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On wave dispersion and stability in nonlocal elasticity with nonlocality in space and time and with second-order displacement gradients

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ABSTRACT

An elastic model is proposed by considering nonlocality in space and time and gradient elasticity with second-order displacement gradients. The theory represents a unification of nonlocal elasticity of Klein–Gordon type developed by Lazar and Agiasofitou with a gradient elasticity model developed by De Domenico and Askes. As a result, a mathematical model is obtained that involves derivatives of the fourth-order with respect to the time variable and of the mixed sixth-order with respect to the spatial and time variable. It incorporates two characteristic length scale parameters, namely the characteristic length ℓ_N of nonlocal elasticity of Klein–Gordon type and the characteristic length ℓ_G of second-order displacement gradient, as well as a characteristic internal time scale parameter in addition to the elastic material parameters. The wave dispersion is investigated within the framework of this nonlocal extended second-order displacement gradient elasticity theory and a qualitative analysis of the phase and group velocities is carried out, in order to examine when normal or anomalous dispersion occurs. The uniqueness and stability of the solution to the initial boundary value problems are investigated using the Lagrange identity method. The result obtained leads to the fact that the proposed model can be well-posed.

1. Introduction

The analysis of wave propagation in solids with microstructure often requires continuum models that go beyond classical elasticity. When the wavelength becomes comparable with the characteristic length of the underlying material structure, the classical assumption of purely local interactions can no longer be sufficient. The works referred to below address this problem from complementary perspectives: De Domenico and Askes (2016) introduce a dynamically consistent gradient-elasticity model with micro-inertia, Lazar and Agiasofitou (2022) develop instead a space-time nonlocal elasticity theory of Klein–Gordon type, Lazar and Agiasofitou (2026) build a sort of bridge between the latter two, in the sense that will be better explained later, while Straughan (2026) provides a mathematical analysis of uniqueness, continuous dependence, and stability for higher-gradient elastodynamic theories. In De Domenico and Askes (2016), the problem is approached from the point of view of gradient elasticity. Rather than adopting an explicitly integral nonlocal constitutive law, the model introduces high-order spatial derivatives into the continuum formulation; more precisely, it combines one strain-gradient contribution with two acceleration-gradient, or micro-inertia, contributions. This simultaneous presence of strain and acceleration gradients is identified as a condition of dynamic

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consistency; for the sake of completeness, it is worth remembering that such concept of dynamic consistency was systematically formulated in Metrikine and Askes (2002). Moreover, a thermodynamically consistent way to take into account higher order inertia effects within continuum mechanics is proposed by Polizzotto (2012), together with applications to wave dispersion. The resulting formulation is a multi-scale elastic model designed to improve the prediction of dispersive wave propagation in materials with heterogeneous microstructure. In Lazar and Agiasofitou (2022), the 'enhancement' is formulated instead within nonlocal elasticity in a stricter sense: the paper develops an elasticity theory with nonlocalities in both space and time, based on nonlocal constitutive equations involving a dynamical scalar kernel. This theory is specified as a nonlocal elasticity of Klein–Gordon type, extending Helmholtz-type nonlocal elasticity by introducing not only an internal length scale but also an internal time scale. The nonlocal kernel is interpreted as the retarded Green function of the Klein–Gordon operator and, as a consequence, the model is able to describe spatial and temporal dispersion and can predict, in addition to acoustic (low-frequency) modes, optic (high-frequency) modes and frequency band-gaps between acoustic and optic branches. The two above approaches are certainly different as for their mathematical structure, but nevertheless closely related in physical motivation. De Domenico and Askes (2016) highlight microstructural effects through higher-order gradients and micro-inertia terms, while Lazar and Agiasofitou (2022) represent them through a space-time nonlocal kernel and the Klein–Gordon operator. In both cases, additional internal scales are introduced into the continuum description: such scales allow the models to reproduce dispersive features that would otherwise require more complex generalized continuum theories. We also emphasize that the recent contribution by Lazar and Agiasofitou (2026) further reinforces this connection by explicitly unifying Helmholtz-type nonlocal elasticity and strain-gradient elasticity within a single framework governed by two characteristic lengths. It is appropriate to recall at this point also the recent contributions by Nobili and Pramanik (2025) and Martin (2025) about the nonlocal elasticity theory in integral form, originally described by Eringen (1972). In this regard, it should be emphasized that our boundary value problem investigated in Section 4 is directly formulated within the context of a differential model, and involves prescribed kinematic boundary data, rather than traction boundary conditions. The compatibility issue between constitutive boundary conditions and natural boundary conditions arising in integral nonlocal formulations is not directly addressed here, representing a separate aspect that lies beyond the scope of the present analysis. In other words, a comparison with Eringen's integral theory over finite domains falls outside the scope of this work.

Straughan (2026), for his part, enters the discussion at a different but complementary level. He studies mathematical properties of higher-gradient elastodynamic theories, with emphasis on logarithmic convexity, uniqueness, and Hölder continuous dependence. In particular, he considers linear elastodynamic problems involving first and second strain gradients, and explicitly establishes Hölder continuous dependence for a sixth-order elastodynamic theory arising from De Domenico and Askes (2016). This makes Straughan (2026) a natural analytical counterpart to De Domenico and Askes (2016): while De Domenico and Askes (2016) show how high-order gradients and micro-inertia may be used to reproduce the behavior of dispersive wave propagation, Straughan (2026) addresses the question of whether such higher-gradient elastodynamic models remain mathematically meaningful when well-posedness and stability issues are taken into account. This is particularly relevant for advanced materials, including auxetic materials, where the elastic coefficients may fail to satisfy classical positive-definiteness assumptions.

This paper proposes a mechanical model that exploits the idea of Lazar and Agiasofitou (2026) in order to unify the two directions presented above. In fact, the model proposed here combines the nonlocal elasticity of Klein–Gordon type developed by Lazar and Agiasofitou (2022) with the multi-scale and micro-inertia theory and with the second-order gradient theory for displacement, as described by De Domenico and Askes (2016). It incorporates three characteristic scale parameters, namely the characteristic length of nonlocal elasticity of Klein–Gordon type, the characteristic length of second-order gradient of displacement and the characteristic internal time scale parameter. The rationale underlying the coexistence of such characteristic parameters is not merely to construct a more general model containing previously proposed theories as particular cases. Rather, our goal is to investigate the cooperation/competition between spatial nonlocality, strain-gradient effects, higher-order micro-inertia and temporal nonlocality within a single continuum framework. For instance, the simultaneous presence of the two characteristic lengths and of the internal time scale parameter makes it possible to identify dispersion regimes governed by combined relations among the latter, which cannot be examined when the corresponding mechanisms are considered separately. This interplay becomes particularly clear when analyzing the results of Section 3.4. In this regard, the dispersion relations for the isotropic and homogeneous media are analytically determined. When the harmonic wave travels in a medium without loss or gain, its dispersion relation is real and this implies that the angular frequency and the wavenumber are assumed real. This leads to a quartic function in terms of the square of the wavenumber which must always be positive. This implies certain constraints on the characteristic internal scale parameters in addition to the elastic material parameters. The phase and group velocities of the acoustic and optic branches are analyzed. Depending on the wave branch, the wavelength regime and the characteristic material parameters, both normal and anomalous dispersion may occur. Moreover, it turns out that the acoustic group velocity may become negative in the short-wavelength limit for suitable relations between the characteristic lengths. All these results reveal a rich dispersive behavior influenced by the internal length and time scales of the model.

On this basis and by using the Lagrange identity method, we also investigate the uniqueness and stability of the solution of the associated initial boundary value problems. In this connection, we manage to establish an inequality that sufficiently expresses the continuous dependence (and implicitly the uniqueness) of the solution with respect to the initial given data. Thus, the mathematical problem modeled by our proposed equations is logically and physically reasonable, that is the model can be well-posed.

2. Mathematical model and governing equations

Let us consider a three-dimensional elastic medium occupying the regular region B, on the time interval $[0,T]$, $T>0$. We assume that the domain B is bounded by the boundary ∂B which is sufficiently regular to allow application of the divergence theorem.

In the nonlocal elasticity theory, the equation of motion in presence of body forces reads (see, e.g., Eringen, 1978, 1983, 1987, 1992, 1999, 2002)

$$\varrho\ddot{u}_i - t_{ji,j} = \varrho f_i, \quad (x,t)\in B\times(0,T), \tag{1}$$

where ϱ is the mass density, u_i are the components of the displacement vector, t_{ij} are the components of the stress tensor of nonlocal elasticity and f_i are the components of the body force density. Moreover, a superposed dot denotes the time derivative and "$,j$" denotes partial differentiation with respect to the spatial coordinates x_j, $j=1,2,3$.

According to the weak nonlocal elasticity theory, we assume

$$Lt_{ij} = \sigma_{ij}, \tag{2}$$

where σ_{ij} are the components of the local Cauchy stress tensor and L is the following differential operator

$$L = 1 - \alpha\ell_N^2\Delta + \beta\ell_N^4\Delta^2 + \gamma\tau^2\frac{\partial^2}{\partial t^2}, \tag{3}$$

where Δ denotes the Laplace operator, $\ell_N>0$ is the characteristic internal length scale parameter due to nonlocality in space (spatial nonlocality) and $\tau>0$ is the characteristic time scale parameter due to nonlocality in time (temporal nonlocality). Moreover, α, β and γ are three coefficients to be calibrated according to the problem being studied. These coefficients adjust the relative magnitudes between the various length scales appearing in the strain gradient term and in the micro-inertia contributions. They are dimensionless scalars, that is quantities that have a magnitude (a numerical value) but no physical units or geometric direction.

If we apply the operator L to Eq. (1), it is written as follows

$$L\left(\varrho\ddot{u}_i\right) - \sigma_{ji,j} = L\left(\varrho f_i\right), \quad (x,t)\in B\times(0,T). \tag{4}$$

We now take into account the microstructural effects via second-order displacement gradients and assume for σ_{ij} the following constitutive equations

$$\sigma_{ij} = \mathcal{L}\left(C_{ijkl}u_{k,l}\right), \tag{5}$$

where the operator $\mathcal{L}$ is defined as

$$\mathcal{L} = 1 - \delta\ell_G^2\Delta, \tag{6}$$

ℓ_G is the characteristic length scale parameter of the second-order displacement gradient elasticity and $\delta\ell_G^2$ (having dimensions of length squared) represents the coefficient entering the one-parameter Aifantis' gradient elasticity theory (see Aifantis, 1992). Further, C_{ijkl} is the elasticity tensor satisfying the following symmetry relation

$$C_{ijkl} = C_{klij} = C_{jikl}. \tag{7}$$

Taking into account the relations (3) to (6), the equations of motion of the model are written in terms of the displacements u_i in the following form

$$\begin{aligned}&\varrho\ddot{u}_i - \alpha\varrho\ell_N^2\ddot{u}_{i,jj} + \beta\varrho\ell_N^4\ddot{u}_{i,jjnn} + \gamma\varrho\tau^2\ddot{\ddot{u}}_i = C_{ijkl}\left(u_{k,lj} - \delta\ell_G^2u_{k,ljnn}\right)\\ &+\left(1-\alpha\ell_N^2\Delta+\beta\ell_N^4\Delta^2+\gamma\tau^2\frac{\partial^2}{\partial t^2}\right)\left(\varrho f_i\right), \quad (x,t)\in B\times(0,T).\end{aligned} \tag{8}$$

For an isotropic and homogeneous elastic medium the elasticity tensor is

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \tag{9}$$

where λ and μ are the classical Lamé constants. Therefore, the basic equations (8) become

$$\begin{aligned}&\varrho\ddot{u}_i - \alpha\varrho\ell_N^2\ddot{u}_{i,jj} + \beta\varrho\ell_N^4\ddot{u}_{i,jjnn} + \gamma\varrho\tau^2\ddot{\ddot{u}}_i = \left(1-\delta\ell_G^2\Delta\right)\left[\mu u_{i,jj} + (\lambda+\mu)u_{j,ji}\right]\\ &+\left(1-\alpha\ell_N^2\Delta+\beta\ell_N^4\Delta^2+\gamma\tau^2\frac{\partial^2}{\partial t^2}\right)\left(\varrho f_i\right), \quad (x,t)\in B\times(0,T).\end{aligned} \tag{10}$$

Let us give some insight to present model mathematically expressed by the operators L and $\mathcal{L}$ and by the basic differential system (8), since it is of principal interest:

- The operator L incorporates nonlocal Klein–Gordon effects described by Agiasofitou and Lazar (2023), Lazar and Agiasofitou (2022) by means of the Klein–Gordon operator

$$T_{KG} = 1 - \alpha\ell_N^2\Delta + \gamma\tau^2\frac{\partial^2}{\partial t^2}, \tag{11}$$

but also some higher-order inertia contributions described by De Domenico and Askes (2016) by means of the operator $\beta\ell_N^4\Delta^2$.

- The operator $\mathcal{L}$ incorporates the microstructural effects by means of the second-order gradient of displacements.

It is worth highlighting that Lazar (2024), Lazar and Agiasofitou (2026) were the first to develop a unification of Eringen's nonlocal elasticity theory of Helmholtz type and the simplified first strain gradient elasticity theory. The model incorporates two characteristic length scale parameters, namely the characteristic length ℓ_N of nonlocal elasticity of Helmholtz type and the characteristic length ℓ_G of simplified first strain gradient elasticity. It corresponds to the situation when in the differential system (8) it is considered that $\beta = \gamma = 0$.

We have to note that by particularizing the coefficients α, β and γ, we obtain as particular cases models recently developed in the literature, using, separately, either the nonlocal theory or the higher-order gradient theory of displacement. Thus:

(i) When we set $\gamma = 0$ in the basic differential system (8), we rediscover the model developed by De Domenico and Askes (2016), relation (3).
(ii) While when we set $\beta = 0$ and $\delta = 0$ in (8), then we rediscover the Klein–Gordon nonlocal elastic model developed by Lazar and Agiasofitou (2022).
(iii) When we set $\beta = 0$ and $\gamma = 0$ in (8), then we rediscover the nonlocal simplified strain gradient elasticity studied by Lazar and Agiasofitou (2026).

3. Wave propagation in nonlocal elasticity with nonlocality in space and time and with second-order displacement gradients

In this Section, the dispersion relations in the framework of nonlocal elasticity with nonlocality in space and time and with second-order displacement gradients are derived for linear plane harmonic waves. The phase and group velocities are calculated in order to explore when normal or anomalous dispersion occurs.

3.1. Dispersion relations and admissibility conditions

We consider a three-dimensional homogeneous isotropic elastic body of infinite extent with vanishing body forces. In our dispersion analysis, for the sake of clarity but without loss of generality, we consider the particular choice $\alpha = \beta = \gamma = \delta = 1$ in order to simplify the algebraic expressions. Then the basic equations (10) reduce to the following partial differential equations

$$\varrho\ddot{u}_i - \varrho\ell_N^2\ddot{u}_{i,jj} + \varrho\ell_N^4\ddot{u}_{i,jjnn} + \varrho\tau^2\ddot{u}_i = \left(1 - \ell_G^2\Delta\right)\left[\mu u_{i,jj} + (\lambda+\mu)u_{j,ji}\right]. \tag{12}$$

We consider linear plane longitudinal harmonic waves propagating in x_1-direction given by the following complex representation

$$u_1(x,t) = A\exp\left[i\left(\varkappa x_1 - \omega t\right)\right], \quad u_2(x,t) = u_3(x,t) = 0, \tag{13}$$

where $\varkappa$ is the wavenumber, ω is the angular or circular frequency, A is the amplitude and i is the complex unit.

Such a wave satisfies the basic equations (12) if ω is a solution of the algebraic equation

$$\tau^2\omega^4 - \left(1 + \ell_N^2\varkappa^2 + \ell_N^4\varkappa^4\right)\omega^2 + c_L^2\varkappa^2\left(1 + \ell_G^2\varkappa^2\right) = 0, \tag{14}$$

where

$$c_L = \sqrt{\frac{\lambda+2\mu}{\varrho}} \tag{15}$$

is the classical longitudinal velocity.

For a linear plane transverse harmonic wave propagating in x_1-direction

$$u_1(x,t) = 0, \quad u_2(x,t) = A\exp\left[i\left(\varkappa x_1 - \omega t\right)\right], \quad u_3(x,t) = 0, \tag{16}$$

the basic equations (12) lead to the dispersion relation

$$\tau^2\omega^4 - \left(1 + \ell_N^2\varkappa^2 + \ell_N^4\varkappa^4\right)\omega^2 + c_T^2\varkappa^2\left(1 + \ell_G^2\varkappa^2\right) = 0, \tag{17}$$

where

$$c_T = \sqrt{\frac{\mu}{\varrho}} \tag{18}$$

is the classical transverse velocity.

Thus, we have

$$\left(\omega_\xi\right)_{O,A}^2 = \frac{1}{2\tau^2}\left[1 + \ell_N^2\varkappa^2 + \ell_N^4\varkappa^4 \pm \sqrt{\left(1 + \ell_N^2\varkappa^2 + \ell_N^4\varkappa^4\right)^2 - 4c_\xi^2\varkappa^2\tau^2\left(1 + \ell_G^2\varkappa^2\right)}\right], \tag{19}$$

where $\xi = L, T$.

Dealing with an elastic model (i.e., a medium *without loss or gain* is considered), there is no energy consumption and so, in continuity with Lazar and Agiasofitou (2026), the angular frequency ω *and* the wavenumber $\varkappa$ have to be assumed real. In this connection we note that, from Eq. (19), the following constraint arises, able to ensure that the frequencies are always real, both in the longitudinal and transverse cases:

$$\ell_N^8 \varkappa^8 + 2\ell_N^6 \varkappa^6 + \left(3\ell_N^4 - 4c_\xi^2 \ell_G^2 \tau^2\right) \varkappa^4 + 2\left(\ell_N^2 - 2c_\xi^2 \tau^2\right) \varkappa^2 + 1 \geq 0 \quad \text{for all real } \varkappa. \tag{20}$$

If we set

$$\zeta = \varkappa^2 \ell_N^2, \quad c = \frac{\ell_N}{\tau}, \quad r = \frac{\ell_G}{\ell_N}, \quad C_\xi = \frac{c_\xi}{c}, \tag{21}$$

we obtain the following quartic function

$$f(\zeta) = \zeta^4 + 2\zeta^3 + \left(3 - 4C_\xi^2 r^2\right) \zeta^2 + 2\left(1 - 2C_\xi^2\right) \zeta + 1 \geq 0, \quad \text{for all} \quad \zeta \geq 0. \tag{22}$$

It should be noted that $c > 0$ represents the characteristic velocity of nonlocality in terms of the length scale ℓ_N and time scale τ (Cf. Lazar & Agiasofitou, 2022). While r represents the characteristic length ℓ_G of the second-order gradient of displacement reported to the characteristic length ℓ_N of nonlocal elasticity.

We have to note that the quartic function $f(\zeta)$ keeps the positive sign constant for any $\zeta \geq 0$ if we impose the constraints (which certainly identify a sufficient, but not necessary, condition)

$$1 - 2C_\xi^2 \geq 0, \quad 3 - 4C_\xi^2 r^2 \geq 0, \tag{23}$$

hence

$$c \geq c_\xi \sqrt{2}, \quad \ell_N \geq \sqrt{\frac{2}{3}}\, \ell_G. \tag{24}$$

The constraint $c \geq c_\xi \sqrt{2}$ agrees with the restriction proven by Lazar and Agiasofitou (2022), $c \geq c_\xi$, whereby the characteristic velocity of nonlocality is higher than the propagation velocities of transverse and longitudinal waves. While the restriction $\ell_N \geq \sqrt{2/3}\, \ell_G$ expresses a ratio between the two characteristic lengths used in the present model.

However, the nature of the roots of the quartic function will be explored in depth in Appendix 1, as well as the restrictions on C_ξ and r given in (23) that will also be deepened.

3.2. Acoustic and optic branches

In this and subsequent Subsections, we will not use the notations given in (21), primarily because of the need to highlight the cut-off frequency as well as the constraints involving the characteristic lengths.

In Lazar and Agiasofitou (2022), Section 4, one can read: *Nonlocal effects are important in high-frequency and short-wavelength regions of waves*. Therefore, it should be emphasized that the small-$\varkappa$ expansions considered below are not used to characterize the short-wavelength regime, where nonlocal and high-gradient effects are expected to play a major role. Their purpose is instead to examine the long-wavelength limit in order to identify the acoustic and optic branches of the dispersion relation.

Let us consider the small-$\varkappa$ expansion of the square root in Eq. (19), that is:

$$1 + \left(\ell_N^2 - 2c_\xi^2 \tau^2\right) \varkappa^2 + \left[\ell_N^4 + 2c_\xi^2 \tau^2 \left(\ell_N^2 - \ell_G^2 - c_\xi^2 \tau^2\right)\right] \varkappa^4 + \mathcal{O}\left(\varkappa^6\right). \tag{25}$$

In agreement with what stated in Lazar and Agiasofitou (2022), Subsect. 4.2,[1] also in this case the roots in Eq. (19) identify acoustic (i.e., low-frequency) and optic (i.e., high-frequency) branches. In fact, in view of (25), the small-$\varkappa$ expansion for the 'minus' branch gives

$$\left(\omega_\xi\right)_A^2 = c_\xi^2 \varkappa^2 - c_\xi^2 \left(\ell_N^2 - \ell_G^2 - c_\xi^2 \tau^2\right) \varkappa^4 + \mathcal{O}(\varkappa^6), \tag{26}$$

and therefore $\left(\omega_\xi\right)_A$ is acoustic because it starts from zero frequency in the long-wavelength limit, namely:

$$\lim_{\varkappa \to 0^+} \left(\omega_\xi\right)_A = 0, \qquad \xi = L, T. \tag{27}$$

On the contrary, the small-$\varkappa$ expansion for the 'plus' branch gives

$$\left(\omega_\xi\right)_O^2 = \frac{1}{\tau^2} + \left(\frac{\ell_N^2}{\tau^2} - c_\xi^2\right) \varkappa^2 + \left[\frac{\ell_N^4}{\tau^2} + c_\xi^2 \left(\ell_N^2 - \ell_G^2 - c_\xi^2 \tau^2\right)\right] \varkappa^4 + \mathcal{O}(\varkappa^6), \tag{28}$$

and therefore $\left(\omega_\xi\right)_O$ is optic because it shows a nonzero cut-off frequency ω_c for $\varkappa \to 0^+$, that is

$$\lim_{\varkappa \to 0^+} \left(\omega_\xi\right)_O = \frac{1}{\tau} = \omega_c, \qquad \xi = L, T, \tag{29}$$

[1] But unlike Lazar and Agiasofitou (2026), Section 3.1, where one can read: *It should be noticed that nonlocal simplified strain gradient elasticity has only acoustic modes*.

i.e., the same as defined in Lazar and Agiasofitou (2022), Eq. (68). Thus, the characteristic time scale parameter τ is responsible for the emergence of (high-frequency) optic modes, while the classical acoustic modes are recovered in the low-frequency and long-wavelength limit.

Remark. Physically, when wavelengths are very large if compared to the internal scales of the material, the microstructure of the medium does not emerge, and therefore it responds as a classical elastic continuum: this is why standard longitudinal and transverse acoustic waves, i.e., $(\omega_\xi)_A \sim c_\xi \varkappa$, are recovered. Conversely, the optic branch is associated with the intrinsic time scale parameter τ: even in the long-wavelength limit, the medium can still sustain a finite frequency time oscillation with cut-off frequency ω_c.

3.3. Phase and group velocities

For each $\xi = L, T$ and for both the acoustic and optic branches, the phase velocity is:

$$\left(v_\xi^P\right)_{O,A} = \varkappa^{-1} \left(\omega_\xi\right)_{O,A} \tag{30}$$

For the sole purpose of writing the corresponding analytical expressions in a sufficiently compact form, let us introduce the notations:

$$\Phi(\varkappa) = 1 + \ell_N^2 \varkappa^2 + \ell_N^4 \varkappa^4 \tag{31}$$

and

$$\Theta_\xi(\varkappa) = \Phi^2(\varkappa) - 4c_\xi^2 \varkappa^2 \tau^2 \left(1 + \ell_G^2 \varkappa^2\right) \tag{32}$$

so that, taking into account Eq. (19), the exact phase velocities are:

$$\left(v_\xi^P\right)_{O,A} = \frac{1}{\sqrt{2}\tau\varkappa} \left[\Phi(\varkappa) \pm \sqrt{\Theta_\xi(\varkappa)}\right]^{1/2}, \qquad \xi = L, T. \tag{33}$$

Such expressions hold for the whole admissible wavenumber range. Anyway, in order to characterize their long-wavelength behavior, we are going to consider their small-$\varkappa$ expansions.

For the acoustic branch, the phase velocity is

$$\left(v_\xi^P\right)_A = c_\xi \left[1 - \frac{1}{2}\left(\ell_N^2 - \ell_G^2 - c_\xi^2 \tau^2\right) \varkappa^2\right] + \mathcal{O}(\varkappa^4) \tag{34}$$

and therefore

$$\lim_{\varkappa \to 0^+} \left(v_\xi^P\right)_A = c_\xi, \qquad \xi = L, T, \tag{35}$$

meaning that, in the long-wavelength limit, the acoustic phase velocities reduce to the classical longitudinal and transverse elastic wave speeds.

For the optic branch, instead, we have

$$\left(v_\xi^P\right)_O = \frac{1}{\tau\varkappa} + \frac{\ell_N^2 - c_\xi^2 \tau^2}{2\tau} \varkappa + \left[\frac{\ell_N^4}{2\tau} + \frac{\tau c_\xi^2}{2}\left(\ell_N^2 - \ell_G^2 - c_\xi^2 \tau^2\right) - \frac{\tau^3}{8}\left(\frac{\ell_N^2}{\tau^2} - c_\xi^2\right)^2\right] \varkappa^3 + \mathcal{O}(\varkappa^5) \tag{36}$$

and consequently

$$\lim_{\varkappa \to 0^+} \left(v_\xi^P\right)_O = +\infty, \qquad \xi = L, T. \tag{37}$$

Such divergence of the optic phase velocity for wavenumbers tending to 0 is expected in view of the existence of a finite cut-off frequency. Evidently, it is not associated with any transport of energy: in a dispersive medium, the relevant[2] propagation velocity of wave packets and energy is the group velocity, which will be examined next.

The group velocity is defined as the derivative of the angular frequency with respect to the wavenumber. Starting again from Eq. (19), and recalling the definitions (31) and (32), it is an easy task to deduce the exact analytical expressions for the group velocities which, in an acceptably compact form, can be written as:

$$\left(v_\xi^G\right)_{O,A} = \frac{1}{2\sqrt{2}\tau} \left[\Phi(\varkappa) \pm \sqrt{\Theta_\xi(\varkappa)}\right]^{-1/2} \left[\Phi'(\varkappa) \pm \Theta_\xi'(\varkappa)\, \Theta_\xi^{-1/2}(\varkappa)/2\right], \tag{38}$$

where the prime $'$ indicates the derivative with respect to $\varkappa$ and provided that, in this case, $\Theta_\xi(\varkappa)$ is strictly positive. The following small-$\varkappa$ expansions are, once again, useful to characterize the long-wavelength behavior.

Still resorting to Maclaurin expansions, again from Eqs. (26) and (28), we have:

$$\left(v_\xi^G\right)_A = c_\xi \left[1 + \frac{3}{2}\left(\ell_G^2 - \ell_N^2 + c_\xi^2 \tau^2\right) \varkappa^2\right] + \mathcal{O}(\varkappa^4) \tag{39}$$

[2] See Lazar and Agiasofitou (2022), Subsect. 4.2.1: *The group velocity is the more important velocity, since it is the relevant quantity for the propagation of the energy into the dispersive medium.*

and

$$\left(v_\xi^G\right)_O = \frac{\ell_N^2 - c_\xi^2\tau^2}{\tau}\varkappa + O(\varkappa^3), \tag{40}$$

which is positive, for sufficiently small wavenumbers, provided (additionally) that $c^2 > \max\left\{c_L^2, c_T^2\right\}$, a restriction consistent with Eq. (57) in Lazar and Agiasofitou (2022). We observe that:

$$\lim_{\varkappa\to 0^+}\left(v_\xi^G\right)_A = c_\xi, \qquad \xi = L, T, \tag{41}$$

meaning that, in the long-wavelength limit, also the acoustic group velocities reduce to the classical longitudinal and transverse elastic wave speeds. In contrast,

$$\lim_{\varkappa\to 0^+}\left(v_\xi^G\right)_O = 0, \qquad \xi = L, T, \tag{42}$$

showing that the optic branch does not describe, in the long-wavelength limit, a perturbation propagating in space, but rather temporal oscillations with cut-off frequency ω_c.

3.4. Normal and anomalous dispersion in the long-wavelength and high-wavenumber limits

We now compare phase and group velocities in order to distinguish between normal and anomalous dispersion. Following a standard terminology (refer again, for instance, to Lazar & Agiasofitou, 2022, 2026), normal dispersion corresponds to the case in which the group velocity is smaller than the phase velocity ($v^G < v^P$), whereas anomalous dispersion corresponds to the opposite case ($v^G > v^P$); all this considering that $v^G - v^P = \varkappa(\mathrm{d}v^P/\mathrm{d}\varkappa)$.

Taking into account relations (34) and (39) we get, for the acoustic branch and in the long-wavelength limit:

$$\left(v_\xi^G\right)_A - \left(v_\xi^P\right)_A = c_\xi\left(\ell_G^2 - \ell_N^2 + c_\xi^2\tau^2\right)\varkappa^2 + \mathcal{O}(\varkappa^4) \tag{43}$$

meaning that the constraint $\ell_N^2 > \ell_G^2 + c_\xi^2\tau^2$, for $\xi = L, T$, returns a condition of normal dispersion, while $\ell_N^2 < \ell_G^2 + c_\xi^2\tau^2$ is associated with an anomalous dispersion. As for the limit case, $\ell_N^2 = \ell_G^2 + c_\xi^2\tau^2$, it is sufficient to increase the expansion orders of $\left(v_\xi^{P,G}\right)_A$ to make a normal dispersion emerge at the fourth power of $\varkappa$. In detail:

$$\left(v_\xi^G\right)_A - \left(v_\xi^P\right)_A = -2c_\xi\ell_N^4\varkappa^4 + \mathcal{O}(\varkappa^6). \tag{44}$$

Evidently, mixed regimes are also possible, depending on the values of c_L and c_T for the considered medium.

As for the optic branch, from Eqs. (36) and (40) we can observe that $\left(v_\xi^G\right)_O - \left(v_\xi^P\right)_O = -(\tau\varkappa)^{-1} + \mathcal{O}(\varkappa)$ meaning that, still in the long-wavelength limit, the optic branch is always normally dispersive.

If we start again from Eq. (19), considered under the restriction (20), we can proceed, this time, with expansions for $\varkappa \to +\infty$. It is not difficult to prove that, in this case:

$$\left(\omega_\xi\right)_A = \frac{c_\xi\ell_G}{\ell_N^2} + \frac{c_\xi\left(\ell_N^2 - \ell_G^2\right)}{2\ell_G\ell_N^4}\frac{1}{\varkappa^2} + \mathcal{O}(\varkappa^{-4}) \tag{45}$$

and so

$$\left(v_\xi^P\right)_A = \frac{c_\xi\ell_G}{\ell_N^2}\frac{1}{\varkappa} + \mathcal{O}(\varkappa^{-3}), \qquad \left(v_\xi^G\right)_A = -\frac{c_\xi\left(\ell_N^2 - \ell_G^2\right)}{\ell_G\ell_N^4}\frac{1}{\varkappa^3} + \mathcal{O}(\varkappa^{-5}), \tag{46}$$

where the sign of the acoustic group velocity clearly depends on the order relation between ℓ_N and ℓ_G. Anyway,

$$\left(v_\xi^G\right)_A - \left(v_\xi^P\right)_A = -\frac{c_\xi\ell_G}{\ell_N^2}\frac{1}{\varkappa} + \mathcal{O}(\varkappa^{-3}). \tag{47}$$

Before proceeding further, it is worth emphasizing what follows. For sufficiently large wavenumbers, the acoustic group velocity is negative when $\ell_N > \ell_G$, whereas it is positive when $\ell_N < \ell_G$. In the limiting case $\ell_N = \ell_G$, higher-order terms must be considered. In any case, the sign of the group velocity and the normal/anomalous character of dispersion are distinct issues. Indeed, even when $(v_\xi^G)_A < 0$, Eq. (47) shows that $(v_\xi^G)_A < (v_\xi^P)_A$, so that the acoustic branch remains normally dispersive in the high-wavenumber limit. A (possibly) negative group velocity in this regime indicates that it is directed oppositely to the phase velocity.

As for the optic branch, instead:

$$\left(\omega_\xi\right)_O = \frac{\ell_N^2}{\tau}\varkappa^2 + \frac{1}{2\tau} + \mathcal{O}(\varkappa^{-2}) \tag{48}$$

and so

$$\left(v_\xi^P\right)_O = \frac{\ell_N^2}{\tau}\varkappa + \frac{1}{2\tau}\frac{1}{\varkappa} + \mathcal{O}(\varkappa^{-3}), \qquad \left(v_\xi^G\right)_O = 2\frac{\ell_N^2}{\tau}\varkappa + \mathcal{O}(\varkappa^{-3}), \tag{49}$$

meaning that, in the short-wavelength limit, the optic branch is always characterized by an anomalous dispersion since

$$\left(v_\xi^G\right)_O - \left(v_\xi^P\right)_O = \frac{\ell_N^2}{\tau}\varkappa + \mathcal{O}(\varkappa^{-1}). \tag{50}$$

It should be noted that Eq. (49) clearly predicts an unbounded growth of both the optic phase and group velocities in the (formal) limit $\varkappa \to +\infty$. This feature represents a limitation of the present continuum model, at least when extended to arbitrarily short wavelengths. Indeed, the same model should be regarded as capable of providing an effective continuum description over a *finite* range of wavelengths, whereas the limit $\varkappa \to +\infty$ explores length scales at which it is no longer expected to remain physically adequate. Evidently, the above asymptotic behavior should not be interpreted as a prediction of physically attainable unbounded propagation velocities. The implications of such high-wavenumber behavior would require a separate in-depth analysis that goes beyond the scope of the present work.

We previously found that the optic branch is normally dispersive in the long-wavelength limit, then continuity implies that it undergoes at least one transition from normal to anomalous dispersion. On the contrary, such a clear conclusion is not possible for the acoustic branch since, in the long-wavelength limit, normal or anomalous dispersion depends on the relation between ℓ_N^2 and $\ell_G^2 + c_\xi^2 \tau^2$.

4. Stability of the model

This Section addresses the stability of solutions of differential equations (8) under small perturbations of initial conditions. In this sense, we define the initial boundary value problem $\mathcal{P}$ consisting of the basic differential system (8) to which we associate the initial conditions

$$u_i(x,0) = u_i^0(x), \quad \dot{u}_i(x,0) = \dot{u}_i^0(x), \quad \ddot{u}_i(x,0) = \ddot{u}_i^0(x), \quad \dddot{u}_i(x,0) = \dddot{u}_i^0(x), \quad x \in B, \tag{51}$$

and the boundary conditions

$$u_i(x,t) = \tilde{u}_i(x,t), \quad \frac{\partial u_i}{\partial n}(x,t) = \tilde{u}_i^*(x,t), \quad (x,t) \in \partial B \times (0,T). \tag{52}$$

Here $u_i^0(x)$, $\dot{u}_i^0(x)$, $\ddot{u}_i^0(x)$, $\dddot{u}_i^0(x)$, $\tilde{u}_i(x,t)$ and $\tilde{u}_i^*(x,t)$ are prescribed continuous functions. Therefore, we investigate the behavior of the solution u_i of the initial boundary value problem $\mathcal{P}$ under small perturbations of the initial conditions (51).

Let $u_i^{(1)}(x,t)$ and $u_i^{(2)}(x,t)$ be solutions of the initial boundary value problem $\mathcal{P}$ for the same boundary data (52) and the same body force f_i, but for different initial conditions $u_i^{0(1)}(x)$, $\dot{u}_i^{0(1)}(x)$, $\ddot{u}_i^{0(1)}(x)$, $\dddot{u}_i^{0(1)}(x)$ and $u_i^{0(2)}(x)$, $\dot{u}_i^{0(2)}(x)$, $\ddot{u}_i^{0(2)}(x)$, $\dddot{u}_i^{0(2)}(x)$. Then, the difference solution

$$u_i(x,t) = u_i^{(1)}(x,t) - u_i^{(2)}(x,t) \tag{53}$$

satisfies the initial boundary value problem $\mathcal{P}_i$ defined by the differential system

$$\rho \ddot{u}_i - \alpha \rho \ell_N^2 \ddot{u}_{i,jj} + \beta \rho \ell_N^4 \ddot{u}_{i,jjnn} + \gamma \rho \tau^2 \dddot{u}_i = C_{ijkl} \left(u_{k,lj} - \delta \ell_G^2 u_{k,ljnn} \right), \quad (x,t) \in B \times (0,T), \tag{54}$$

the null boundary conditions

$$u_i(x,t) = 0, \quad \frac{\partial u_i}{\partial n}(x,t) = 0, \quad (x,t) \in \partial B \times (0,T), \tag{55}$$

and the initial conditions

$$u_i(x,0) = p_i(x), \quad \dot{u}_i(x,0) = q_i(x), \quad \ddot{u}_i(x,0) = r_i(x), \quad \dddot{u}_i(x,0) = s_i(x), \quad x \in B, \tag{56}$$

where

$$\begin{aligned} &p_i(x) = u_i^{0(1)}(x) - u_i^{0(2)}(x), \quad q_i(x) = \dot{u}_i^{0(1)}(x) - \dot{u}_i^{0(2)}(x), \\ &r_i(x) = \ddot{u}_i^{0(1)}(x) - \ddot{u}_i^{0(2)}(x), \quad s_i(x) = \dddot{u}_i^{0(1)}(x) - \dddot{u}_i^{0(2)}(x), \quad x \in B. \end{aligned} \tag{57}$$

To study the continuous dependence of the solution on the initial data, we will use the following identity

$$\frac{\partial}{\partial s} \left[f(t+s)\dot{f}(t-s) + \dot{f}(t+s)f(t-s) \right] = f(t-s)\ddot{f}(t+s) - f(t+s)\ddot{f}(t-s), \tag{58}$$

for any continuous differentiable function f and for all $s \in (0,t)$, $t \in (0, T/2)$. By an integration over $s \in (0,t)$, we get the basic form of the Lagrange identity as (see, e.g., Brun, 1969)

$$\begin{aligned} &2f(t)\dot{f}(t) = \int_0^t \left[f(t+s)\ddot{f}(t-s) - f(t-s)\ddot{f}(t+s) \right] ds \\ &+ f(2t)\dot{f}(0) + \dot{f}(2t)f(0), \quad t \in (0, T/2). \end{aligned} \tag{59}$$

Let us first put $f(t) \to \sqrt{\rho}\, u_i(x,t)$ in the Lagrange identity (59). Then by using the basic differential system (54) and by means of the divergence theorem, an integration by parts and the use of the null boundary condition (55), this Lagrange identity implies

$$
\begin{aligned}
&2\int_B \varrho u_i(t)\dot{u}_i(t)dv = \int_B \varrho\left[u_i(2t)\dot{u}_i(0)+\dot{u}_i(2t)u_i(0)\right]dv\\
&+\int_0^t\int_B \alpha\varrho\ell_N^2\left[u_{i,j}(t-s)\ddot{u}_{i,j}(t+s)-u_{i,j}(t+s)\ddot{u}_{i,j}(t-s)\right]dvds\\
&+\int_0^t\int_B \beta\varrho\ell_N^4\left[u_{i,j}(t+s)\ddot{u}_{i,jnn}(t-s)-u_{i,j}(t-s)\ddot{u}_{i,jnn}(t+s)\right]dvds\\
&+\int_0^t\int_B \gamma\varrho\tau^2\left[u_i(t-s)\dddot{u}_i(t+s)-u_i(t+s)\dddot{u}_i(t-s)\right]dvds\\
&+\int_0^t\int_B \delta\ell_G^2 C_{ijkl}\left[u_{i,j}(t+s)u_{k,lnn}(t-s)-u_{i,j}(t-s)u_{k,lnn}(t+s)\right]dv,\quad t\in(0,T/2).
\end{aligned}
\tag{60}
$$

In this relation, and throughout the rest of the Section, the dependence of the displacement on the spatial variable x is tacitly understood.

At this point we should note that the null boundary conditions (55) imply that

$$
u_{i,j}(x,t)=0,\quad \text{for all}\quad (x,t)\in\partial B\times(0,T).
\tag{61}
$$

On this basis, and by using an identity of type (59), we get

$$
\begin{aligned}
&E_1(t)=E_1(0)+\int_0^t\int_B \varrho\left[u_i(2s)\dot{u}_i(0)+u_i(0)\dot{u}_i(2s)\right]dvds+\int_0^t\int_B \alpha\varrho\ell_N^2\left[u_{i,j}(0)\dot{u}_{i,j}(2s)\right.\\
&\left.+u_{i,j}(2s)\dot{u}_{i,j}(0)\right]dvds+\int_0^t\int_B \beta\varrho\ell_N^4\left[u_{i,jn}(0)\dot{u}_{i,jn}(2s)+u_{i,jn}(2s)\dot{u}_{i,jn}(0)\right]dvds\\
&+\int_0^t\int_B \gamma\varrho\tau^2\left[u_i(0)\ddot{u}_i(2s)+u_i(2s)\ddot{u}_i(0)+\dot{u}_i(2s)\ddot{u}_i(0)+\dot{u}_i(0)\ddot{u}_i(2s)\right]dvds,\quad t\in(0,T/2),
\end{aligned}
\tag{62}
$$

where

$$
\begin{aligned}
&E_1(t)=\int_B\left[\varrho\left(u_i(t)+\gamma\tau^2\ddot{u}_i(t)\right)\left(u_i(t)+\gamma\tau^2\ddot{u}_i(t)\right)+\alpha\varrho\ell_N^2u_{i,j}(t)u_{i,j}(t)\right.\\
&\left.+\beta\varrho\ell_N^4u_{i,jn}(t)u_{i,jn}(t)-\varrho\gamma^2\tau^4\ddot{u}_i(t)\ddot{u}_i(t)\right]dv,\quad t\in(0,T).
\end{aligned}
\tag{63}
$$

On the other side, if we set $f(t)\to\sqrt{\gamma\varrho}\tau\ddot{u}_i(x,t)$ into identity (59), we obtain

$$
\begin{aligned}
&2\int_B \gamma\varrho\tau^2\ddot{u}_i(t)\dddot{u}_i(t)dv=\int_B \gamma\varrho\tau^2\left[\ddot{u}_i(2t)\ddot{u}_i(0)+\ddot{u}_i(0)\ddot{u}_i(2t)\right]dv\\
&+\int_0^t\int_B \gamma\varrho\tau^2\left[\ddot{u}_i(t+s)\dddot{u}_i(t-s)-\ddot{u}_i(t-s)\dddot{u}_i(t+s)\right]dvds,\quad t\in(0,T/2).
\end{aligned}
\tag{64}
$$

By virtue of relation (54), the null boundary conditions (55) and an integration by parts, from (64) we deduce

$$
\begin{aligned}
&E_2(t)=E_2(0)+\int_0^t\int_B \gamma\varrho\tau^2\left[\ddot{u}_i(2s)\ddot{u}_i(0)+\ddot{u}_i(0)\ddot{u}_i(2s)\right]dvds\\
&-\int_0^t\int_B C_{ijkl}\left[u_{i,j}(2s)\dot{u}_{k,l}(0)+u_{i,j}(0)\dot{u}_{k,l}(2s)\right]dvds\\
&-\int_0^t\int_B \delta\ell_G^2C_{ijkl}\left[u_{i,jn}(2s)\dot{u}_{k,ln}(0)+u_{i,jn}(0)\dot{u}_{k,ln}(2s)\right]dvds,\quad t\in(0,T/2),
\end{aligned}
\tag{65}
$$

where

$$
E_2(t)=\int_B\left[\gamma\varrho\tau^2\ddot{u}_i(t)\ddot{u}_i(t)-C_{ijkl}u_{i,j}(t)u_{k,l}(t)-\delta\ell_G^2C_{ijkl}u_{i,jn}(t)u_{k,ln}(t)\right]dv,\quad t\in(0,T).
\tag{66}
$$

By combining the two identities (62) and (65), we obtain

$$
\begin{aligned}
&\mathcal{E}(t)=\mathcal{E}(0)+\int_0^t\int_B \varrho\left[u_i(2s)\dot{u}_i(0)+u_i(0)\dot{u}_i(2s)\right]dvds+\int_0^t\int_B \alpha\varrho\ell_N^2\left[u_{i,j}(0)\dot{u}_{i,j}(2s)\right.\\
&\left.+u_{i,j}(2s)\dot{u}_{i,j}(0)\right]dvds+\int_0^t\int_B \beta\varrho\ell_N^4\left[u_{i,jn}(0)\dot{u}_{i,jn}(2s)+u_{i,jn}(2s)\dot{u}_{i,jn}(0)\right]dvds\\
&+\int_0^t\int_B \gamma\varrho\tau^2\left[u_i(0)\ddot{u}_i(2s)+u_i(2s)\ddot{u}_i(0)+\dot{u}_i(2s)\ddot{u}_i(0)+\dot{u}_i(0)\ddot{u}_i(2s)\right]dvds\\
&+\int_0^t\int_B \varrho\gamma^2\tau^4\left[\ddot{u}_i(2s)\ddot{u}_i(0)+\ddot{u}_i(0)\ddot{u}_i(2s)\right]dvds\\
&-\int_0^t\int_B \gamma\tau^2C_{ijkl}\left[u_{i,j}(2s)\dot{u}_{k,l}(0)+u_{i,j}(0)\dot{u}_{k,l}(2s)\right]dvds\\
&-\int_0^t\int_B \gamma\tau^2\delta\ell_G^2C_{ijkl}\left[u_{i,jn}(2s)\dot{u}_{k,ln}(0)+u_{i,jn}(0)\dot{u}_{k,ln}(2s)\right]dvds,\quad t\in(0,T/2)
\end{aligned}
\tag{67}
$$

where

$$\mathcal{E}(t) = E_1(t) + \gamma\tau^2 E_2(t) = \int_B \varrho \left(u_i(t) + \gamma\tau^2 \ddot{u}_i(t)\right)\left(u_i(t) + \gamma\tau^2 \ddot{u}_i(t)\right) dv + \mathcal{F}(t), \tag{68}$$

and

$$\begin{aligned}\mathcal{F}(t) = &\int_B \left[\alpha\varrho\ell_N^2 u_{i,j}(t)u_{i,j}(t) + \beta\varrho\ell_N^4 u_{i,jn}(t)u_{i,jn}(t)\right] dv\\ &-\gamma\tau^2 \int_B C_{ijkl}u_{i,j}(t)u_{k,l}(t)dv - \gamma\delta\tau^2 \int_B \ell_G^2 C_{ijkl}u_{i,jn}(t)u_{k,ln}(t)dv.\end{aligned} \tag{69}$$

It is now time to show that $\mathcal{E}(t)$ is a measure of the solution in the sense that $\mathcal{E}(t) \geq 0$ for all $t \in (0, T/2)$ and $\mathcal{E}(t) = 0$ implies $u_i(x,t) = 0$ for all $(x,t) \in B \times (0, T/2)$. To this end, we have to see from (69) that

$$\begin{aligned}\mathcal{F}(t) \geq &\alpha\varrho\tau^2\left(\frac{\ell_N^2}{\tau^2} - \frac{\gamma}{\alpha}C_{max}^2\right)\int_B u_{i,j}(t)u_{i,j}(t)dv\\ &+\beta\varrho\tau^2\ell_G^2\left(\frac{\ell_N^4}{\tau^2\ell_G^2} - \frac{\gamma\delta}{\beta}C_{max}^2\right)\int_B u_{i,jn}(t)u_{i,jn}(t)dv,\end{aligned} \tag{70}$$

where

$$C_{max}^2 = \max_B \left(\frac{C_{ijkl}C_{ijkl}}{\varrho^2}\right)^{1/2} \tag{71}$$

intercepts the maximum speed of propagation corresponding to C_{ijkl} and ϱ (see, e.g., Gurtin, 1972, p. 245).

It now becomes clear that, under the assumptions

$$c \geq \sqrt{\frac{\gamma}{\alpha}}\, C_{max}, \quad \frac{c}{r} \geq \sqrt{\frac{\delta\gamma}{\beta}}C_{max}, \tag{72}$$

we have

$$\begin{aligned}\mathcal{F}(t) \geq &\alpha\varrho\tau^2\left(c^2 - \frac{\gamma}{\alpha}C_{max}^2\right)\int_B u_{i,j}(t)u_{i,j}(t)dv\\ &+\beta\varrho\tau^2\ell_G^2\left(\frac{\ell_N^4}{\tau^2\ell_G^2} - \frac{\gamma\delta}{\beta}C_{max}^2\right)\int_B u_{i,jn}(t)u_{i,jn}(t)dv \geq 0,\end{aligned} \tag{73}$$

which, when used in relation (68), shows our conclusion that $\mathcal{E}(t)$ represents a measure of the solution. It is worth noting that the constraints (72) are in line with the assumptions (23) discussed in relation to plane wave propagation, but also with the analysis developed in Appendix 1.

Now we use the Cauchy–Schwarz inequality in (67) in order to obtain the following estimate

$$\mathcal{E}(t) \leq \mathcal{E}(0) + \mathcal{M}_0[u_i(0)] \cdot \mathcal{N}_T[u_i(t)] = 2\sum_{p=1}^{16} M_{0p}N_{Tp}, \quad t \in (0, T/2), \tag{74}$$

where M_{0p} and N_{Tp} are measures related to the initial given data and to the values at time t for u_i, whose explicit values are listed in the Appendix 2.

If now the solution $u_i(x,t)$ belongs to the class where $N_{Tp} < \infty$, $p = 1, 2, \ldots, 16$, at time T, then (Cf. John (1960)) the estimate (74) shows that it depends continuously upon the initial given data on compact sub-intervals of $(0, T/2)$, in the measure $\mathcal{E}(t)$ as defined in (68).

Remark 1 (*Uniqueness*)**.** It is worth noting that when the initial data are zero, from inequality (74) it follows that $\mathcal{E}(t) = 0$ and, based on the assumptions described in (72), it follows that $u_i(x,t) = 0$, i.e. the uniqueness of the solution to the initial boundary value problem $\mathcal{P}$.

Remark 2. Our analysis in Section 4 is restricted to initial boundary value problems with prescribed kinematic boundary conditions. The derivation and mechanical interpretation of the corresponding higher-order natural boundary conditions constitute a challenging but distinct problem, not addressed in this work. Moreover, we have to note that other types of boundary conditions (different from ours in (52)) could be formulated by means of the transformations involving the divergence theorem applied to our differential system: however, their physical interpretation is not straightforward. The uniqueness question can be established under such boundary conditions, but the continuous dependence does not seem to be achievable.

5. Final comments

We have presented a mechanical model that combines the features of the nonlocal Klein–Gordon theory with those of second-order gradients for elastic displacement and with the multi-scale and micro-inertia theory. Strictly speaking, the motivation behind this investigation is not to improve the underlying earlier sub-models, but rather the possibility of investigating within a single

framework the interaction among spatial nonlocality, gradient elasticity, higher-order micro-inertia and temporal nonlocality. The propagation of waves in the considered mechanical system leads to a rather complex scenario, which highlights the relationship between the two characteristic internal length scales, ℓ_N and ℓ_G, and the time scale τ. Our analysis in Section 3 provides a picture of the optical and acoustic branches, as well as of normal and anomalous dispersion behavior with respect to the wavenumber. An objective limitation of the proposed model lies in the fact that the optic phase and group velocities become unbounded in the formal high-wavenumber limit. This feature indicates that the same model should not be extended *tout court* to arbitrarily short wavelengths, but rather motivates the necessity for it – in this sense – of further in-depth study.

Under constraints on the characteristic internal length scale parameters and on characteristic time scale parameter, as well as on the elastic characteristics, constraints in any case compatible with those deduced investigating the harmonic wave propagation, it is shown that the presented model can be well-posed.

Our paper presents the significant dispersion relation (19), along with an analysis thereof in Section 3. However, to evaluate the range of applicability of the model described by the system of differential equations (12), it is necessary to experimentally identify the characteristic scale and material parameters involved. A combination of numerical and experimental methods for determining these parameters, together with the results from Section 3, could confirm the model's applicability.

Incorporating higher-order effects into the mechanical approach can lead to an even more complex model, by introducing a contribution that is proportional to the sixth-order spatial derivative of the displacement in the potential energy density expression and to a sixth-order spatial derivative in the kinetic energy density. This would result in a sixth-order spatial derivative of the displacements in the right-hand side of the corresponding equations of motion as shown below

$$\begin{aligned}&\rho\ddot{u}_i - \alpha\rho\ell_N^2\ddot{u}_{i,jj} + \beta\rho\ell_N^4\ddot{u}_{i,jjnn} + \sigma\rho\ell_N^6\ddot{u}_{i,jjnnmm} + \gamma\rho\tau^2\dddot{u}_i\\ &= C_{ijkl}\left(u_{k,lj} - \delta\ell_G^2 u_{k,ljnn} + \eta\ell_G^4 u_{k,ljnnmm}\right)\\ &+ \left(1 - \alpha\ell_N^2\Delta + \beta\ell_N^4\Delta^2 + \sigma\ell_N^6\Delta^3 + \gamma\tau^2\frac{\partial^2}{\partial t^2}\right)\left(\rho f_i\right), \quad (x,t)\in B\times(0,T).\end{aligned} \tag{75}$$

This would severely complicate the formulation in terms of additional boundary conditions and initial conditions, but its mathematical treatment is enough similar to that of the present analysis.

As already pointed out by Lazar and Agiasofitou (2022), in nonlocal elasticity, one distinguishes between strong (integral operators) nonlocal elasticity and weak (differential operators) nonlocal elasticity (see, also, Eringen, 1992; Lazar et al., 2020). These two types of nonlocal models (integral and differential) are not equivalent, as it was established by Kaplunov et al. (2023). We must emphasize that the models described in our paper fall into the weak nonlocal elasticity type.

Finally, we note that a comprehensive investigation of generalized models involves the following steps: (a) constitutive modeling; (b) investigation of physical behavior (e.g., wave propagation); (c) mathematical analysis of the associated initial boundary value problems; (d) analytical, numerical, and engineering applications. In our paper we have followed the first three steps, while the fourth remains open for future research.

CRediT authorship contribution statement

Stan Chiriţă: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Vittorio Zampoli:** Writing – review & editing, Supervision, Software, Resources, Methodology, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix 1

Recalling the definitions (21), from Eq. (22) we have

$$f(\zeta) = \mathfrak{a}\zeta^4 + \mathfrak{b}\zeta^3 + \mathfrak{c}\zeta^2 + \mathfrak{d}\zeta + \mathfrak{e}, \qquad \xi = L, T, \tag{76}$$

with

$$\begin{aligned}&\mathfrak{a} = 1,\\ &\mathfrak{b} = 2,\\ &\mathfrak{c} = 3 - 4C_\xi^2 r^2,\\ &\mathfrak{d} = 2\left(1 - 2C_\xi^2\right),\\ &\mathfrak{e} = 1.\end{aligned} \tag{77}$$

We aim to intercept restrictions *excluding* real zeros for $f(\zeta)$, even if this represents, of course, only a sufficient condition. Since $f(0) = 1 > 0$, a quartic polynomial (i.e., continuous) without real zeros will satisfy the condition $f(\zeta) > 0$ for every $\zeta \in \mathbb{R}$ (for us, $\zeta \in \mathbb{R}^+$).

If we define, for a quartic function, the discriminant

$$\begin{aligned}\Delta_\xi = {} & 256\mathfrak{a}^3\mathfrak{e}^3 - 192\mathfrak{a}^2\mathfrak{b}\mathfrak{d}\mathfrak{e}^2 - 128\mathfrak{a}^2\mathfrak{c}^2\mathfrak{e}^2 + 144\mathfrak{a}^2\mathfrak{c}\mathfrak{d}^2\mathfrak{e} - 27\mathfrak{a}^2\mathfrak{d}^4 \\ & + 144\mathfrak{a}\mathfrak{b}^2\mathfrak{c}\mathfrak{e}^2 - 6\mathfrak{a}\mathfrak{b}^2\mathfrak{d}^2\mathfrak{e} - 80\mathfrak{a}\mathfrak{b}\mathfrak{c}^2\mathfrak{d}\mathfrak{e} + 18\mathfrak{a}\mathfrak{b}\mathfrak{c}\mathfrak{d}^3 + 16\mathfrak{a}\mathfrak{c}^4\mathfrak{e} - 4\mathfrak{a}\mathfrak{c}^3\mathfrak{d}^2 \\ & - 27\mathfrak{b}^4\mathfrak{e}^2 + 18\mathfrak{b}^3\mathfrak{c}\mathfrak{d}\mathfrak{e} - 4\mathfrak{b}^3\mathfrak{d}^3 - 4\mathfrak{b}^2\mathfrak{c}^3\mathfrak{e} + \mathfrak{b}^2\mathfrak{c}^2\mathfrak{d}^2\end{aligned} \tag{78}$$

and also

$$P_\xi = 8\mathfrak{a}\mathfrak{c} - 3\mathfrak{b}^2, \qquad D_\xi = 64\mathfrak{a}^3\mathfrak{e} - 16\mathfrak{a}^2\mathfrak{c}^2 + 16\mathfrak{a}\mathfrak{b}^2\mathfrak{c} - 16\mathfrak{a}^2\mathfrak{b}\mathfrak{d} - 3\mathfrak{b}^4, \tag{79}$$

the coexistence of conditions $P_\xi > 0$ and $\Delta_\xi > 0$ (or even $D_\xi > 0$ and $\Delta_\xi > 0$) ensures that two pairs of non-real complex conjugate roots exist.

As for $P_\xi = 8\left(3 - 4C_\xi^2 r^2\right) - 12 > 0$, we have

$$P_\xi > 0 \quad \Longleftrightarrow \quad 0 < s_\xi^P < 1, \tag{80}$$

where $s_\xi^P = 8C_\xi^2 r^2/3$, while for $D_\xi = 128C_\xi^2\left(1 + r^2 - 2C_\xi^2 r^4\right) > 0$, we have

$$D_\xi > 0 \quad \Longleftrightarrow \quad 0 < s_\xi^D < 1, \tag{81}$$

where $s_\xi^D = 2C_\xi^2 r^4/(1 + r^2)$. The above restrictions can also be recast as

$$P_\xi > 0 \quad \Longleftrightarrow \quad C_\xi^2 < \frac{3}{8r^2} \qquad \text{and} \qquad D_\xi > 0 \quad \Longleftrightarrow \quad C_\xi^2 < \frac{1 + r^2}{2r^4}, \tag{82}$$

comparing which we realize that condition $D_\xi > 0$ is less restrictive than $P_\xi > 0$; nevertheless, $\Delta_\xi > 0$ and $D_\xi > 0$ ensure the absence of real zeros for $f(\zeta)$. For this reason, in our search for a sufficient condition ensuring $f(\zeta) > 0$, we will leave aside the constraint $P_\xi > 0$ and focus only on $\Delta_\xi > 0$ under $D_\xi > 0$.

With the support of software Wolfram Mathematica, we propose the following alternative form for the discriminant Δ_ξ:

$$\Delta_\xi(r, s_\xi^D) = \frac{16}{r^{16}}\left(1 + r^2\right)^2\left(s_\xi^D\right)^2 \Psi_\xi(r, s_\xi^D), \tag{83}$$

where

$$\begin{aligned}\Psi_\xi(r, s_\xi^D) = {} & 36r^8\left(1 - r^2 + r^4\right) + 2r^4(1 + r^2)\left(35 - 69r^2 + 72r^4 - 40r^6\right)s_\xi^D \\ & + (1 + r^2)^2\left(-27 + 36r^2 - 32r^4 - 16r^6 + 16r^8\right)(s_\xi^D)^2 + 8r^2(1 + r^2)^3(s_\xi^D)^3.\end{aligned} \tag{84}$$

Under the hypothesis $D_\xi > 0$, we realize that

$$\Delta_\xi > 0 \quad \Longleftrightarrow \quad \Psi_\xi(r, s_\xi^D) > 0, \tag{85}$$

and therefore, in order to investigate the sign of Δ_ξ, it will be sufficient to study the sign of the function $\Psi_\xi(r, s_\xi^D)$ in the region $\mathcal{D} = \left\{(r, s_\xi^D) \in \mathbb{R}^2 : r > 0,\ 0 < s_\xi^D < 1\right\}$. We do this through the *Plot3D* tool made available by Wolfram Mathematica (see Fig. 1).

Recalling that we are evaluating only a sufficient condition for $f(\zeta) > 0$, we can record that, by direct substitution, we get $\Psi_\xi(1, s_\xi^D) = 4(s_\xi^D - 1)^2(16s_\xi^D + 9)$, meaning that $\Psi_\xi(1, s_\xi^D) > 0$ for every possible choice of s_ξ^D in $(0, 1)$; in particular, $\Psi_\xi(1, 1) = 0$.

Appendix 2

The quantities M_{0p} and N_{Tp}, $p = 1, 2, \ldots, 16$, are defined as

$$\begin{aligned}
M_{01} &= \left(\int_B \rho u_i(0)u_i(0)dv\right)^{1/2}, \quad N_{T1} = \int_0^T \left(\int_B \rho \dot{u}_i(s)\dot{u}_i(s)dv\right)^{1/2} ds, \\
M_{02} &= \left(\int_B \rho \dot{u}_i(0)\dot{u}_i(0)dv\right)^{1/2}, \quad N_{T2} = \int_0^T \left(\int_B \rho u_i(s)u_i(s)dv\right)^{1/2} ds, \\
M_{03} &= \left(\int_B \alpha\rho\ell_N^2 u_{i,j}(0)u_{i,j}(0)dv\right)^{1/2}, \quad N_{T3} = \int_0^T \left(\int_B \alpha\rho\ell_N^2 \dot{u}_{i,j}(s)\dot{u}_{i,j}(s)dv\right)^{1/2} ds, \\
M_{04} &= \left(\int_B \alpha\rho\ell_N^2 \dot{u}_{i,j}(0)\dot{u}_{i,j}(0)dv\right)^{1/2}, \quad N_{T4} = \int_0^T \left(\int_B \alpha\rho\ell_N^2 u_{i,j}(s)u_{i,j}(s)dv\right)^{1/2} ds,
\end{aligned}$$

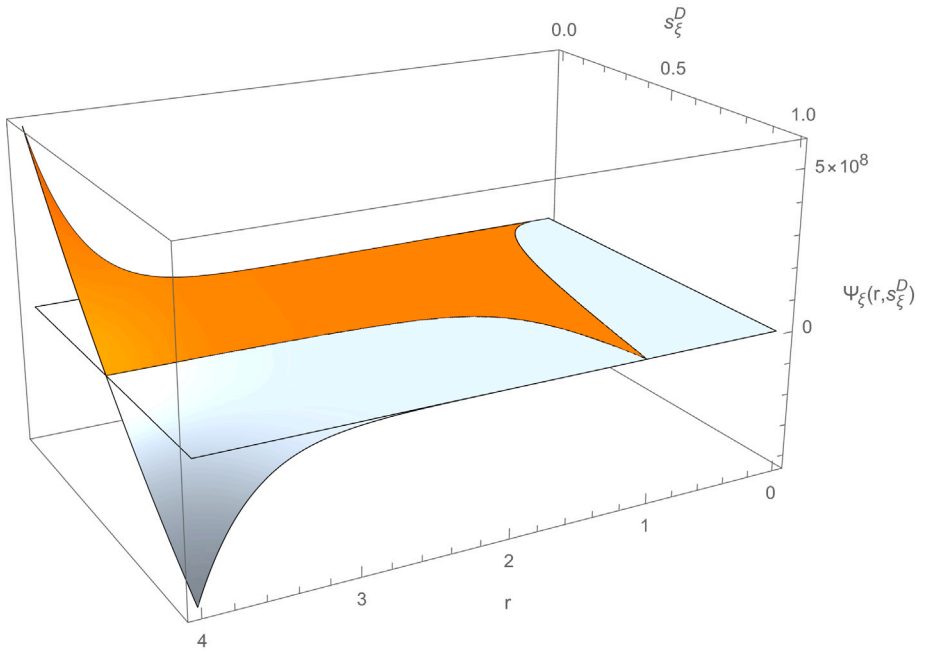

Fig. 1. Three-dimensional representation of $\Psi_\xi(r, s_\xi^D)$ for $0 < r < 4$ and $0 < s_\xi^D < 1$. The orange surface above the zero plane identifies the pairs (r, s_ξ^D) for which $\Psi_\xi > 0$, and consequently $\Delta_\xi > 0$..

$$
\begin{aligned}
M_{05} &= \left(\int_B \beta\varrho\ell_N^4 u_{i,jn}(0)u_{i,jn}(0)dv\right)^{1/2}, \quad N_{T5} = \int_0^T \left(\int_B \beta\varrho\ell_N^4 \dot{u}_{i,jn}(s)\dot{u}_{i,jn}(s)dv\right)^{1/2} ds,\\
M_{06} &= \left(\int_B \beta\varrho\ell_N^4 \dot{u}_{i,jn}(0)\dot{u}_{i,jn}(0)dv\right)^{1/2}, \quad N_{T6} = \int_0^T \left(\int_B \beta\varrho\ell_N^4 u_{i,jn}(s)u_{i,jn}(s)dv\right)^{1/2} ds,\\
M_{07} &= \left(\int_B \gamma\varrho\tau^2 u_i(0)u_i(0)dv\right)^{1/2}, \quad N_{T7} = \int_0^T \left(\int_B \gamma\varrho\tau^2 \ddot{u}_i(s)\ddot{u}_i(s)dv\right)^{1/2} ds,\\
M_{08} &= \left(\int_B \gamma\varrho\tau^2 \ddot{u}_i(0)\ddot{u}_i(0)dv\right)^{1/2}, \quad N_{T8} = \int_0^T \left(\int_B \gamma\varrho\tau^2 u_i(s)u_i(s)dv\right)^{1/2} ds,\\
M_{09} &= \left(\int_B \gamma\varrho\tau^2 \dot{u}_i(0)\dot{u}_i(0)dv\right)^{1/2}, \quad N_{T9} = \int_0^T \left(\int_B \gamma\varrho\tau^2 \ddot{u}_i(s)\ddot{u}_i(s)dv\right)^{1/2} ds,\\
M_{010} &= \left(\int_B \gamma\varrho\tau^2 \ddot{u}_i(0)\ddot{u}_i(0)dv\right)^{1/2}, \quad N_{T10} = \int_0^T \left(\int_B \gamma\varrho\tau^2 \dot{u}_i(s)\dot{u}_i(s)dv\right)^{1/2} ds,\\
M_{011} &= \left(\int_B \varrho\gamma^2\tau^4 \ddot{u}_i(0)\ddot{u}_i(0)dv\right)^{1/2}, \quad N_{T11} = \int_0^T \left(\int_B \varrho\gamma^2\tau^4 \ddot{u}_i(s)\ddot{u}_i(s)dv\right)^{1/2} ds,\\
M_{012} &= \left(\int_B \varrho\gamma^2\tau^4 \ddot{u}_i(0)\ddot{u}_i(0)dv\right)^{1/2}, \quad N_{T12} = \int_0^T \left(\int_B \varrho\gamma^2\tau^4 \ddot{u}_i(s)\ddot{u}_i(s)dv\right)^{1/2} ds,
\end{aligned}
\tag{86}
$$

$$
\begin{aligned}
M_{013} &= \gamma\tau^2 C_{max}^2 \left(\int_B \varrho u_{i,j}(0)u_{i,j}(0)dv\right)^{1/2}, \quad N_{T13} = \int_0^T \left(\int_B \varrho\dot{u}_{k,l}(s)\dot{u}_{k,l}(s)dv\right)^{1/2} ds,\\
M_{014} &= \gamma\tau^2 C_{max}^2 \left(\int_B \varrho\dot{u}_{i,j}(0)\dot{u}_{i,j}(0)dv\right)^{1/2}, \quad N_{T14} = \int_0^T \left(\int_B \varrho u_{i,j}(s)u_{i,j}(s)dv\right)^{1/2} ds,
\end{aligned}
\tag{87}
$$

$$
\begin{aligned}
M_{015} &= \gamma\tau^2\delta\ell_G^2 C_{max}^2 \left(\int_B \varrho u_{i,jn}(0)u_{i,jn}(0)dv\right)^{1/2}, \quad N_{T15} = \int_0^T \left(\int_B \varrho\dot{u}_{k,ln}(s)\dot{u}_{k,ln}(s)dv\right)^{1/2} ds,\\
M_{016} &= \gamma\tau^2\delta\ell_G^2 C_{max}^2 \left(\int_B \varrho\dot{u}_{i,jn}(0)\dot{u}_{i,jn}(0)dv\right)^{1/2}, \quad N_{T16} = \int_0^T \left(\int_B \varrho u_{i,jm}(s)u_{i,jm}(s)dv\right)^{1/2} ds.
\end{aligned}
\tag{88}
$$

Data availability

No data was used for the research described in the article.

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