(\sigma_2, \psi) + \sigma_2 \Lambda_{22}(\sigma_2, \psi) - \frac{i\beta_2 T_0}{c_{22}} \Lambda_{32}(\sigma_2, \psi), \\
\psi_{53} &= \frac{c_{12}}{c_{22}} \Lambda_{13}(\sigma_3, \psi) + \sigma_3 \Lambda_{23}(\sigma_3, \psi) - \frac{i\beta_2 T_0}{c_{22}} \Lambda_{33}(\sigma_3, \psi), \\
\psi_{54} &= \frac{c_{12}}{c_{22}} \Lambda_{11}(\sigma_4, \psi) + \sigma_4 \Lambda_{21}(\sigma_4, \psi) - \frac{i\beta_2 T_0}{c_{22}} \Lambda_{31}(\sigma_4, \psi), \\
\psi_{55} &= \frac{c_{12}}{c_{22}} \Lambda_{12}(\sigma_5, \psi) + \sigma_5 \Lambda_{22}(\sigma_5, \psi) - \frac{i\beta_2 T_0}{c_{22}} \Lambda_{32}(\sigma_5, \psi), \\
\psi_{56} &= \frac{c_{12}}{c_{22}} \Lambda_{13}(\sigma_6, \psi) + \sigma_6 \Lambda_{23}(\sigma_6, \psi) - \frac{i\beta_2 T_0}{c_{22}} \Lambda_{33}(\sigma_6, \psi), \\
\psi_{57} &= -\delta \left[\frac{c_{12}^*}{c_{22}^*} \Lambda_{11}^*(\sigma_1^*, \psi) + \sigma_1^* \Lambda_{21}^*(\sigma_1^*, \psi) - \frac{i\beta_2^* T_0}{c_{22}^*} \Lambda_{31}^*(\sigma_1^*, \psi) \right], \\
\psi_{58} &= -\delta \left[\frac{c_{12}^*}{c_{22}^*} \Lambda_{12}^*(\sigma_2^*, \psi) + \sigma_2^* \Lambda_{22}^*(\sigma_2^*, \psi) - \frac{i\beta_2^* T_0}{c_{22}^*} \Lambda_{32}^*(\sigma_2^*, \psi) \right], \\
\psi_{59} &= -\delta \left[\frac{c_{12}^*}{c_{22}^*} \Lambda_{13}^*(\sigma_3^*, \psi) + \sigma_3^* \Lambda_{23}^*(\sigma_3^*, \psi) - \frac{i\beta_2^* T_0}{c_{22}^*} \Lambda_{33}^*(\sigma_3^*, \psi) \right],
\end{aligned} \tag{101}$$

$$\begin{aligned}
\psi_{61} &= \sigma_1 \Lambda_{31}(\sigma_1, \psi), & \psi_{62} &= \sigma_2 \Lambda_{32}(\sigma_2, \psi), & \psi_{63} &= \sigma_3 \Lambda_{33}(\sigma_3, \psi), \\
\psi_{64} &= \sigma_4 \Lambda_{31}(\sigma_4, \psi), & \psi_{65} &= \sigma_5 \Lambda_{32}(\sigma_5, \psi), & \psi_{66} &= \sigma_6 \Lambda_{33}(\sigma_6, \psi), \\
\psi_{67} &= -\zeta \sigma_1^* \Lambda_{31}^*(\sigma_1^*, \psi), & \psi_{68} &= -\zeta \sigma_2^* \Lambda_{32}^*(\sigma_2^*, \psi), & \psi_{69} &= -\zeta \sigma_3^* \Lambda_{33}^*(\sigma_3^*, \psi),
\end{aligned} \tag{102}$$

$$\begin{aligned}
\psi_{71} &= \psi_{41} e^{-i\kappa \sigma_1 h}, & \psi_{72} &= \psi_{42} e^{-i\kappa \sigma_2 h}, & \psi_{73} &= \psi_{43} e^{-i\kappa \sigma_3 h}, \\
\psi_{74} &= \psi_{44} e^{-i\kappa \sigma_4 h}, & \psi_{75} &= \psi_{45} e^{-i\kappa \sigma_5 h}, & \psi_{76} &= \psi_{46} e^{-i\kappa \sigma_6 h}, \\
\psi_{77} &= 0, & \psi_{78} &= 0, & \psi_{79} &= 0,
\end{aligned} \tag{103}$$

$$\begin{aligned}
\psi_{81} &= \psi_{51} e^{-i\kappa \sigma_1 h}, & \psi_{82} &= \psi_{52} e^{-i\kappa \sigma_2 h}, & \psi_{83} &= \psi_{53} e^{-i\kappa \sigma_3 h}, \\
\psi_{84} &= \psi_{54} e^{-i\kappa \sigma_4 h}, & \psi_{85} &= \psi_{55} e^{-i\kappa \sigma_5 h}, & \psi_{86} &= \psi_{56} e^{-i\kappa \sigma_6 h}, \\
\psi_{87} &= 0, & \psi_{88} &= 0, & \psi_{89} &= 0,
\end{aligned} \tag{104}$$

$$\begin{aligned}
\psi_{91} &= \psi_{61} e^{-i\kappa \sigma_1 h}, & \psi_{92} &= \psi_{62} e^{-i\kappa \sigma_2 h}, & \psi_{93} &= \psi_{63} e^{-i\kappa \sigma_3 h}, \\
\psi_{94} &= \psi_{64} e^{-i\kappa \sigma_4 h}, & \psi_{95} &= \psi_{65} e^{-i\kappa \sigma_5 h}, & \psi_{96} &= \psi_{66} e^{-i\kappa \sigma_6 h}, \\
\psi_{97} &= 0, & \psi_{98} &= 0, & \psi_{99} &= 0.
\end{aligned} \tag{105}$$