

Rayleigh waves in a layered thermoelastic structure

Mathematics and Mechanics of Solids
1–18

© The Author(s) 2025

Article reuse guidelines:

sagepub.com/journals-permissions

DOI: 10.1177/10812865241309911

journals.sagepub.com/home/mms**Stan Chiriță** *Alexandru Ioan Cuza University, Iași, Romania*

Received 4 July 2024; accepted 10 December 2024

Abstract

This paper aims to establish a general exact secular equation for Rayleigh surface waves propagating in an isotropic thermoelastic half-space coated by an isotropic thermoelastic layer of arbitrary thickness. The contact between the layer and the half-space is assumed to be welded on the interface. The outer surface of the layer is free of traction, and no heat transfer is allowed with the surrounding environment. When passing through the contact surface, the displacements and temperature fields, as well as the stresses and heat flux fields, are supposed to vary continuously, no jump being allowed. The explicit expressions of free Rayleigh waves in both the layer and the half-space are first presented, and then they are used to establish the general exact secular equation. Numerical illustrations are presented for a layered structure consisting of an aluminum layer welded on a half-space made of various thermoelastic materials outlining the main characteristics of Rayleigh wave propagation: the dimensionless propagation speed and the dimensionless damping in time of the wave's amplitude. The exact effective boundary conditions for the Rayleigh waves are established in the thermoelastic half-spaces covered with thin films. Some limiting cases are also discussed: (1) when the thermoelastic layer is absent and (2) when the thermal effects are neglected.

Keywords

Rayleigh waves, exact secular equation, isotropic thermoelastic half-space, isotropic thermoelastic layer, welded contact, exact effective boundary conditions

1. Introduction

Coated materials are widely used in engineering devices, especially for solid structures such as in the technologies of space vehicles (like space rockets (shuttles), satellites, etc.), which work in extreme conditions or have specific surface requirements. The use of a thin layer covering a half-space helps to improve the engineering properties of the substrate material and to prevent external damage caused by weather conditions, friction, chemical corrosion, or thermal barriers. In this line, there are good reasons for a continuing revival of interest to study the influence of the characteristic properties of the layer on the material substrate. An efficient tool in this study is represented by guided waves because their use presents advantages such as their non-destructive nature in testing materials, laboratory simulations, measurements of

Corresponding author:

Stan Chiriță, Faculty of Mathematics, Alexandru Ioan Cuza University Mathematics, 700506 Iași, Romania.

Email: schirita@uaic.ro

the mechanical properties of the covering layer, reduced material costs, as well as less time required for inspections.

In the light of what was said above, there was an intense concern for the study of the half-space filled with an elastic material (isotropic or anisotropic) on which an elastic layer (also isotropic or anisotropic) is overlaid (welded, smoothing, or with sliding contact). We mention here the studies carried out by Sezawa [1], Sezawa and Kanai [2], Haskell [3], Achenbach and Epstein [4], Tiersten [5], Makarov et al. [6], Kuchler and Richter [7], Cai and Fu [8], Sotiropoulos [9], Every [10], and Chattopadhyay and Raju [11]. Mathematical modeling of the elastic surface waves guided by thin films in layered structures was described in the books by Nayfeh [12] and Rose [13]. A comprehensive review of established and promising technologies under development in the emerging area of structural health monitoring of aerospace composite structures, as well as a study of guided waves in thin-wall composite structures, is developed in the excellent textbooks written by Nowinski [14] and Giurgiutiu [15].

A systematic approximation to the linear equations for small-amplitude surface waves in an elastic half space, interacting with a residually stressed thin film of different material bonded to its plane boundary, is developed by Steigmann and Ogden [16]. Recently, interesting results have been obtained both in terms of establishing effective boundary conditions and obtaining the exact secular equation for the propagation of Rayleigh waves in half-spaces on which an elastic layer is laid in the various forms of contact. In this connection, we mention the research developed by Dai et al. [17], Vinh and Anh [18], [19], [20], [21], Vinh et al. [22], [23], [24], Kaplunov and Prikazchikov [25], Erbaş et al. [26], Kaplunov et al. [27], [28], Phan et al. [29], Mubarak et al. [30], and Kaur et al. [31].

On the contrary, the effect of thermal properties on the propagation of Rayleigh waves in a thermoelastic half-space was investigated by various researchers, e.g., Lockett [32], Chadwick [33], Deresiewicz [34], Brock [35], Lykotrafitis and Georgiadis [36], [37], Chiriță [38], [39], and the Chapter *Rayleigh's Waves* of the Encyclopedia of Thermal Stresses [40]. Recently, it was shown by Zampoli and Chiriță [41] that by taking into account the thermal effects, the propagation speed of the Rayleigh waves tends to decrease in relation to the purely elastic situation and, more important, the presence of damping in time of the wave amplitude is signaled.

In this paper, the motion of Rayleigh surface waves in a layered structure modeled as an isotropic thermoelastic layer overlying an isotropic thermoelastic half-space is investigated. The explicit expressions of free Rayleigh waves in both the layer and the half-space are established. Using the boundary conditions on the free surface and at the interface, the exact secular equation of Rayleigh waves is provided. An illustrative numerical analysis is carried out on the model of a layered structure consisting of a layer of aluminum welded over a half-space made of a different material (copper, gold, iron, lead, nickel, or steel), highlighting the behavior of the two characteristic parameters of Rayleigh wave propagation: the dimensionless speed of wave propagation and the rate of damping in time of the wave amplitude.

The so-called exact effective boundary conditions are presented in terms of values of the displacements and temperature fields, as well as the stresses and the heat flux of the half-space at its surface. These boundary conditions replace the entire action of the layer on the half-space, i.e., the wave can be imagined as a Rayleigh wave propagating in the thermoelastic half-space, without the coating thermoelastic layer, but subjected to the effective boundary conditions.

We also show that our analysis in this paper provides, by neglecting the thermal effects, an alternative representation of the exact secular equation for an isotropic elastic layer overlaid on an isotropic elastic half-space with respect to that established by Vinh et al. [24].

On the contrary, by expanding the displacements and temperature fields, as well as the stresses and the heat flux vector, at the upper surface of the layer into Taylor series of the thickness of the layer, one can obtain approximate secular equations of Rayleigh waves propagating in thermoelastic half-space coated by a thin thermoelastic layer. Furthermore, they can be employed as theoretical bases for extracting the mechanical properties of the thin films from experimental data.

The plan of the work is the following. Section 2 presents the basic systems of differential equations describing the evolutionary thermoelastic behavior of the materials in the layer and the half-space, respectively. Section 3 introduces the classes of wave solutions that allow us to solve the Rayleigh surface wave problem. Section 4 treats the propagating Rayleigh wave problem and establishes the exact secular equation. Section 5 provides a numerical simulation exemplifying the analysis developed in the previous

sections. Section 6 establishes the exact effective boundary conditions. Section 7 treats the limiting case when the layer and half-space are both elastic, i.e., the thermal effects are absent.

2. The layered thermoelastic structure

Consider the isotropic thermoelastic half-space $0 < x_2 < \infty$ overlaid by an isotropic thermoelastic layer with uniform thickness h occupying the domain $-h < x_2 < 0$. The layer is assumed to be perfectly bonded to the half-space, and they are both compressible. Note that same quantities related to the half-space and the layer have the same symbol but are systematically distinguished by a star if pertaining to the half-space $0 < x_2 < \infty$. In accordance with what was said above and with the linear theory of isotropic thermoelasticity, in the half-space $x_2 > 0$, we have the basic motion equations:

$$\begin{aligned} S_{rs,r}^* &= \varrho^* \ddot{u}_s^*, \\ q_{r,r}^* &= T_0 \varrho^* \dot{\eta}^*, \end{aligned} \quad (1)$$

with the constitutive equations:

$$\begin{aligned} S_{rs}^* &= \lambda^* u_{p,p}^* \delta_{rs} + \mu^* (u_{r,s}^* + u_{s,r}^*) + \beta^* \theta^* \delta_{rs}, \\ \varrho^* \eta^* &= -\beta^* u_{r,r}^* + a^* \theta^*, \\ q_r^* &= k^* \theta_{,r}^*, \end{aligned} \quad (2)$$

while in the layer $-h < x_2 < 0$, we have:

$$\begin{aligned} S_{rs,r} &= \varrho \ddot{u}_s, \\ q_{r,r} &= T_0 \varrho \dot{\eta}, \end{aligned} \quad (3)$$

and

$$\begin{aligned} S_{rs} &= \lambda u_{p,p} \delta_{rs} + \mu (u_{r,s} + u_{s,r}) + \beta \theta \delta_{rs}, \\ \varrho \eta &= -\beta u_{r,r} + a \theta, \\ q_r &= k \theta_{,r}. \end{aligned} \quad (4)$$

In the above relations, we have used the following notations: λ and μ are the elastic Lamé constants, $k > 0$ is the constant thermal conductivity, $a > 0$ is the constant mass specific heat, $\beta = -(3\lambda + 2\mu)\alpha$ is the stress-temperature constant modulus, α is the coefficient of thermal expansion, ϱ is the density mass, while u_r are the components of the displacement vector and θ is the temperature variation from the absolute temperature $T_0 > 0$, S_{rs} are the components of the stress tensor, q_r are the components of the heat flux vector, and η is the entropy density per unit mass.

In terms of the displacements and temperature variation fields, the above equations can be written as follows:

$$\begin{aligned} \mu^* u_{r,ss}^* + (\lambda^* + \mu^*) u_{s,sr}^* + \beta^* \theta_{,r}^* &= \varrho^* \ddot{u}_r^*, \\ k^* \theta_{,ss}^* + T_0 \beta^* \dot{u}_{s,s}^* &= T_0 a^* \dot{\theta}^*, \end{aligned} \quad (5)$$

within the half-space $x_2 > 0$ and

$$\begin{aligned} \mu u_{r,ss} + (\lambda + \mu) u_{s,sr} + \beta \theta_{,r} &= \varrho \ddot{u}_r, \\ k \theta_{,ss} + T_0 \beta \dot{u}_{s,s} &= T_0 a \dot{\theta}, \end{aligned} \quad (6)$$

in the layer $-h < x_2 < 0$.

The contact between the thermoelastic layer and the half-space is supposed to be welded, and there are no temperature changes or heat transfer at the contact interface. This means that the displacements and temperature fields, as well as the stresses and the heat flux, are continuous through the interface of the layer and the half-space. Therefore, we must have the following conditions fulfilled:

$$\begin{aligned} u_s^*(x_1, 0, x_3, t) &= u_s(x_1, 0, x_3, t), \quad S_{s2}^*(x_1, 0, x_3, t) = S_{s2}(x_1, 0, x_3, t), \\ \theta^*(x_1, 0, x_3, t) &= \theta(x_1, 0, x_3, t), \quad q_2^*(x_1, 0, x_3, t) = q_2(x_1, 0, x_3, t), \quad \text{for all } x_1, x_3 \in \mathbb{R}, t > 0. \end{aligned} \quad (7)$$

Furthermore, we assume that the surface $x_2 = -h$ of the thermoelastic layer is free of traction and that there are no thermal exchanges with the outside, i.e., we have:

$$\begin{aligned} S_{12}(x_1, -h, x_3, t) = 0, \quad S_{22}(x_1, -h, x_3, t) = 0, \quad S_{32}(x_1, -h, x_3, t) = 0, \\ q_2(x_1, -h, x_3, t) = 0, \quad \text{for all } x_1, x_3 \in \mathbb{R}, t > 0. \end{aligned} \quad (8)$$

Alternatively, we can consider that the free surface of the layer is thermally insulated, when the condition (8)₂ is replaced by the following one:

$$\theta(x_1, -h, x_3, t) = 0, \quad \text{for all } x_1, x_3 \in \mathbb{R}, t > 0. \quad (9)$$

Finally, because we intend to talk about surface waves, we will have to impose zero asymptotic conditions on the mechanical and thermal quantities in the depth of the half-space, i.e., we will assume that:

$$\begin{aligned} \lim_{x_2 \rightarrow \infty} u_r^*(x_1, x_2, x_3, t) = 0, \quad \lim_{x_2 \rightarrow \infty} S_{rs}^*(x_1, x_2, x_3, t) = 0, \\ \lim_{x_2 \rightarrow \infty} \theta^*(x_1, x_2, x_3, t) = 0, \quad \lim_{x_2 \rightarrow \infty} q_r^*(x_1, x_2, x_3, t) = 0, \quad \text{for all } x_1, x_3 \in \mathbb{R}, t > 0. \end{aligned} \quad (10)$$

The following dimensionless coefficients are important in the study of thermoelastic phenomena:

$$\varepsilon^* = \frac{\beta^{*2}}{a^* \varrho^* c_1^{*2}}, \quad \varepsilon = \frac{\beta^2}{a \varrho c_1^2}, \quad (11)$$

where

$$c_1^{*2} = \frac{\lambda^* + 2\mu^*}{\varrho^*}, \quad c_1^2 = \frac{\lambda + 2\mu}{\varrho}. \quad (12)$$

Moreover, we also need the following notations:

$$c_2^{*2} = \frac{\mu^*}{\varrho^*}, \quad c_2^2 = \frac{\mu}{\varrho}. \quad (13)$$

3. Free wave solutions of the basic equations

Throughout this paper, we will study the propagation of a Rayleigh wave, traveling along the interface between the layer and the half-space with velocity $c = c_1 \psi$ and wave-number $\varkappa > 0$ in the x_1 -direction and decaying in the x_2 -direction. To solve the Rayleigh surface wave problem for the considered layered structure, we are looking for wave solutions of basic equation (6) in the following form:

$$\begin{aligned} u_1(x_1, x_2, x_3, t) &= \text{Re} \left\{ \frac{1}{\varkappa} \mathcal{U}_1(x_2) e^{i\varkappa(x_1 - c_1 \psi t)} \right\}, \\ u_2(x_1, x_2, x_3, t) &= \text{Re} \left\{ \frac{i}{\varkappa} \mathcal{U}_2(x_2) e^{i\varkappa(x_1 - c_1 \psi t)} \right\}, \\ u_3(x_1, x_2, x_3, t) &= 0, \\ \theta(x_1, x_2, x_3, t) &= \text{Re} \left\{ T_0 \mathcal{T}(x_2) e^{i\varkappa(x_1 - c_1 \psi t)} \right\}, \end{aligned} \quad (14)$$

leading to the following differential system for the unknown dimensionless vector $(\mathcal{U}_1(x_2), \mathcal{U}_2(x_2), \mathcal{T}(x_2))$:

$$\begin{aligned} \left[\mu \left(\frac{\partial^2}{\partial x_2^2} - \varkappa^2 \right) + \varrho \varkappa^2 c_1^2 \psi^2 \right] (\mathcal{U}_{1,2} + \varkappa \mathcal{U}_2) &= 0, \\ \left[(\lambda + 2\mu) \left(\frac{\partial^2}{\partial x_2^2} - \varkappa^2 \right) + \varrho \varkappa^2 c_1^2 \psi^2 \right] (\varkappa \mathcal{U}_1 + \mathcal{U}_{2,2}) - i\varkappa \beta T_0 (\mathcal{T}_{,22} - \varkappa^2 \mathcal{T}) &= 0, \\ \beta c_1 \psi (\varkappa \mathcal{U}_1 + \mathcal{U}_{2,2}) + k \left(\mathcal{T}_{,22} - \varkappa^2 \mathcal{T} + \frac{i\varkappa a c_1 T_0 \psi}{k} \mathcal{T} \right) &= 0. \end{aligned} \quad (15)$$

Furthermore, we seek for solutions of the differential system (15) in the form:

$$(\mathcal{U}_1(x_2), \mathcal{U}_2(x_2), \mathcal{T}(x_2)) = (U_1, U_2, T) e^{i\kappa\sigma x_2}, \quad (16)$$

with (U_1, U_2, T) being a non-zero constant complex vector and σ a dimensionless constant parameter. Consequently, the differential system (15) leads to the homogeneous algebraic system:

$$\begin{aligned} \left(s^2 - \frac{c_1^2}{c_2^2} \psi^2\right) (i\sigma U_1 + U_2) &= 0, \\ (s^2 - \psi^2)(U_1 + i\sigma U_2) - \frac{iT_0\beta}{\varrho c_1^2} s^2 T &= 0, \\ -\frac{\beta c_1 \psi}{k\kappa} (U_1 + i\sigma U_2) + \left(s^2 - \frac{i\psi}{K}\right) T &= 0, \end{aligned} \quad (17)$$

where

$$\begin{aligned} s^2 &= \sigma^2 + 1, \\ K &= \frac{\kappa k}{T_0 a c_1}. \end{aligned} \quad (18)$$

Since we have $(U_1, U_2, T) \neq 0$, it is necessary that the determinant of the algebraic system (17) to be zero, i.e.,

$$\left(s^2 - \frac{c_1^2}{c_2^2} \psi^2\right) \left\{ s^4 - \left[\psi^2 + \frac{i\psi(1+\varepsilon)}{K} \right] s^2 + \frac{i\psi^3}{K} \right\} = 0, \quad (19)$$

with the following roots:

$$\begin{aligned} s_1^2 &= \frac{c_1^2}{c_2^2} \psi^2, \\ s_2^2 &= \frac{\psi}{2} \left\{ \psi + \frac{i(\varepsilon+1)}{K} + \sqrt{\left(\psi + \frac{i(\varepsilon-1)}{K}\right)^2 - \frac{4\varepsilon}{K^2}} \right\}, \\ s_3^2 &= \frac{\psi}{2} \left\{ \psi + \frac{i(\varepsilon+1)}{K} - \sqrt{\left(\psi + \frac{i(\varepsilon-1)}{K}\right)^2 - \frac{4\varepsilon}{K^2}} \right\}. \end{aligned} \quad (20)$$

Consequently, there exist the following admissible values for the dimensionless parameter σ :

$$\begin{aligned} \sigma_1 &= i\xi_1, & \sigma_2 &= i\xi_2, & \sigma_3 &= i\xi_3, \\ \sigma_4 &= -i\xi_1, & \sigma_5 &= -i\xi_2, & \sigma_6 &= -i\xi_3, \end{aligned} \quad (21)$$

with

$$\xi_1 = \sqrt{1 - s_1^2}, \quad \xi_2 = \sqrt{1 - s_2^2}, \quad \xi_3 = \sqrt{1 - s_3^2}. \quad (22)$$

Then, the corresponding vector solutions of the differential system (15) are:

$$(\mathcal{U}_1^{(n)}, \mathcal{U}_2^{(n)}, \mathcal{T}^{(n)})(x_2) = (U_1^{(n)}, U_2^{(n)}, T^{(n)}) e^{i\kappa\sigma_n x_2}, \quad n = 1, 2, \dots, 6, \quad (23)$$

where

$$\begin{aligned} (U_1^{(1)}, U_2^{(1)}, T^{(1)}) &= (\xi_1, 1, 0), \\ (U_1^{(2)}, U_2^{(2)}, T^{(2)}) &= \left(s_2^2 - \frac{i\psi}{K}, \xi_2 \left(s_2^2 - \frac{i\psi}{K}\right), N\psi s_2^2\right) \\ (U_1^{(3)}, U_2^{(3)}, T^{(3)}) &= (iM, iM\xi_3, s_3^2 - \psi^2), \end{aligned} \quad (24)$$

$$\begin{aligned}
(U_1^{(4)}, U_2^{(4)}, T^{(4)}) &= (-\xi_1, 1, 0), \\
(U_1^{(5)}, U_2^{(5)}, T^{(5)}) &= \left(s_2^2 - \frac{i\psi}{K}, -\xi_2 \left(s_2^2 - \frac{i\psi}{K} \right), N\psi s_2^2 \right), \\
(U_1^{(6)}, U_2^{(6)}, T^{(6)}) &= (iM, -iM\xi_3, s_3^2 - \psi^2),
\end{aligned} \tag{25}$$

and the dimensionless parameters M and N are defined by:

$$M = \frac{\beta T_0}{\rho c_1^2}, \quad N = \frac{\beta c_1}{\kappa k}. \tag{26}$$

Finally, we note that:

$$(\mathcal{U}_1, \mathcal{U}_2, \mathcal{T})(x_2) = \sum_{n=1}^6 B_n \left(U_1^{(n)}, U_2^{(n)}, T^{(n)} \right) e^{i\kappa \sigma_n x_2}, \tag{27}$$

with $B_1, B_2, \dots, B_6$ being the arbitrary dimensionless constant parameters, represents a solution of the differential system (15). Consequently, in view of relations (14) and (27), we have that:

$$\begin{aligned}
u_1(x_1, x_2, x_3, t) &= \text{Re} \left\{ \frac{1}{\kappa} e^{i\kappa(x_1 - c_1 \psi t)} \sum_{n=1}^6 B_n U_1^{(n)} e^{i\kappa \sigma_n x_2} \right\}, \\
u_2(x_1, x_2, x_3, t) &= \text{Re} \left\{ \frac{i}{\kappa} e^{i\kappa(x_1 - c_1 \psi t)} \sum_{n=1}^6 B_n U_2^{(n)} e^{i\kappa \sigma_n x_2} \right\}, \\
u_3(x_1, x_2, x_3, t) &= 0, \\
\theta(x_1, x_2, x_3, t) &= \text{Re} \left\{ T_0 e^{i\kappa(x_1 - c_1 \psi t)} \sum_{n=1}^6 B_n T^{(n)} e^{i\kappa \sigma_n x_2} \right\},
\end{aligned} \tag{28}$$

is a solution of the differential system (6) in the layer $-h < x_2 < 0$. In view of the constitutive equations (4), from equation (28), we get:

$$\begin{aligned}
S_{12}(x_1, x_2, x_3, t) &= \text{Re} \left\{ \mu e^{i\kappa(x_1 - c_1 \psi t)} \sum_{n=1}^6 B_n S_{12}^{(n)} e^{i\kappa \sigma_n x_2} \right\}, \\
S_{22}(x_1, x_2, x_3, t) &= \text{Re} \left\{ i\mu e^{i\kappa(x_1 - c_1 \psi t)} \sum_{n=1}^6 B_n S_{22}^{(n)} e^{i\kappa \sigma_n x_2} \right\}, \\
S_{32}(x_1, x_2, x_3, t) &= 0, \\
q_2(x_1, x_2, x_3, t) &= \text{Re} \left\{ i\kappa k T_0 e^{i\kappa(x_1 - c_1 \psi t)} \sum_{n=1}^6 B_n Q_2^{(n)} e^{i\kappa \sigma_n x_2} \right\},
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
S_{12}^{(n)} &= i\sigma_n U_1^{(n)} - U_2^{(n)}, \\
S_{22}^{(n)} &= \frac{c_1^2}{c_2^2} \left(U_1^{(n)} + i\sigma_n U_2^{(n)} \right) - 2U_1^{(n)} - \frac{i\beta T_0}{\rho c_2^2} T^{(n)}, \\
Q_2^{(n)} &= \sigma_n T^{(n)}, \quad n = 1, 2, \dots, 6.
\end{aligned} \tag{30}$$

Furthermore, on the basis of the relations (24) and (25), we have:

$$\begin{aligned} S_{12}^{(1)} &= \psi^2 - 2, \quad S_{22}^{(1)} = -2\xi_1, \quad Q_2^{(1)} = 0, \\ S_{12}^{(2)} &= -2\xi_2 \left(s_2^2 - \frac{i\psi}{K} \right), \quad S_{22}^{(2)} = \left(\frac{c_1^2}{c_2^2} s_2^2 - 2 \right) \left(s_2^2 - \frac{i\psi}{K} \right) - \frac{i\varepsilon c_1^2}{c_2^2 K} \psi s_2^2, \quad Q_2^{(2)} = -N \psi \xi_2 s_2^2, \\ S_{12}^{(3)} &= -2iM \xi_3, \quad S_{22}^{(3)} = iM \left(\frac{c_1^2}{c_2^2} \psi^2 - 2 \right), \quad Q_2^{(3)} = -\xi_3 (s_3^2 - \psi^2), \end{aligned} \quad (31)$$

and

$$\begin{aligned} S_{12}^{(4)} &= \psi^2 - 2, \quad S_{22}^{(4)} = 2\xi_1, \quad Q_2^{(4)} = 0, \\ S_{12}^{(5)} &= 2\xi_2 \left(s_2^2 - \frac{i\psi}{K} \right), \quad S_{22}^{(5)} = \left(\frac{c_1^2}{c_2^2} s_2^2 - 2 \right) \left(s_2^2 - \frac{i\psi}{K} \right) - \frac{i\varepsilon c_1^2}{c_2^2 K} \psi s_2^2, \quad Q_2^{(5)} = N \psi \xi_2 s_2^2, \\ S_{12}^{(6)} &= 2iM \xi_3, \quad S_{22}^{(6)} = iM \left(\frac{c_1^2}{c_2^2} \psi^2 - 2 \right), \quad Q_2^{(6)} = \xi_3 (s_3^2 - \psi^2). \end{aligned} \quad (32)$$

Herewith, we write the corresponding wave solutions for the differential system (5), in the half-space $x_2 > 0$, and we take into account the asymptotic conditions (10), as:

$$\begin{aligned} u_1^*(x_1, x_2, x_3, t) &= \text{Re} \left\{ \frac{1}{\varkappa} e^{i\varkappa(x_1 - c_1 \psi t)} \sum_{p=1}^3 D_p U_1^{*(p)} e^{i\varkappa \sigma_p^* x_2} \right\}, \\ u_2^*(x_1, x_2, x_3, t) &= \text{Re} \left\{ \frac{i}{\varkappa} e^{i\varkappa(x_1 - c_1 \psi t)} \sum_{p=1}^3 D_p U_2^{*(p)} e^{i\varkappa \sigma_p^* x_2} \right\}, \\ u_3^*(x_1, x_2, x_3, t) &= 0, \\ \theta^*(x_1, x_2, x_3, t) &= \text{Re} \left\{ T_0 e^{i\varkappa(x_1 - c_1 \psi t)} \sum_{p=1}^3 D_p T^{*(p)} e^{i\varkappa \sigma_p^* x_2} \right\}, \end{aligned} \quad (33)$$

where D_1 , D_2 , and D_3 are complex arbitrary dimensionless parameters, not all zero. Further, in view of the constitutive equation (2) and by using relation (33), we have:

$$\begin{aligned} S_{12}^*(x_1, x_2, x_3, t) &= \text{Re} \left\{ \mu^* e^{i\varkappa(x_1 - c_1 \psi t)} \sum_{p=1}^3 D_p S_{12}^{*(p)} e^{i\varkappa \sigma_p^* x_2} \right\}, \\ S_{22}^*(x_1, x_2, x_3, t) &= \text{Re} \left\{ i\mu^* e^{i\varkappa(x_1 - c_1 \psi t)} \sum_{p=1}^3 D_p S_{22}^{*(p)} e^{i\varkappa \sigma_p^* x_2} \right\}, \\ S_{32}^*(x_1, x_2, x_3, t) &= 0, \\ q_2^*(x_1, x_2, x_3, t) &= \text{Re} \left\{ i\varkappa k^* T_0 e^{i\varkappa(x_1 - c_1 \psi t)} \sum_{p=1}^3 D_p Q_2^{*(p)} e^{i\varkappa \sigma_p^* x_2} \right\}, \end{aligned} \quad (34)$$

where

$$\begin{aligned} s_{12}^{*(p)} &= i\sigma_p^* U_1^{*(p)} - U_2^{*(p)}, \\ s_{22}^{*(p)} &= \frac{c_1^{*2}}{c_2^{*2}} \left(U_1^{*(p)} + i\sigma_p^* U_2^{*(p)} \right) - 2U_1^{*(p)} - \frac{i\beta^* T_0}{\varrho^* c_2^{*2}} T^{*(p)}, \\ Q_2^{*(p)} &= \sigma_p^* T^{*(p)}, \quad p = 1, 2, 3. \end{aligned} \quad (35)$$

At this instant, we must note that the quantities s_1^* , s_2^* , and s_3^* , as well as σ_1^* , σ_2^* , and σ_3^* are calculated by means of relations (20) and (21), respectively, where the material characteristics must be taken as those corresponding to the material of the half-space $x_2 > 0$. Furthermore, with these values thus calculated, we obtain the values $U_1^{*(p)}$, $U_2^{*(p)}$, and $T^{*(p)}$, $p = 1, 2, 3$, based on the relation (24), which are involved in relations (33) and (35).

4. Explicit secular equation of Rayleigh waves

In the previous section, we established the free wave solutions in the thermoelastic layer as well as in the thermoelastic half-space in the form of relations (28) and (33), respectively. Now, we will use these wave solutions to establish the secular equation for a Rayleigh wave propagating in the considered medium: thermoelastic layer of thickness h welded over the thermoelastic half-space. For this purpose, we will replace the relations (28), (29), (33), and (34) in the compatibility relations described by (7), as well as in the boundary conditions described by equation (8). Thus, we obtain for the arbitrary constants $B_1, B_2, \dots, B_6$ and D_1, D_2 , and D_3 , the following homogeneous algebraic system:

$$\begin{aligned} \sum_{n=1}^6 B_n U_1^{(n)} - \sum_{p=1}^3 D_p U_1^{*(p)} &= 0, \quad \sum_{n=1}^6 B_n U_2^{(n)} - \sum_{p=1}^3 D_p U_2^{*(p)} = 0, \\ \sum_{n=1}^6 B_n T^{(n)} - \sum_{p=1}^3 D_p T^{*(p)} &= 0, \end{aligned} \quad (36)$$

$$\begin{aligned} \sum_{n=1}^6 B_n s_{12}^{(n)} - \gamma \sum_{p=1}^3 D_p s_{12}^{*(p)} &= 0, \quad \sum_{n=1}^6 B_n s_{22}^{(n)} - \gamma \sum_{p=1}^3 D_p s_{22}^{*(p)} = 0, \\ \sum_{n=1}^6 B_n Q_2^{(n)} - \delta \sum_{p=1}^3 D_p Q_2^{*(p)} &= 0, \end{aligned} \quad (37)$$

$$\sum_{n=1}^6 B_n s_{12}^{(n)} = 0, \quad \sum_{n=1}^6 B_n s_{22}^{(n)} = 0, \quad \sum_{n=1}^6 B_n p_2^{(n)} = 0, \quad (38)$$

where

$$s_{12}^{(n)} = S_{12}^{(n)} e^{-i\kappa \sigma_n h}, \quad s_{22}^{(n)} = S_{22}^{(n)} e^{-i\kappa \sigma_n h}, \quad p_2^{(n)} = Q_2^{(n)} e^{-i\kappa \sigma_n h}, \quad n = 1, 2, \dots, 6, \quad (39)$$

and

$$\gamma = \frac{\mu^*}{\mu}, \quad \delta = \frac{k^*}{k}. \quad (40)$$

Since not all arbitrary constants $B_1, B_2, \dots, B_6$ and D_1, D_2 , and D_3 can be zero, it follows that the determinant of the homogeneous algebraic system described by relations (36) to (38) must be vanishing, i.e.,

$$\text{Det } \Lambda_{(9 \times 9)} = 0, \quad (41)$$

where

$$\Lambda_{(9 \times 9)} = \begin{pmatrix} U_1^{(1)} & U_1^{(2)} & U_1^{(3)} & U_1^{(4)} & U_1^{(5)} & U_1^{(6)} & -U_1^{*(1)} & -U_1^{*(2)} & -U_1^{*(3)} \\ U_2^{(1)} & U_2^{(2)} & U_2^{(3)} & U_2^{(4)} & U_2^{(5)} & U_2^{(6)} & -U_2^{*(1)} & -U_2^{*(2)} & -U_2^{*(3)} \\ T^{(1)} & T^{(2)} & T^{(3)} & T^{(4)} & T^{(5)} & T^{(6)} & -T^{*(1)} & -T^{*(2)} & -T^{*(3)} \\ S_{12}^{(1)} & S_{12}^{(2)} & S_{12}^{(3)} & S_{12}^{(4)} & S_{12}^{(5)} & S_{12}^{(6)} & -\gamma S_{12}^{*(1)} & -\gamma S_{12}^{*(2)} & -\gamma S_{12}^{*(3)} \\ S_{22}^{(1)} & S_{22}^{(2)} & S_{22}^{(3)} & S_{22}^{(4)} & S_{22}^{(5)} & S_{22}^{(6)} & -\gamma S_{22}^{*(1)} & -\gamma S_{22}^{*(2)} & -\gamma S_{22}^{*(3)} \\ Q_2^{(1)} & Q_2^{(2)} & Q_2^{(3)} & Q_2^{(4)} & Q_2^{(5)} & Q_2^{(6)} & -\delta Q_2^{*(1)} & -\delta Q_2^{*(2)} & -\delta Q_2^{*(3)} \\ s_{12}^{(1)} & s_{12}^{(2)} & s_{12}^{(3)} & s_{12}^{(4)} & s_{12}^{(5)} & s_{12}^{(6)} & 0 & 0 & 0 \\ s_{22}^{(1)} & s_{22}^{(2)} & s_{22}^{(3)} & s_{22}^{(4)} & s_{22}^{(5)} & s_{22}^{(6)} & 0 & 0 & 0 \\ p_2^{(1)} & p_2^{(2)} & p_2^{(3)} & p_2^{(4)} & p_2^{(5)} & p_2^{(6)} & 0 & 0 & 0 \end{pmatrix}. \quad (42)$$

Relations (41) and (42) represent explicitly the exact secular equation of the Rayleigh wave in an isotropic thermoelastic half-space overlaid by an isotropic thermoelastic layer for determining the dimensionless parameter ψ . In general, the parameter ψ is a complex number so that $\text{Re}(c_1 \psi) \geq 0$ will represent the wave speed and $\text{Im}(c_1 \psi) \leq 0$ will be related to the rate of decaying in time of the wave amplitude. We must note that for $\text{Re}(\psi) > 0$, we will have a genuine wave, while for $\text{Re}(\psi) = 0$, we will have a standing mode. Moreover, when $\text{Im}(\psi) > 0$, we will have the phenomenon of damping in time of the amplitude, while for $\text{Im}(\psi) = 0$, we will have undamped in time waves. Therefore, there have to get solutions of the exact secular equation (41) satisfying conditions:

$$\text{Re}(\psi) \geq 0, \quad \text{Im}(\psi) \leq 0. \quad (43)$$

A general analytical result concerning the existence and uniqueness of solutions of the secular equation (41), under restrictions generated by the inequalities described in equation (43), there seems to be not possible at this stage, and so it remains an open problem to be studied in future research. However, for particular materials with numerical values for the thermoelastic characteristics, the secular equation can be solved by using numerical methods.

Remark 1. We must note that if the boundary condition $q_2(x_1, -h, x_3, t) = 0$ is substituted by the boundary condition (9), then $p_2^{(n)}$ in the last line of the matrix Λ described by equation (42) must be replaced by $p_2^{(n)} = T^{(n)} e^{-i\kappa \sigma_n h}$, $n = 1, 2, \dots, 6$.

Remark 2. A closer examination of the exact secular equation (41), through the Λ matrix defined by the relation (42), shows the influence of the thermoelastic layer on the propagation of the Rayleigh wave through: (1) the system of the first six lines and six columns, i.e., matrix comprising the six wave solutions $(U_1^{(n)}, U_2^{(n)}, T^{(n)})$, $n = 1, 2, \dots, 6$, as described by relations (24) and (25) and the corresponding $(S_{12}^{(n)}, S_{22}^{(n)}, Q_2^{(n)})$, $n = 1, 2, \dots, 6$, as defined by relations (31) and (32), which also involve the material characteristics of the layer and (2) the thickness h of the thermoelastic layer, as well as the boundary data $(s_{12}^{(n)}, s_{22}^{(n)}, p_2^{(n)})$, $n = 1, 2, \dots, 6$, as given by the relation (39).

Remark 3. (Secular equation of a Rayleigh wave in an isotropic thermoelastic half-space). When the thickness of the thermoelastic layer tends to zero ($h \rightarrow 0$), i.e., there is no one thermoelastic layer, then

the secular equation (41) reduces to:

$$\left[\text{Det } \mathcal{A}_{(6 \times 6)} \right] \times \begin{vmatrix} S_{12}^{*(1)} & S_{12}^{*(2)} & S_{12}^{*(3)} \\ S_{22}^{*(1)} & S_{22}^{*(2)} & S_{22}^{*(3)} \\ Q_2^{*(1)} & Q_2^{*(2)} & Q_2^{*(3)} \end{vmatrix} = 0, \quad (44)$$

where

$$\mathcal{A}_{(6 \times 6)} = \begin{pmatrix} U_1^{(1)} & U_1^{(2)} & U_1^{(3)} & U_1^{(4)} & U_1^{(5)} & U_1^{(6)} \\ U_2^{(1)} & U_2^{(2)} & U_2^{(3)} & U_2^{(4)} & U_2^{(5)} & U_2^{(6)} \\ T^{(1)} & T^{(2)} & T^{(3)} & T^{(4)} & T^{(5)} & T^{(6)} \\ S_{12}^{(1)} & S_{12}^{(2)} & S_{12}^{(3)} & S_{12}^{(4)} & S_{12}^{(5)} & S_{12}^{(6)} \\ S_{22}^{(1)} & S_{22}^{(2)} & S_{22}^{(3)} & S_{22}^{(4)} & S_{22}^{(5)} & S_{22}^{(6)} \\ Q_2^{(1)} & Q_2^{(2)} & Q_2^{(3)} & Q_2^{(4)} & Q_2^{(5)} & Q_2^{(6)} \end{pmatrix}, \quad (45)$$

is a non-singular matrix, being the six wave solutions linearly independent.

By means of the relations (24) and (35), we get:

$$\begin{aligned} S_{12}^{*(1)} &= -\left(2 - \frac{c_1^2}{\tilde{c}_2^2} \psi^2\right), \quad S_{12}^{*(2)} = -2\xi_2^* \left(s_2^{*2} - \frac{ic_1}{c_1^* K^*} \psi\right), \quad S_{12}^{*(3)} = -2iM^* \xi_3^*, \\ S_{22}^{*(1)} &= -2\xi_1^*, \quad S_{22}^{*(2)} = \left(s_2^{*2} - \frac{ic_1}{c_1^* K^*} \psi\right) \left(\frac{c_1^{*2}}{c_2^{*2}} s_2^{*2} - 2\right) - \frac{i\varepsilon^* c_1 c_1^*}{K^* c_2^{*2}} \psi s_2^{*2}, \\ S_{22}^{*(3)} &= -iM^* \left(2 - \frac{c_1^2}{c_2^{*2}} \psi^2\right), \quad Q_2^{*(1)} = 0, \quad Q_2^{*(2)} = iN^* \psi \xi_2^* s_2^{*2}, \quad Q_2^{*(3)} = i\xi_3^* \left(s_3^{*2} - \frac{c_1^2}{c_1^{*2}} \psi^2\right), \end{aligned} \quad (46)$$

and hence, the secular equation (44) can be brought to the form:

$$\begin{aligned} &\xi_3^* \left(s_3^{*2} - \frac{c_1^2}{c_1^{*2}} \psi^2\right) \left(s_2^{*2} - \frac{ic_1}{c_1^* K^*} \psi\right) \left[4\xi_1^* \xi_2^* - \left(\frac{c_1^2}{c_2^{*2}} \psi^2 - 2\right) \left(\frac{c_1^{*2}}{c_2^{*2}} s_2^{*2} - 2\right)\right] \\ &+ \frac{ic_1 s_2^{*2}}{c_1^* K^*} \psi \varepsilon^* \left\{ \xi_3^* \frac{c_1^{*2}}{c_2^{*2}} \left(s_3^{*2} - \frac{c_1^2}{c_1^{*2}} \psi^2\right) \left(\frac{c_1^2}{c_2^{*2}} \psi^2 - 2\right) + \xi_2^* \left[\left(\frac{c_1^2}{c_2^{*2}} \psi^2 - 2\right)^2 - 4\xi_1^* \xi_3^*\right] \right\} = 0, \end{aligned} \quad (47)$$

which resonates with the secular equation presented by Chiriță and Zampoli [41] for the propagation of Rayleigh waves in isotropic linear thermoelastic half-space.

5. Numerical analysis

The illustrative analysis in this section is based on a variety of materials whose useful characteristics are presented in Table 1. We consider that the thermoelastic layer is made of aluminum, and the half-space consists of one of the other materials shown in Table 1: copper, gold, iron, lead, nickel, or steel. We are interested in how the two characteristics of the Rayleigh wave, dimensionless wave speed $\text{Re}(\psi)$ and dimensionless rate of decaying of wave amplitude $\text{Im}(\psi)$, behave in relation to the thickness of

Table 1. The materials in study together their characteristics useful to our analysis.

Material	$c_1 (m/s)$	$c_2 (m/s)$	$\varepsilon (\times 10^{-2})$	$K (\times 10^{-8})$	$k (w/mK)$	$\varrho (kg/m^3)$	$\alpha (\times 10^{-6}/K)$
Aluminum	6317.01	3110.28	2.65	1.272	237	2700	23
Copper	4641.67	2285.40	1.90	2.108	401	8940	17
Gold	3212.02	1192.91	3.01	3.871	317	19300	14
Iron	5778.29	3137.25	1.01	0.372	80	7874	11.1
Lead	1999.35	654.44	4.74	1.171	36	11400	29
Nickel	5650.40	2965.04	1.25	0.345	94	8907	13
Steel	5533.04	3105.75	0.77	0.069	54	7850	11

Table 2. The dimensionless speed wave, $Re(\psi)$, depending upon the thickness h of the layer for the considered layered structures.

h	0	$25 \times 10^{-4} m$	$5 \times 10^{-3} m$	$75 \times 10^{-4} m$	$10^{-2} m$
Copper	0.33943	0.339009	0.338584	0.338156	0.337723
Gold	0.178601	0.178279	0.177951	0.177618	0.177277
Iron	0.467482	0.467126	0.466769	0.466411	0.466053
Lead	0.0985793	0.0976307	0.0965591	0.095356	0.0940126
Nickel	0.443142	0.442827	0.442511	0.442195	0.441878
Steel	0.462512	0.462161	0.461809	0.461456	0.461102

Table 3. The dimensionless rate of damping in time of the wave amplitude, $Im(\psi)$, depending upon the thickness h of the layer for the considered layered structures.

h	0	$25 \times 10^{-4} m$	$5 \times 10^{-3} m$	$75 \times 10^{-4} m$	$10^{-2} m$
Copper	-0.00250732	-0.00254988	-0.00259312	-0.00263707	-0.00268173
Gold	-0.000441715	-0.000454409	-0.000467424	-0.000480768	-0.000494449
Iron	-0.00286191	-0.00289574	-0.00293019	-0.00296526	-0.00300095
Lead	-0.00030109	-0.00035035	-0.0004064	-0.000470151	-0.000542662
Nickel	-0.00233527	-0.00236366	-0.00239252	-0.00242185	-0.00245165
Steel	-0.000799118	-0.000815468	-0.000832104	-0.00084903	-0.000866252

the covering layer. Throughout this section, we take the wave number to be $\kappa = 1 m^{-1}$ and the initial temperature to be $T_0 = 293 K$.

Tables 2 and 3 consider a thermoelastic layered structure consisting of an aluminum layer coating a half-space made of one of the rest of materials presented in Table 1. Table 2 presents the variation of the dimensionless speed wave, $Re(\psi)$, of the Rayleigh thermoelastic wave with respect to the thickness h of the layer, the Table 3 provides the variation of the dimensionless rate of damping in time of the wave amplitude, $Im(\psi)$, with respect to the same thickness. The results in Table 2 lead to the idea that by increasing the thickness of the covering layer, the dimensionless speed of the wave decreases, whatever the material of which the half-space is made of. Instead, Table 3 suggests that the dimensionless rate of damping in time of the wave amplitude, $Im(\psi)$, increases in relation to the increase in the thickness of the layer.

In the last part of our numerical analysis, we considered a hypothetical layered structure consisting of an aluminum layer with a thermal conductivity of $237 w/mK$ and covering a half-space made of a material with the same elastic characteristics and thermal like aluminum, only the thermal conductivity being different, here k^* varying in $[220, 260] w/mK$. We wanted to see how the characteristics of the Rayleigh wave in the layered structure vary in relation to the variation of the thermal conductivity, taking in this

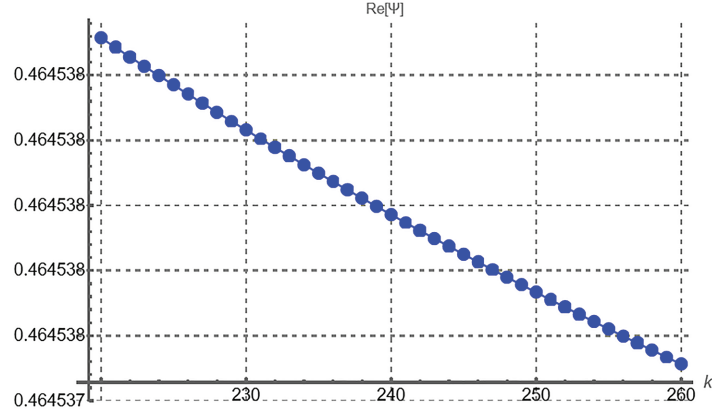


Figure 1. The variation of the dimensionless propagation speed $\text{Re}(\psi) > 0$ with respect to the thermal conductivity $k^* \in [200, 260] \text{ w/mK}$, (the thickness of the layer is fixed at $h = 10^{-3} \text{ m}$).

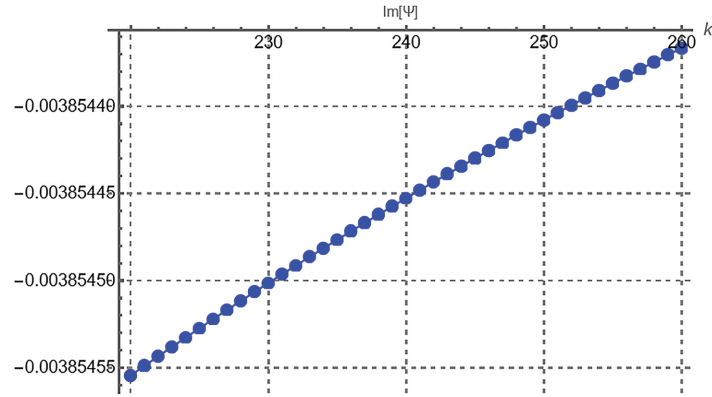


Figure 2. The variation of the dimensionless rate of damping of amplitude, $\text{Im}(\psi) < 0$, with respect to the thermal conductivity $k^* \in [200, 260] \text{ w/mK}$, (the thickness of the layer is fixed at $h = 10^{-3} \text{ m}$).

sense the thickness of the layer to be 10^{-3} m . Thus, Figures 1 and 2 answer this question and show that the dimensionless speed of propagation of the Rayleigh wave, $\text{Re}(\psi)$, decreases in relation to the increase in thermal conductivity, corroborated with an increase in the dimensionless rate of damping of the wave amplitude $\text{Im}(\psi)$.

6. Exact effective boundary conditions

We proceed here to establish the effective boundary conditions that could be very useful in deriving the approximate explicit secular equations for the Rayleigh waves in thermoelastic half-spaces covered with a thin film. We have to mention that the entire effect of the thermoelastic layer on the half-space is replaced by the exact effective boundary conditions at the interface. Consequently, the wave can be imagined as a Rayleigh wave propagating in the thermoelastic half-space, without the coating thermoelastic layer, but subjected to the effective boundary conditions. Following this idea, we should establish boundary surface conditions on $x_2 = 0$ that look like some linear combinations including the values calculated at $x_2 = 0$ for the displacements and temperature fields, as well as for traction and heat flux.

In this connection, we consider the propagation of a Rayleigh wave, traveling along surface $x_2 = 0$ of the isotropic thermoelastic half-space with velocity $c = c_1 \psi$ and wave number $\varkappa$ in the x_1 -direction, decaying in the x_2 -direction. Consequently, for $x_2 > 0$, we will have:

$$\begin{aligned} u_1^*(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ \frac{1}{\varkappa} e^{i\varkappa(x_1 - c_1 \psi t)} \mathcal{U}_1^*(x_2) \right\}, \\ u_2^*(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ \frac{i}{\varkappa} e^{i\varkappa(x_1 - c_1 \psi t)} \mathcal{U}_2^*(x_2) \right\}, \\ u_3^*(x_1, x_2, x_3, t) &= 0, \\ \theta^*(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ T_0 e^{i\varkappa(x_1 - c_1 \psi t)} \mathcal{T}^*(x_2) \right\}, \end{aligned} \quad (48)$$

and

$$\begin{aligned} S_{12}^*(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ \mu^* e^{i\varkappa(x_1 - c_1 \psi t)} \mathcal{S}_{12}^*(x_2) \right\}, \\ S_{22}^*(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ i\mu^* e^{i\varkappa(x_1 - c_1 \psi t)} \mathcal{S}_{22}^*(x_2) \right\}, \\ S_{32}^*(x_1, x_2, x_3, t) &= 0, \\ q_2^*(x_1, x_2, x_3, t) &= \operatorname{Re} \left\{ i\varkappa k^* T_0 e^{i\varkappa(x_1 - c_1 \psi t)} \mathcal{Q}^*(x_2) \right\}. \end{aligned} \quad (49)$$

In view of the continuity conditions (7), from relations (28), (29) and (48) and (49), we obtain the matrix equation:

$$\mathcal{A}_{(6 \times 6)} \mathbf{B}_{(6 \times 1)} = \mathbf{D}_{(6 \times 1)}, \quad (50)$$

where

$$\begin{aligned} \mathbf{B}_{(6 \times 1)} &= (B_1, B_2, B_3, B_4, B_5, B_6)^T, \\ \mathbf{D}_{(6 \times 1)} &= (\mathcal{U}_1^*(0), \mathcal{U}_2^*(0), \mathcal{T}^*(0), \gamma \mathcal{S}_{12}^*(0), \gamma \mathcal{S}_{22}^*(0), \delta \mathcal{Q}_2^*(0))^T. \end{aligned} \quad (51)$$

On the contrary, the boundary condition on the free surface of the layer in the form (38) can be written in the following matrix form:

$$\Gamma_{(3 \times 6)} \mathbf{B}_{(6 \times 1)} = \mathbf{0}_{(3 \times 1)}, \quad (52)$$

where

$$\Gamma_{(3 \times 6)} = \begin{pmatrix} s_{12}^{(1)} & s_{12}^{(2)} & s_{12}^{(3)} & s_{12}^{(4)} & s_{12}^{(5)} & s_{12}^{(6)} \\ s_{22}^{(1)} & s_{22}^{(2)} & s_{22}^{(3)} & s_{22}^{(4)} & s_{22}^{(5)} & s_{22}^{(6)} \\ p_2^{(1)} & p_2^{(2)} & p_2^{(3)} & p_2^{(4)} & p_2^{(5)} & p_2^{(6)} \end{pmatrix}. \quad (53)$$

Consequently, from relations (50) and (52), we can deduce that on the interface $x_2 = 0$ we have to fulfill the following condition:

$$\Gamma_{(3 \times 6)} \mathcal{A}_{(6 \times 6)}^{-1} \mathbf{D}_{(6 \times 1)} = \mathbf{0}_{(3 \times 1)}. \quad (54)$$

As it clearly results from relation (51), the matrix $\mathbf{D}_{(6 \times 1)}$ involves the values of the displacements and temperature fields, as well as those of the traction and the heat flux vector on the surface $x_2 = 0$ of the half-space $x_2 > 0$. This means that in the Rayleigh wave propagation problem for the half-space $x_2 > 0$, the boundary conditions on $x_2 = 0$ must be specified as a linear combination, as described in relation (54), relating the values of the displacements and temperature fields, as well as the values of traction and heat flux.

7. The isotropic elastic layered structure

We start here by studying the secular equation for Rayleigh wave propagating in an isotropic elastic half-space on which an isotropic elastic layer is welded. In this connection, we will use our analysis from the previous sections, in which we will neglect the thermal effects. This shows that:

$$\begin{aligned} s_1^2 &= \frac{c_1^2}{c_2^2} \psi^2, \quad s_2^2 = \psi^2, \\ r_1 &= i\xi, \quad r_2 = i\zeta, \quad r_3 = -i\xi, \quad r_4 = -i\zeta, \\ \xi &= \sqrt{1 - \frac{c_1^2}{c_2^2} \psi^2}, \quad \zeta = \sqrt{1 - \psi^2}, \end{aligned} \quad (55)$$

so that we have:

$$\begin{aligned} u_1(x_1, x_2, x_3, t) &= \text{Re} \left\{ \frac{1}{\varkappa} e^{i\varkappa(x_1 - c_1 \psi t)} \left[B_1 \xi e^{-\varkappa \xi x_2} + B_2 e^{-\varkappa \zeta x_2} - B_3 \xi e^{\varkappa \xi x_2} + B_4 e^{\varkappa \zeta x_2} \right] \right\}, \\ u_2(x_1, x_2, x_3, t) &= \text{Re} \left\{ \frac{i}{\varkappa} e^{i\varkappa(x_1 - c_1 \psi t)} \left[B_1 e^{-\varkappa \xi x_2} + B_2 \zeta e^{-\varkappa \zeta x_2} + B_3 e^{\varkappa \xi x_2} - B_4 \zeta e^{\varkappa \zeta x_2} \right] \right\}, \\ u_3(x_1, x_2, x_3, t) &= 0, \end{aligned} \quad (56)$$

and

$$\begin{aligned} S_{12}(x_1, x_2, x_3, t) &= \text{Re} \left\{ \mu e^{i\varkappa(x_1 - c_1 \psi t)} \left[-B_1 (1 + \xi^2) e^{-\varkappa \xi x_2} - 2B_2 \zeta e^{-\varkappa \zeta x_2} \right. \right. \\ &\quad \left. \left. - B_3 (1 + \xi^2) e^{\varkappa \xi x_2} + 2B_4 \zeta e^{\varkappa \zeta x_2} \right] \right\}, \\ S_{22}(x_1, x_2, x_3, t) &= \text{Re} \left\{ i\mu e^{i\varkappa(x_1 - c_1 \psi t)} \left[-2B_1 \xi e^{-\varkappa \xi x_2} - B_2 (1 + \xi^2) e^{-\varkappa \zeta x_2} \right. \right. \\ &\quad \left. \left. + 2B_3 \xi e^{\varkappa \xi x_2} - B_4 (1 + \xi^2) e^{\varkappa \zeta x_2} \right] \right\}, \\ S_{32}(x_1, x_2, x_3, t) &= 0. \end{aligned} \quad (57)$$

Furthermore, we have:

$$\begin{aligned} u_1^*(x_1, x_2, x_3, t) &= \text{Re} \left\{ \frac{1}{\varkappa} e^{i\varkappa(x_1 - c_1 \psi t)} \left[D_1 \xi^* e^{-\varkappa \xi^* x_2} + D_2 e^{-\varkappa \zeta^* x_2} \right] \right\}, \\ u_2^*(x_1, x_2, x_3, t) &= \text{Re} \left\{ \frac{i}{\varkappa} e^{i\varkappa(x_1 - c_1 \psi t)} \left[D_1 e^{-\varkappa \xi^* x_2} + D_2 \zeta^* e^{-\varkappa \zeta^* x_2} \right] \right\}, \\ u_3^*(x_1, x_2, x_3, t) &= 0, \end{aligned} \quad (58)$$

and

$$\begin{aligned} S_{12}^*(x_1, x_2, x_3, t) &= \text{Re} \left\{ \mu^* e^{i\varkappa(x_1 - c_1 \psi t)} \left[-D_1 (1 + \xi^{*2}) e^{-\varkappa \xi^* x_2} - 2D_2 \zeta^* e^{-\varkappa \zeta^* x_2} \right] \right\}, \\ S_{22}^*(x_1, x_2, x_3, t) &= \text{Re} \left\{ i\mu^* e^{i\varkappa(x_1 - c_1 \psi t)} \left[-2D_1 \xi^* e^{-\varkappa \xi^* x_2} - D_2 (1 + \xi^{*2}) e^{-\varkappa \zeta^* x_2} \right] \right\}, \\ S_{32}^*(x_1, x_2, x_3, t) &= 0, \end{aligned} \quad (59)$$

where

$$s_1^{*2} = \frac{c_1^2}{c_2^{*2}} \psi^2, \quad s_2^{*2} = \frac{c_1^2}{c_1^{*2}} \psi^2, \quad \xi^* = \sqrt{1 - \frac{c_1^2}{c_2^{*2}} \psi^2}, \quad \zeta^* = \sqrt{1 - \frac{c_1^2}{c_1^{*2}} \psi^2}. \quad (60)$$

Consequently, the conditions on the free surface of the elastic layer, together with the continuity conditions when passing through the contact surface between the layer and the half-space, lead to the following exact secular equation for the Rayleigh wave:

$$\begin{vmatrix} \xi & 1 & -\xi & 1 & -\xi^* & -1 \\ 1 & \zeta & 1 & -\zeta & -1 & -\zeta^* \\ -(1+\xi^2) & -2\zeta & -(1+\xi^2) & 2\zeta & \gamma(1+\xi^{*2}) & 2\gamma\zeta^* \\ -2\xi & -(1+\xi^2) & 2\xi & -(1+\xi^2) & 2\gamma\xi^* & \gamma(1+\xi^{*2}) \\ -(1+\xi^2)e^{\varkappa\xi h} & -2\zeta e^{\varkappa\zeta h} & -(1+\xi^2)e^{-\varkappa\xi h} & 2\zeta e^{-\varkappa\zeta h} & 0 & 0 \\ -2\xi e^{\varkappa\xi h} & -(1+\xi^2)e^{\varkappa\zeta h} & 2\xi e^{-\varkappa\xi h} & -(1+\xi^2)e^{-\varkappa\zeta h} & 0 & 0 \end{vmatrix} = 0. \quad (61)$$

The relation (61) offers an alternative to the exact secular equation obtained by Vinh et al. [24]. Of course, a development of the determinant described in equation (61) after the last two rows should lead to an exact secular equation similar to that established by Vinh et al. [24]. In fact, we have the following non-zero minors of the second order, formed with the elements of the last two rows of the determinant:

$$\begin{aligned} M_{12} &= \begin{vmatrix} -(1+\xi^2)e^{\xi\kappa h} & -2\zeta e^{\zeta\kappa h} \\ -2\xi e^{\xi\kappa h} & -(1+\xi^2)e^{\zeta\kappa h} \end{vmatrix} = [(1+\xi^2)^2 - 4\xi\zeta]e^{(\xi+\zeta)\kappa h}, \\ M_{13} &= -4\xi(1+\xi^2), \quad M_{14} = [(1+\xi^2)^2 + 4\xi\zeta]e^{(\xi-\zeta)\kappa h} \\ M_{23} &= -[(1+\xi^2)^2 + 4\xi\zeta]e^{(\zeta-\xi)\kappa h}, \quad M_{24} = 4\zeta(1+\xi^2), \\ M_{34} &= [(1+\xi^2)^2 - 4\xi\zeta]e^{-(\xi+\zeta)\kappa h}, \end{aligned} \quad (62)$$

and hence, the exact secular equation takes the following form:

$$\begin{aligned} &4(1+\xi^2)(\xi N_{13} - \zeta N_{24}) + [(1+\xi^2)^2 - 4\xi\zeta][N_{12}e^{(\xi+\zeta)\kappa h} + N_{34}e^{-(\xi+\zeta)\kappa h}] \\ &+ [(1+\xi^2)^2 + 4\xi\zeta][N_{14}e^{(\xi-\zeta)\kappa h} - N_{23}e^{(\zeta-\xi)\kappa h}] = 0, \end{aligned} \quad (63)$$

where N_{12} , N_{13} , N_{14} , N_{23} , N_{24} , and N_{34} are the cofactor of the minors M_{12} , M_{13} , M_{14} , M_{23} , M_{24} , and M_{34} , respectively.

The effective boundary conditions are:

$$\begin{aligned} &[(1+\xi^2)^2 \sinh(\xi\kappa h) - 4\xi\zeta \sinh(\zeta\kappa h)]\mathcal{U}_1^*(0) \\ &- 2\xi(1+\xi^2)[\cosh(\xi\kappa h) - \cosh(\zeta\kappa h)]\mathcal{U}_2^*(0) \\ &- \gamma\xi[(1+\xi^2) \cosh(\xi\kappa h) - 2 \cosh(\zeta\kappa h)]\mathcal{S}_{12}^*(0) \\ &+ \gamma[(1+\xi^2) \sinh(\xi\kappa h) - 2\xi\zeta \sinh(\zeta\kappa h)]\mathcal{S}_{22}^*(0) = 0, \end{aligned} \quad (64)$$

$$\begin{aligned} &2\zeta(1+\xi^2)[\cosh(\xi\kappa h) - \cosh(\zeta\kappa h)]\mathcal{U}_1^*(0) \\ &- [4\xi\zeta \sinh(\xi\kappa h) - (1+\xi^2)^2 \sinh(\zeta\kappa h)]\mathcal{U}_2^*(0) \\ &- \gamma[2\xi\zeta \sinh(\xi\kappa h) - (1+\xi^2) \sinh(\zeta\kappa h)]\mathcal{S}_{12}^*(0) \\ &+ \gamma\zeta[2 \cosh(\xi\kappa h) - (1+\xi^2) \cosh(\zeta\kappa h)]\mathcal{S}_{22}^*(0) = 0. \end{aligned} \quad (65)$$

These are the exact effective boundary conditions that replaces exactly the entire influence of the elastic layer on the elastic half-space.

When the thickness of the elastic layer tends to zero ($h \rightarrow 0$), i.e., there is no one elastic layer, then the secular equation (61) reduces to:

$$(1+\xi^{*2})^2 = 4\xi^*\zeta^*, \quad (66)$$

which, in view of relation (55), takes the form:

$$\left(2 - \frac{c_1^2}{c_2^{*2}} \psi^2\right)^2 = 4 \sqrt{1 - \frac{c_1^2}{c_1^{*2}} \psi^2} \sqrt{1 - \frac{c_1^2}{c_2^{*2}} \psi^2}, \quad (67)$$

i.e., the secular equation deduced by Rayleigh [42] for surface waves propagating at the surface of the isotropic elastic half-space. Furthermore, the effective boundary conditions (64) are reduced to:

$$(\xi^2 - 1)S_{12}^*(0) = 0, \quad (1 - \xi^2)S_{22}^*(0) = 0, \quad (68)$$

i.e., they are coincident with the free traction on the surface of the half-space:

$$S_{12}^*(0) = 0, \quad S_{22}^*(0) = 0. \quad (69)$$

It seems that the effective boundary conditions established in Vinh et al. [24] do not fulfill this property of being able to be reduced to the conditions of free traction on the surface of the half-space, when the thickness of the layer tends to zero.

8. Conclusion

This paper highlights the importance of knowing the system of free wave solutions in the half-space $x_2 > 0$ described by the relations (28) and (29) and in the thermoelastic layer $-h < x_2 < 0$ described by the relations (33) and (34), which determines the exact secular equation (41) for the propagation of the Rayleigh wave in a thermoelastic layered structure. They are therefore the key factor, the main purpose of the investigations of Rayleigh waves propagating in half-spaces covered by a thin layer.

On the contrary, the numerical simulation of our analysis regarding the secular equation (41) reveals the behavior in relation to the thickness of the covering layer, h , for the two characteristics of the Rayleigh wave propagating in the layered thermoelastic structure. This suggests that the dimensionless speed propagation of the wave, $\text{Re}(\psi) > 0$, tends to decrease in relation to the increase of h , corroborated with an increase in the rate of damping in time of the wave amplitude, $\text{Im}(\psi)$.

By establishing the effective boundary conditions (54), the entire influence of the thermoelastic layer on the substrate is known. These relations together with the expansion of the stresses and of the heat flux at the top surface of the layer into Taylor series in its thickness furnish approximate secular equations of the waves supported by thin-film/substrate interactions that can be employed as theoretical bases for extracting the thermo-mechanical properties of the thin films from experimental data.

Finally, the relation (61) provides an explicit alternative to the exact secular equation for the propagation of Rayleigh waves supported by isotropic elastic layer/elastic half-space interactions, previously presented in Vinh et al. [24].

Declaration of conflicting interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding

The author(s) received no financial support for the research, authorship, and/or publication of this article.

ORCID

Stan Chiriță  <https://orcid.org/0000-0001-7560-0367>

References

- [1] Sezawa, K. Dispersion of elastic waves propagated on the surface of stratified bodies and on curved surfaces. *Bull Earthquake Res Inst* 1927; 3: 1–18.
- [2] Sezawa, K, and Kanai, K. Discontinuity in dispersion curves of Rayleigh-waves. *Bull Earthquake Res Inst* 1935; 13(2): 237–244.
- [3] Haskell, NA. The dispersion of surface waves on multilayered media. *Bull Seismol Soc Am* 1953; 43(1): 17–34.
- [4] Achenbach, JD, and Epstein, HI. Dynamic interaction of a layer and a half-space. *J Eng Mech Div* 1967; 93(5): 27–42.
- [5] Tiersten, HF. Elastic surface waves guided by thin films. *J Appl Phys* 1969; 40(2): 770–789.
- [6] Makarov, S, Chilla, E, and Frohlich, HJ. Determination of elastic constants of thin films from phase velocity dispersion of different surface acoustic wave modes. *J Appl Phys* 1995; 78(8): 5028–5034.
- [7] Kuchler, K, and Richter, E. Ultrasonic surface waves for studying the properties of thin films. *Thin Solid Film* 1998; 315(1–2): 29–34.
- [8] Cai, Z, and Fu, Y. On the imperfection sensitivity of a coated elastic half-space. *Proc Math Phys Eng Sci* 1999; 455(1989): 3285–3309.
- [9] Sotiropoulos, DA. The effect of anisotropy on guided elastic waves in a layered half-space. *Mech Mat* 1999; 31(3): 215–233.
- [10] Every, AG. Measurement of the near-surface elastic properties of solids and thin supported films. *Meas Sci Technol* 2002; 13(5): R21–R39.
- [11] Chattopadhyay, DK, and Raju, K. Structural engineering of polyurethane coatings for high performance applications. *Prog Polym Sci* 2007; 32(3): 352–418.
- [12] Nayfeh, AH. *Wave propagation in layered anisotropic media: with application to composites*. Amsterdam: North Holland, 1995.
- [13] Rose, JL. *Ultrasonic guided waves in solid media*. New York: Cambridge University Press, 2014.
- [14] Nowinski, JL. *Theory of thermoelasticity with applications*. Alphen aan den Rijn: Sijthoff & Noordhoff International Publishers, 2011.
- [15] Giurgiutiu, V. *Structural health monitoring of aerospace composites*. Saint Louis, MI: Elsevier Science, 2015.
- [16] Steigmann, DJ, and Ogden, RW. Surface waves supported by thin-film/substrate interactions. *IMA J Appl Math* 2007; 72(6): 730–747.
- [17] Dai, HH, Kaplunov, J, and Prikazchikov, DA. A long-wave model for the surface elastic wave in a coated half-space. *Proc Royal Soc A* 2010; 466(2122): 3097–3116.
- [18] Vinh, PC, and Anh, VTN. Rayleigh waves in an orthotropic half-space coated by a thin orthotropic layer with sliding contact. *Int J Eng Sci* 2014; 75: 154–164.
- [19] Vinh, PC, and Anh, VTN. Effective boundary condition method and Rayleigh waves in orthotropic half-spaces coated by a thin layer with sliding contact. *Arch Mech* 2015; 67(6): 477–498.
- [20] Vinh, PC, and Anh, VTN. Effective boundary condition method and approximate secular equations of Rayleigh waves in orthotropic half-spaces coated by a thin layer. *J Mech Mat Struct* 2016; 11(3): 259–277.
- [21] Vinh, PC, and Anh, VTN. Rayleigh waves in a layered orthotropic elastic half-space with sliding contact. *J Vib Control* 2018; 24(10): 2070–2079.
- [22] Vinh, PC, Anh, VTN, and Thanh, VP. Rayleigh waves in an isotropic elastic half-space coated by a thin isotropic elastic layer with smooth contact. *Wave Motion* 2014; 51(3): 496–504.
- [23] Vinh, PC, Anh, VTN, and Linh, NTK. On a technique for deriving the explicit secular equation of Rayleigh waves in an orthotropic half-space coated by an orthotropic layer. *Waves Random Complex Media* 2016; 26(2): 176–188.
- [24] Vinh, PC, Anh, VTN, and Linh, NTK. Exact secular equations of Rayleigh waves in an orthotropic elastic half-space overlaid by an orthotropic elastic layer. *Int J Solids Struct* 2016; 83, 65–72.
- [25] Kaplunov, J, and Prikazchikov, DA. Chapter One—Asymptotic theory for Rayleigh and Rayleigh-type waves. *Adv Appl Mech* 2017; 50: 1–106.
- [26] Erbaş, B, Kaplunov, J, Prikazchikov, DA, and Şahin, O. The near-resonant regimes of a moving load in a three-dimensional problem for a coated elastic half-space. *Math Mech Solid* 2017; 22(1): 89–100.
- [27] Kaplunov, J, Prikazchikov, DA, and Sultanova, L. On higher order effective boundary conditions for a coated elastic half-space. In: Andrianov I, Manevich A, Mikhlin Y, et al. (eds) *Problems of nonlinear mechanics and physics of materials*. Cham: Springer, 2019, pp. 449–462.
- [28] Kaplunov, J, Prikazchikov, DA, and Sultanova, L. Rayleigh-type waves on a coated elastic half-space with a clamped surface. *Philos Trans R Soc A* 2019; 377(2156): 20190111.
- [29] Phan, H, Cho, Y, Pham, CV, Nguyen, H, and Bui, TQ. A theoretical approach for guided waves in layered structures. *AIP Conf Proc* 2019; 2102(1): 050011.
- [30] Mubarak, AM, Prikazchikov, DA, and Kudaibergenov, AK. Explicit model for surface waves on an elastic half-space coated by a thin vertically inhomogeneous layer. In: Awrejcewicz J (ed.) *Perspectives in dynamical systems I: mechatronics and life sciences. DSTA 2019. Springer proceedings in mathematics & Statistics*, Vol. 362, Łódź, 2–5 December. Cham: Springer, 2021.

- [31] Kaur, S, Khurana, A, and Tomar, SK. An approximate secular equation of Rayleigh-like waves in coated elastic half-space containing voids. *Int J Eng Sci* 2024; 196: 104016.
- [32] Lockett, FJ. Effect of thermal properties of a solid on the velocity of Rayleigh waves. *J Mech Phys Solid* 1958; 7(1): 71–75.
- [33] Chadwick, P. Thermoelasticity. The dynamical theory. In: Sneddon IN and Hill R (eds) *Progress in solid mechanics*, Vol. 1. Amsterdam: North-Holland Publishing Company, 1960, pp. 263–328.
- [34] Deresiewicz, HA. A note on thermoelastic Rayleigh waves. *J Mech Phys Solid* 1961; 9(3): 191–195.
- [35] Brock, LM. Transient three-dimensional Rayleigh and Stoneley signal effects in thermoelastic solids. *Int J Solid Struct* 1997; 34(12): 1463–1478.
- [36] Lykotrafitis, G, and Georgiadis, HG. The three-dimensional steady-state thermo-elastodynamic problem of moving sources over a half-space. *Int J Solid Struct* 2003; 40(4): 899–940.
- [37] Georgiadis, HG, and Lykotrafitis, G. Rayleigh waves generated by a thermal source: a 3D transient thermoelasticity solution. *ASME J Appl Mech* 2005; 72(1): 129–138.
- [38] Chiriță, S. On the Rayleigh surface waves on an anisotropic homogeneous thermoelastic half space. *Acta Mech* 2013; 224(3): 657–674.
- [39] Chiriță, S. Thermoelastic surface waves on an exponentially graded half-space. *Mech Res Comm* 2013; 49: 27–35.
- [40] Hetnarski, RB (ed.). *Encyclopedia of thermal stresses, Chapter: Rayleigh's waves*. Dordrecht: Springer, 2014.
- [41] Zampoli, V, and Chiriță, S. Elastic and thermal effects upon the propagation of waves with assigned wavelength. *Proc Royal Soc A* 2022; 478: 20220478.
- [42] Rayleigh, L. On waves propagated along the plane surface of an elastic solid. *Proc London Math Soc* 1885; 17(1): 4–11.