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Vector variational principles for set-valued functions

Christiane TAMMER* and Constantin ZĂLINESCU†

Abstract

Ekeland's variational principle (EVP) has many equivalent formulations and generalizations. In this paper we present new minimal point theorems in product spaces and the corresponding vector variational principles for set-valued functions. As special cases we derive many of the existing variational principles of Ekeland's type. Moreover, we use our new approach to get extensions of EVPs of Isac–Tammer and Ha types, as well as extensions of EVPs for bi-functions.

Key words. Ekeland's variational principle, minimal points in product spaces, multifunction, bi-multifunction, cs-complete set

Mathematics subject classification. 49J53, 90C48, 65K10

1 Introduction

The celebrated Ekeland variational principle (see [9]) has many equivalent formulations and generalizations. Quite rapidly after the publication of the Ekeland variational principle (EVP) in 1974 there were formulated extensions to functions $f : (X, d) \to Y$, where Y is a real (topological) vector space. A systematization of such results was done in [11] (see also [10]), where instead of a function f it was considered a subset of $X \times Y$; said differently, it was considered a multifunction from X to Y. The common feature of these results is the presence of a certain term $d(x, x')k^0$ in the perturbed objective function, where K is the convex ordering cone in Y and $k^0 \in K \setminus \{0\}$. Very recently $d(x, x')k^0$ was replaced by $d(x, x')H$ with H a bounded convex subset of K (see [5]) or by $F(x, x') \subset K$, F being a so called K-metric (see [12]); in both papers one deals with functions $f : X \to Y$.

In this paper we present new results with proofs very similar to the corresponding ones in [11], which have as particular cases most part of the existing EVPs, or they are very close to them. Moreover, we use the same approach to get extensions of EVPs of Isac–Tammer and Ha types, as well as extensions of EVPs for bi-functions.

In the sequel (X, d) is a complete metric space, Y is a real topological vector space, Y^* is its topological dual, and $K \subset Y$ is a proper convex cone; as usual,

$$K^+ = \{y^* \in Y^* \mid y^*(y) \geq 0 \; \forall \, y \in K\}$$

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is the dual cone of K and

$$K^{\#} = \{y^* \in Y^* \mid y^*(y) > 0 \; \forall y \in K \setminus \{0\}\}$$

is the quasi interior of K^+.

If Y is just a real linear space we endow it with the finest locally convex topology, that is, the topology defined by all the seminorms on Y.

As in [2] and [3], we say that $E \subset Y$ is *quasi bounded* (from below) if there exists a bounded set $B \subset Y$ such that $E \subset B + K$; as in [12], we say that E is K-*bounded* (by scalarization) if $y^*(E)$ is bounded from below for every $y^* \in K^+$. It is clear that any quasi bounded set is K-bounded.

Let $F : X \times X \rightrightarrows K$ satisfy the conditions:

(F1) $0 \in F(x, x)$ for all $x \in X$,

(F2) $F(x_1, x_2) + F(x_2, x_3) \subset F(x_1, x_3) + K$ for all $x_1, x_2, x_3 \in X$.

Using F we introduce a preorder on $X \times Y$, denoted by $\preceq_F$, in the following manner:

$$(x_1, y_1) \preceq_F (x_2, y_2) \iff y_2 \in y_1 + F(x_1, x_2) + K. \tag{1}$$

Indeed, $\preceq_F$ is reflexive by (F1). If $(x_1, y_1) \preceq_F (x_2, y_2)$ and $(x_2, y_2) \preceq_F (x_3, y_3)$, then

$$y_2 = y_1 + v_1 + k_1, \quad y_3 = y_2 + v_2 + k_2 \tag{2}$$

with $v_1 \in F(x_1, x_2)$, $v_2 \in F(x_2, x_3)$ and $k_1, k_2 \in K$. By (F2) we have that $v_1 + v_2 = v_3 + k_3$ for some $v_3 \in F(x_1, x_3)$ and $k_3 \in K$, and so

$$y_3 = y_1 + v_1 + k_1 + v_2 + k_2 = y_1 + v_3 + k_1 + k_2 + k_3 \in y_1 + F(x_1, x_3) + K;$$

hence $(x_1, y_1) \preceq_F (x_3, y_3)$, and so $\preceq_F$ is transitive. Of course,

$$(x_1, y_1) \preceq_F (x_2, y_2) \Rightarrow y_1 \leq_K y_2; \tag{3}$$

moreover, by (F1), we have that

$$(x, y_1) \preceq_F (x, y_2) \iff y_2 \in y_1 + K \iff y_1 \leq_K y_2. \tag{4}$$

Besides conditions (F1) and (F2) we shall assume to be true the condition

(F3) there exists $z^* \in K^+$ such that

$$\eta(\delta) := \inf \left\{ z^*(v) \mid v \in \cup_{d(x,x') \geq \delta} F(x, x') \right\} > 0 \quad \forall \delta > 0. \tag{5}$$

Clearly, by (F3) we have that $0 \notin \text{cl conv} F(x, x')$ for $x \neq x'$.

A sufficient condition for (5) is

$$\inf_{z \in F(x,x')} z^*(z) \geq d(x, x') \quad \forall x, x' \in X. \tag{6}$$

If (6) holds then

$$(x_1, y_1) \preceq_F (x_2, y_2) \Rightarrow d(x_1, x_2) \leq z^*(y_2) - z^*(y_1). \tag{7}$$

Indeed, since $F(x_1, x_2) \subset K$, from (3) we get first that $y_1 \leq_K y_2$; then from (2)

$$z^*(y_2) = z^*(y_1) + z^*(v_1) + z^*(k_1) \geq z^*(y_1) + \inf_{v \in F(x_1,x_2)} z^*(v) \geq z^*(y_1) + d(x_1, x_2),$$

and so (7) holds.

Using (7) we obtain that

$$[(x_1, y_1) \preceq_F (x_2, y_2),\ (x_2, y_2) \preceq_F (x_1, y_1)] \Rightarrow [x_1 = x_2,\ z^*(y_1) = z^*(y_2)]\,; \tag{8}$$

moreover, if $z^* \in K^\#$ then $\preceq_F$ is anti-symmetric, and so $\preceq_F$ is a partial order.

For F satisfying conditions (F1)–(F3), z^* being that from (F3), we introduce the order relation $\preceq_{F,z^*}$ on $X \times Y$ by

$$(x_1, y_1) \preceq_{F,z^*} (x_2, y_2) \iff \begin{cases} (x_1, y_1) = (x_2, y_2) \text{ or} \\ (x_1, y_1) \preceq_F (x_2, y_2) \text{ and } z^*(y_1) < z^*(y_2). \end{cases}$$

It is easy to verify that $\preceq_{F,z^*}$ is reflexive, transitive, and antisymmetric.

2 Minimal points in product spaces

We take X, Y, K, F as above, that is, F satisfies conditions (F1)–(F3), z^* being that from (F3).

Consider a nonempty set $A \subset X \times Y$. In the sequel we shall use the following condition on A:

(H1) for every $\preceq_F$–decreasing sequence $((x_n, y_n)) \subset A$ with $x_n \to x \in X$ there exists $y \in Y$ such that $(x, y) \in A$ and $(x, y) \preceq_F (x_n, y_n)$ for every $n \in \mathbb{N}$.

The next theorem is the main result of this section.

Theorem 1 *Assume that (X, d) is a complete metric space, Y is a real topological vector space and $K \subset Y$ is a proper convex cone. Let $F : X \times X \rightrightarrows Y$ satisfy conditions (F1)–(F3) and $A \subset X \times Y$ satisfy (H1). Furthermore, suppose that z^* (from (F3)) is bounded from below on $\operatorname{Pr}_Y(A)$. Then for every $(x_0, y_0) \in A$ there exists a minimal element $(\bar{x}, \bar{y})$ of A with respect to $\preceq_{F,z^*}$ such that $(\bar{x}, \bar{y}) \preceq_{F,z^*} (x_0, y_0)$.*

Proof. Let

$$\alpha := \inf \{z^*(y) \mid \exists x \in X : (x, y) \in A,\ (x, y) \preceq_{F,z^*} (x_0, y_0)\} \in \mathbb{R}.$$

Let us denote by $A(x, y)$ the set of those $(x', y') \in A$ with $(x', y') \preceq_{F,z^*} (x, y)$. We construct a sequence $((x_n, y_n))_{n \geq 0} \subset A$ as follows: Having $(x_n, y_n) \in A$, we take $(x_{n+1}, y_{n+1}) \in A(x_n, y_n)$ such that

$$z^*(y_{n+1}) \leq \inf\{z^*(y) \mid (x, y) \in A(x_n, y_n)\} + 1/(n + 1).$$

Of course, $((x_n, y_n))$ is $\preceq_{F,z^*}$–decreasing. It follows that $(y_n)_{n \geq 0}$ is $\leq_K$–decreasing, and so the sequence $(z^*(y_n))_{n \geq 0}$ is nonincreasing and bounded from below; hence $\gamma := \lim z^*(y_n) \in \mathbb{R}$.

If $A(x_{n_0}, y_{n_0})$ is a singleton (that is, $\{(x_{n_0}, y_{n_0})\}$) for some $n_0 \in \mathbb{N}$, then clearly $(\overline{x}, \overline{y}) := (x_{n_0}, y_{n_0})$ is the desired element. In the contrary case the sequence $(z^*(y_n))$ is (strictly) decreasing; moreover, $\gamma < z^*(y_n)$ for every $n \in \mathbb{N}$.

Assume that (x_n) is not a Cauchy sequence. Then there exist $\delta > 0$ and the sequences (n_k), (p_k) from $\mathbb{N}^*$ such that $n_k \to \infty$ and $d(x_{n_k}, x_{n_k+p_k}) \geq \delta$ for every k. Since $(x_{n_k+p_k}, y_{n_k+p_k}) \preceq_{F,z^*} (x_{n_k}, y_{n_k})$ we obtain that

$$z^*(y_{n_k}) - z^*(y_{n_k+p_k}) \geq \inf \{z^*(v) \mid v \in F(x_{n_k+p_k}, x_{n_k})\} \geq \eta(\delta) \quad \forall k \in \mathbb{N}.$$

Since $\eta(\delta) > 0$ and $(z^*(y_n))$ is convergent, this is a contradiction. Therefore, (x_n) is a Cauchy sequence in the complete metric space (X, d), and so (x_n) converges to some $\bar{x} \in X$. Since $((x_n, y_n))$ is $\preceq_F$–decreasing, by (H1) there exists some $\bar{y} \in Y$ such that $(\bar{x}, \bar{y}) \in A$ and $(\bar{x}, \bar{y}) \preceq_F (x_n, y_n)$ for every $n \in \mathbb{N}$. It follows that $z^*(\bar{y}) \leq \lim z^*(y_n)$, and so $z^*(\bar{y}) < z^*(y_n)$ for every $n \in \mathbb{N}$. Therefore $(\bar{x}, \bar{y}) \preceq_{F,z^*} (x_n, y_n)$ for every $n \in \mathbb{N}$. Let $(x', y') \in A$ be such that $(x', y') \preceq_{F,z^*} (\bar{x}, \bar{y})$. Since $(\bar{x}, \bar{y}) \preceq_{F,z^*} (x_n, y_n)$, we have that $(x', y') \preceq_{F,z^*} (x_n, y_n)$ for every $n \in \mathbb{N}$. It follows that

$$0 \leq z^*(\bar{y}) - z^*(y') \leq z^*(y_n) - z^*(y') \leq 1/n \quad \forall n \geq 1,$$

whence $z^*(y') = z^*(\bar{y})$. By the definition of $\preceq_{F,z^*}$ we obtain that $(x', y') = (\bar{x}, \bar{y})$. □

As seen from the proof, for a fixed $(x_0, y_0) \in A$ it is sufficient that z^* be bounded from below on the set $\{y \in Y \mid \exists x \in X : (x, y) \in A, \ (x, y) \preceq_{F,z^*} (x_0, y_0)\}$ instead of being bounded from below on $\mathrm{Pr}_Y(A)$.

Remark 1 When $k^0 \in K$ and $F(x, x') := \{d(x, x')k^0\}$ we have that F satisfies conditions (F1) and (F2); moreover, if Y is a separated locally convex space and $-k^0 \notin \mathrm{cl}\, K$, then there exists $z^* \in K^+$ with $z^*(k^0) = 1$, and so (F3) is also satisfied (even (6) is satisfied). In this case condition (H1) becomes condition (H1) in [10, p. 199]. So Theorem 1 extends [10, Th. 3.10.7] to this framework, using practically the same proof.

In [12] one considers for a proper pointed convex cone $D \subset Y$ a so called set-valued D-metric, that is, a multifunction $F : X \times X \rightrightarrows D$ satisfying the following conditions:

(a) $F(x, y) \neq \emptyset$ and $F(x, x) = \{0\}$ $\forall x, y \in X$, and $0 \notin F(x, y)$ $\forall x \neq y$,

(b) $F(x, y) = F(y, x)$ $\forall x, y \in X$,

(c) $F(x, y) + F(y, z) \subset F(x, z) + D$ $\forall x, y, z \in X$.

The basic supplementary assumptions on D and F are:

(A1) D is w-normal and D_F is based,

(A2) $0 \notin \mathrm{cl}_w \left(\cup_{d(x,y)\geq\delta} F(x, y)\right)$ $\forall \delta > 0$.

Here $K_F := \mathrm{cone}(\mathrm{conv}\,(\cup\{F(x, y) \mid x, y \in X\}))$ and $D_F := (K_F \setminus \{0\} + D) \cup \{0\}$.

As observed in [12], D is w-normal iff $D^+ - D^+ = Y^*$, and D_F is based iff $D^+ \cap K_F^\# \neq \emptyset$.

Comparing with our assumptions on F, we see that (F1) is weaker than (a) because we ask just $0 \in F(x, x)$ for every $x \in X$, and we don't ask the symmetry condition (b). From (F3) we obtain that (A2) is verified and that $z^* \in K_F^\#$ and so $(K_F \setminus \{0\} + K) \cup \{0\}$ is based, but we don't need either K be w-normal or even K be pointed.

Another possible choice for F, considered also in [12], is $F(x,x') := d(x,x')H$ with $H \subset K \setminus \{0\}$ a nonempty set such that $H+K$ is convex. Clearly (F1), (a), and (b) are satisfied (for $D = K$). From the convexity of $H+K$ we obtain easily that (F2) (and (c)) holds. When Y is a separated locally convex space condition (F3) is equivalent to $0 \notin \mathrm{cl}(H+K)$. In order to have that (A1) holds one needs $K^+ - K^+ = Y^*$ and the existence of $z^* \in K^+$ with $z^*(v) > 0$ for every $v \in H$, while for (A2) one needs $0 \notin \mathrm{cl}_w H$ (see [12, Lem. 5.9 (d)]); of course, if $H = H+K$, the last condition is equivalent to $0 \notin \mathrm{cl}(H+K)$. So it seems that our condition (F3) is more convenient than (A1) and (A2).

For H as above, that is, $H \subset K$ is a nonempty set such that $H+K$ is convex and $0 \notin \mathrm{cl}(H+K)$, we consider $F_H(x,x') := d(x,x')H$ for $x,x' \in X$, and we set $\preceq_H := \preceq_{F_H}$; moreover, if $z^* \in K^+$ is such that $\inf z^*(H) > 0$ we set $\preceq_{H,z^*} := \preceq_{F_H,z^*}$.

An immediate consequence of the preceding theorem is the next result.

Corollary 2 *Assume that (X,d) is a complete metric space, Y is a real topological vector space and $K \subset Y$ is a proper convex cone. Let $A \subset X \times Y$ satisfy (H1) with respect to $\preceq_H$ and suppose that there exists $z^* \in K^+$ such that $\inf z^*(H) > 0$ and $\inf z^*(\mathrm{Pr}_Y(A)) > -\infty$. Then for every $(x_0,y_0) \in A$ there exists $(\bar{x},\bar{y}) \in A$ a minimal element of A with respect to $\preceq_{H,z^*}$ such that $(\bar{x},\bar{y}) \preceq_{H,z^*} (x_0,y_0)$.*

A condition related to (H1) is the next one.

(H2) for every sequence $\big((x_n,y_n)\big) \subset A$ with $x_n \to x \in X$ and (y_n) $\leq_K$–decreasing there exists $y \in Y$ such that $(x,y) \in A$ and $y \leq_K y_n$ for every $n \in \mathbb{N}$.

Remark 2 Note that (H2) holds if A is closed with $\mathrm{Pr}_Y(A) \subset y_0 + K$ for some $y_0 \in Y$ and every $\leq_K$–decreasing sequence in K is convergent (i.e., K is a sequentially Daniell cone). In fact, instead of asking that A is closed we may assume that

$$\forall \left((x_n,y_n)\right)_{n\geq 1} \subset A : [x_n \to x,\ y_n \to y,\ (y_n)\ \leq_K \text{–decreasing } \Rightarrow (x,y) \in A].$$

Remark 3 Moreover, note that (H1) is verified whenever A satisfies (H2) and

$$\forall u \in X,\ \forall X \supset (x_n) \to x \in X\ :\ \cap_{n\in\mathbb{N}} \left(F(x_n,u) + K\right) \subset F(x,u) + K.$$

Indeed, let $\big((x_n,y_n)\big) \subset A$ be $\preceq_F$–decreasing with $x_n \to x$. It is obvious that (y_n) is $\leq_K$–decreasing. By (H2), there exists $y \in Y$ such that $(x,y) \in A$ and $y \leq_K y_n$ for every $n \in \mathbb{N}$. It follows that

$$y_n \in y_{n+p} + F(x_{n+p},x_n) + K \subset y + F(x_{n+p},x_n) + K \quad \forall n,p \in \mathbb{N}.$$

Fix n; then $y_n - y \in F(x_{n+p},x_n) + K$ for every $p \in \mathbb{N}$, and so, by our hypothesis, $y_n - y \in F(x,x_n) + K$ because $\lim_{p\to\infty} x_{n+p} = x$. Therefore, $(x,y) \preceq_F (x_n,y_n)$.

Remark 4 In the case $F = F_H$, (H1) is verified whenever A satisfies (H2) and $H+K$ is closed.

Indeed, let $\big((x_n,y_n)\big) \subset A$ be a $\preceq_H$–decreasing sequence with $x_n \to x$. It is obvious that (y_n) is $\leq_K$–decreasing. By (H2), there exists $y \in Y$ such that $(x,y) \in A$ and $y \leq_K y_n$ for every $n \in \mathbb{N}$.

Fix n. If $x_n = x$ then clearly $(x,y) = (x_n, y) \preceq_H (x_n, y_n)$. Else, because $d(x_{n+p}, x_n) \to d(x, x_n) > 0$ for $p \to \infty$, we get $d(x_{n+p}, x_n) > 0$ for sufficiently large p, and so

$$y_n \in y_{n+p} + d(x_{n+p}, x_n)H \subset y + d(x_{n+p}, x_n)H + K = y + d(x_{n+p}, x_n)(H + K)$$

for sufficiently large p. Since $H + K$ is closed we obtain that

$$y_n \in y + d(x_n, x)(H + K) = y + d(x_n, x)H + K,$$

that is, $(x,y) \preceq_H (x_n, y_n)$.

Another condition to be added to (H2) in order to have (H1) is suggested by the hypotheses of [5, Th. 4.1]. Recall that a set $C \subset Y$ is cs-complete (see [19, p. 9]) if for all sequences $(\lambda_n)_{n\geq 1} \subset [0,\infty)$ and $(y_n)_{n\geq 1} \subset C$ such that $\sum_{n\geq 1} \lambda_n = 1$ and the sequence $\left(\sum_{m=1}^{n} \lambda_m y_m\right)_{n\geq 1}$ is Cauchy, the series $\sum_{n\geq 1} \lambda_n y_n$ is convergent and its sum belongs to C. One says that $C \subset Y$ is cs-closed if the sum of the series $\sum_{n\geq 1} \lambda_n y_n$ belongs to C whenever $\sum_{n\geq 1} \lambda_n y_n$ is convergent and $(y_n) \subset C$, $(\lambda_n)_{n\geq 1} \subset [0,\infty)$ and $\sum_{n\geq 1} \lambda_n = 1$. Of course, any cs-complete set is cs-closed; if Y is complete then the converse is true. Moreover, notice that any cs-closed set is convex.

Note that the sequence $\left(\sum_{m=1}^{n} \lambda_m y_m\right)_{n\geq 1}$ is Cauchy whenever $(\lambda_n)_{n\geq 1} \subset [0,\infty)$ is such that the series $\sum_{n\geq 1} \lambda_n$ is convergent and $(y_n)_{n\geq 1} \subset Y$ is such that $\operatorname{conv}\{y_n \mid n \geq 1\}$ is bounded; of course, if Y is a locally convex space then $B \subset Y$ is bounded iff $\operatorname{conv} B$ is bounded. Indeed, let $(\lambda_n)_{n\geq 1} \subset [0,\infty)$ with $\sum_{n\geq 1} \lambda_n$ convergent and $(y_n)_{n\geq 1} \subset Y$ with $B := \operatorname{conv}\{y_n \mid n \geq 1\}$ bounded. Fix $V \subset Y$ a balanced neighborhood of 0. Because B is bounded, there exists $\alpha > 0$ such that $B \subset \alpha V$. Since the series $\sum_{n\geq 1} \lambda_n$ is convergent there exists $n_0 \geq 1$ such that $\sum_{k=n}^{n+p} \lambda_k \leq \alpha^{-1}$ for all $n, p \in \mathbb{N}$ with $n \geq n_0$. Then for such n, p and some $b_{n,p} \in B$ we have

$$\sum_{k=n}^{n+p} \lambda_k y_k = \left(\sum_{k=n}^{n+p} \lambda_k\right) b_{n,p} \in [0, \alpha^{-1}]B \subset [0, \alpha^{-1}]\alpha V = V.$$

Proposition 3 *Assume that (X,d) is a complete metric space, Y is a real topological vector space and $K \subset Y$ is a proper closed convex cone. Furthermore, suppose that $H \subset K$ is a nonempty cs-complete bounded set with $0 \notin \operatorname{cl}(H+K)$. If A satisfies (H2) then A satisfies (H1), too.*

Proof. Let $\left((x_n, y_n)\right)_{n\geq 1} \subset A$ be a $\preceq_H$–decreasing sequence with $x_n \to x$. It follows that (y_n) is $\leq_K$–decreasing. By (H2), there exists $y \in Y$ such that $(x,y) \in A$ and $y \leq_K y_n$ for every $n \in \mathbb{N}$.

Because $\left((x_n, y_n)\right)_{n\geq 1}$ is $\preceq_H$–decreasing we have that

$$y_n = y_{n+1} + d(x_n, x_{n+1})h_n + k_n \tag{9}$$

with $h_n \in H$ and $k_n \in K$ for $n \geq 1$. If $x_n = x_{\overline{n}}$ for $n \geq \overline{n} \geq 1$ we take $x := x_{\overline{n}}$; then $(x,y) \preceq_H (x_n, y_n)$ for every $n \in \mathbb{N}$. Indeed, for $n \leq \overline{n}$ we have that $(x_{\overline{n}}, y_{\overline{n}}) \preceq_H (x_n, y_n)$; because $y \leq_K y_{\overline{n}}$, by (4) we get $(x,y) \preceq_H (x, y_{\overline{n}}) = (x_{\overline{n}}, y_{\overline{n}})$, and so $(x,y) \preceq (x_n, y_n)$. If $n > \overline{n}$, using again (4), we have $(x,y) = (x_n, y) \preceq_H (x_n, y_n)$.

Assume that (x_n) is not constant for large n. Fix $n \geq 1$. From (9), for $p \geq 0$, we have

$$y_n = y_{n+p+1} + \sum_{l=n}^{n+p} d(x_l, x_{l+1})h_l + \sum_{l=n}^{n+p} k_l = y_{n+p+1} + \left(\sum_{l=n}^{n+p} d(x_l, x_{l+1})\right) h_{n,p} + \sum_{l=n}^{n+p} k_l$$
$$= y + k'_{n,p} + \left(\sum_{l=n}^{n+p} d(x_l, x_{l+1})\right) h_{n,p} \qquad (10)$$

for some $h_{n,p} \in H$ and $k'_{n,p} \in K$. Assuming that $\sum_{l\geq n} d(x_l, x_{l+1}) = \infty$, from

$$\left(\sum_{l=n}^{n+p} d(x_l, x_{l+1})\right)^{-1} (y_n - y) = h_{n,p} + \left(\sum_{l=n}^{n+p} d(x_l, x_{l+1})\right)^{-1} k'_{n,p} \in H + K,$$

we get the contradiction $0 \in \operatorname{cl}(H + K)$ taking the limit for $p \to \infty$. Hence $0 < \mu := \sum_{l\geq n} d(x_l, x_{l+1}) < \infty$. Set $\lambda_l := \mu^{-1} d(x_l, x_{l+1})$ for $l \geq n$. Since H is cs-complete and $\operatorname{conv}\{h_l \mid l \geq n\}$ $(\subset H)$ is bounded we obtain that the series $\sum_{l\geq n} \lambda_l h_l$ is convergent and its sum $\overline{h}_n$ belongs to H. It follows that $\sum_{l\geq n} d(x_l, x_{l+1})h_l = \mu \overline{h}_n$, and so

$$\overline{k}_n := \lim_{p\to\infty} k'_p = y_n - y - \mu \overline{h}_n \in K$$

because K is closed. Since $d(x_n, x_{n+p}) \leq \sum_{l=n}^{n+p-1} d(x_l, x_{l+1})$, we obtain that $d(x_n, x) \leq \mu$, and so

$$y_n = y + d(x_n, x)\overline{h}_n + \overline{k}_n + (\mu - d(x_n, x))\, \overline{h}_n \in y + d(x_n, x)H + K.$$

Hence $(x, y) \preceq_H (x_n, y_n)$ for every $n \in \mathbb{N}$. □

The most part of vector EVP type results are established for Y a separated locally convex space. However, there are topological vector spaces Y whose topological dual reduce to $\{0\}$. In such a case it is not possible to find z^* satisfying the conditions of Corollary 2. In [2, Th. 1], in the case H is a singleton, the authors consider such a situation.

Theorem 4 *Assume that (X, d) is a complete metric space, Y is a real topological vector space. Let $K \subset Y$ be a proper closed convex cone and $H \subset K$ be a nonempty cs-complete bounded set with $0 \notin \operatorname{cl}(H + K)$. Suppose that $A \subset X \times Y$ satisfies (H2) and that $\operatorname{Pr}_Y(A)$ is quasi bounded. Then for every $(x_0, y_0) \in A$ there exists $(\bar{x}, \bar{y}) \in A$ such that $(\bar{x}, \bar{y}) \preceq_H (x_0, y_0)$ and $(x, y) \in A$, $(x, y) \preceq_H (\bar{x}, \bar{y})$ imply $x = \bar{x}$.*

Proof. First observe that A satisfies condition (H1) by Proposition 3. Moreover, because $\operatorname{Pr}_Y(A)$ is quasi bounded, there exists a bounded set $B \subset Y$ such that $\operatorname{Pr}_Y(A) \subset B + K$.

Note that for $(x, y) \in A$ the set $\operatorname{Pr}_X(A(x, y))$ is bounded, where $A(x, y) := \{(x', y') \in A \mid (x', y') \preceq_H (x, y)\}$. In the contrary case there exists a sequence $((x_n, y_n))_{n\geq 1} \subset A(x, y)$ with $d(x_n, x) \to \infty$. Hence $y = y_n + d(x_n, x)h_n + k_n = b_n + d(x_n, x)h_n + k'_n$ with $h_n \in H$, $b_n \in B$, $k_n, k'_n \in K$. It follows that $d(x_n, x)^{-1}(y - b_n) \in H + K$, whence the contradiction $0 \in \operatorname{cl}(H + K)$.

Let us construct a sequence $((x_n, y_n))_{n\geq 0} \subset A$ in the following way: Having $(x_n, y_n) \in A$, where $n \in \mathbb{N}$, because $D_n := \operatorname{Pr}_X(A(x_n, y_n))$ is bounded, there exists $(x_{n+1}, y_{n+1}) \in A(x_n, y_n)$ such that

$$d(x_{n+1}, x_n) \geq \tfrac{1}{2} \sup\{d(x, x_n) \mid x \in D_n\} \geq \tfrac{1}{4} \operatorname{diam} D_n.$$

We obtain in this way the sequence $((x_n, y_n))_{n\geq 0} \subset A$, which is $\preceq_H$–decreasing. Since $A(x_{n+1}, y_{n+1}) \subset A(x_n, y_n)$, we have that $D_{n+1} \subset D_n$ for every $n \in \mathbb{N}$. Of course, $x_n \in D_n$. Let us show that $\operatorname{diam} D_n \to 0$. In the contrary case there exists $\delta > 0$ such that $\operatorname{diam} D_n \geq 4\delta$, and so $d(x_{n+1}, x_n) \geq \delta$ for every $n \in \mathbb{N}$. As in the proof of Proposition 3, for every $p \in \mathbb{N}$, we obtain that

$$\begin{aligned} y_0 &= y_{p+1} + \left(\sum_{l=0}^{p} d(x_l, x_{l+1})\right) h_p + \sum_{l=0}^{p} k_l = b_p + \left(\sum_{l=0}^{p} d(x_l, x_{l+1})\right) h_p + k_p' \\ &= b_p + (p+1)\delta h_p + k_p'', \end{aligned}$$

where $h_p \in H$, $b_p \in B$, $k_l, k_p', k_p'' \in K$. It follows that $[(p+1)\delta]^{-1}(y_0 - b_p) \in H + K$ for every $p \in \mathbb{N}$. Since (b_p) is bounded we obtain the contradiction $0 \in \operatorname{cl}(H+K)$. Thus we have that the sequence $(\operatorname{cl} D_n)$ is a decreasing sequence of nonempty closed subsets of the complete metric space (X, d), whose diameters tend to 0. By Cantor's theorem, $\bigcap_{n\in\mathbb{N}} \operatorname{cl} D_n = \{\bar{x}\}$ for some $\bar{x} \in X$. Of course, $x_n \to \bar{x}$. Since $((x_n, y_n)) \subset A$ is a $\preceq_H$–decreasing sequence, from (H1) we get an $\bar{y} \in Y$ such that $(\bar{x}, \bar{y}) \preceq_H (x_n, y_n)$ for every $n \in \mathbb{N}$; $(\bar{x}, \bar{y})$ is the desired element. Indeed, $(\bar{x}, \bar{y}) \preceq_H (x_0, y_0)$. Let $(x', y') \in A(\bar{x}, \bar{y})$. It follows that $(x', y') \in A(x_n, y_n)$, and so $x' \in D_n \subset \operatorname{cl} D_n$ for every n. Thus $x' = \bar{x}$. □

If Y is a separated locally convex space, the preceding result follows immediately from Corollary 2.

Of course, the set $A \subset X \times Y$ can be viewed as the graph of a multifunction $\Gamma : X \rightrightarrows Y$; then $\operatorname{Pr}_X(A) = \operatorname{dom} \Gamma$ and $\operatorname{Pr}_Y(A) = \operatorname{Im} \Gamma$. In [2] one assumes that Γ is *level-closed*, that is,

$$\begin{aligned} L(b) :=& \{x \in X \mid \exists y \in \Gamma(x) : y \leq_K b\} = \{x \in X \mid b \in \Gamma(x) + K\} \\ =& \{x \in X \mid \Gamma(x) \cap (b - K) \neq \emptyset\} \end{aligned}$$

is closed for every $b \in Y$.

For the nonempty set $E \subset Y$ let us set

$$\operatorname{BMMin} E := \{\overline{y} \in E \mid E \cap (\overline{y} - K) = \{\overline{y}\}\}$$

(see [3, (1.2)]); note that this set is different of the usual set

$$\operatorname{Min} E := \{\overline{y} \in E \mid E \cap (\overline{y} - K) \subset \overline{y} + K\},$$

but they coincide if K is pointed, that is, $K \cap (-K) = \{0\}$. As in [3, Def. 3.2], we say that $\Gamma : X \rightrightarrows Y$ satisfies the *limiting monotonicity condition* at $\overline{x} \in \operatorname{dom} \Gamma$ if for every sequence $((x_n, y_n))_{n\geq 1} \subset \operatorname{gph} \Gamma$ with $x_n \to \overline{x}$ and (y_n) $\leq_K$–decreasing, there exists $\overline{y} \in \operatorname{BMMin} \Gamma(\overline{x})$ such that $\overline{y} \leq y_n$ for every $n \geq 1$. As observed in [3], if Γ satisfies the limiting monotonicity condition at $\overline{x} \in \operatorname{dom} \Gamma$ then $\Gamma(\overline{x}) \subset \operatorname{BMMin} \Gamma(\overline{x}) + K$, that is, $\Gamma(\overline{x})$ satisfies the domination property.

In [3, Prop. 3.3], in the case Y a Banach space, there are mentioned sufficient conditions in order that Γ satisfy the limiting monotonicity condition at $\overline{x} \in \operatorname{dom} \Gamma$.

When X and Y are Banach spaces and H is a singleton the next result is practically [3, Th. 3.5].

Corollary 5 *Assume that (X, d) is a complete metric space, Y is a real topological vector space. Let $K \subset Y$ be a proper closed convex cone and $H \subset K$ be a nonempty cs-complete* *bounded set with* $0 \notin \operatorname{cl}(H+K)$. *Suppose that* $\Gamma : X \rightrightarrows Y$ *is level-closed, satisfies the limiting monotonicity condition on* $\operatorname{dom}\Gamma$ *and* $\operatorname{Im}\Gamma$ *is quasi-bounded. Then for every* $(x_0, y_0) \in \operatorname{gph}\Gamma$ *there exist* $\bar{x} \in \operatorname{dom}\Gamma$ *and* $\bar{y} \in \operatorname{BMMin}\Gamma(\bar{x})$ *such that* $(\bar{x}, \bar{y}) \preceq_H (x_0, y_0)$ *and* $(x, y) \in \operatorname{gph}\Gamma$, $(x, y) \preceq_H (\bar{x}, \bar{y})$ *imply* $x = \bar{x}$.

Proof. In order to apply Theorem 4 for $A := \operatorname{gph}\Gamma$ we have only to show that A verifies condition (H2). For this consider the sequence $((x_n, y_n))_{n \geq 1} \subset A$ such that (y_n) is $\leq_K$–decreasing and $x_n \to \overline{x}$. Clearly, $x_n \in L(y_1)$ for every n; since Γ is level-closed, we have that $\overline{x} \in L(y_1) \subset \operatorname{dom}\Gamma$. Since Γ satisfies the limiting monotonicity condition at $\overline{x}$, we find $\overline{y} \in \operatorname{BMMin}\Gamma(\overline{x}) \subset \Gamma(\overline{x})$ such that $\overline{y} \leq y_n$ for every n. Hence (H2) holds. By Theorem 4 there exists $(x, y) \in A$ such that $(x, y) \preceq_H (x_0, y_0)$ and $(x', y') \in \operatorname{gph}\Gamma$, $(x', y') \preceq_H (x, y)$ imply $x' = x$. Set $\bar{x} := x$ and take $\bar{y} \in \operatorname{BMMin}\Gamma(\bar{x})$ such that $\bar{y} \leq_K y$. By (4) we have that $(\bar{x}, \bar{y}) \preceq_H (x_0, y_0)$. Let now $(x', y') \in \operatorname{gph}\Gamma = A$ with $(x', y') \preceq_H (\bar{x}, \bar{y})$. Since $(\bar{x}, \bar{y}) = (x, \bar{y}) \preceq_H (x, y)$, we have that $(x', y') \preceq_H (x, y)$, and so $x' = x = \bar{x}$. The proof is complete. □

In the case when H is a singleton the next result is practically [2, Th. 1] under the supplementary hypothesis that $\operatorname{Min}\Gamma(x)$ is compact for every $x \in X$; it seems that this condition has to be added in order that [2, Th. 1] be true.

Corollary 6 *Assume that* (X, d) *is a complete metric space,* Y *is a real topological vector space. Let* $K \subset Y$ *be a proper closed convex cone and* $H \subset K$ *be a nonempty cs-complete bounded set with* $0 \notin \operatorname{cl}(H+K)$. *Suppose that* $\Gamma : X \rightrightarrows Y$ *is level-closed,* $\operatorname{Min}\Gamma(x)$ *is compact and* $\Gamma(x) \subset K + \operatorname{Min}\Gamma(x)$ *for every* $x \in \operatorname{dom}\Gamma$, *and* $\operatorname{Im}\Gamma$ *is quasi-bounded. Then for every* $(x_0, y_0) \in \operatorname{gph}\Gamma$ *there exist* $\bar{x} \in \operatorname{dom}\Gamma$ *and* $\bar{y} \in \operatorname{Min}\Gamma(\bar{x})$ *such that* $(\bar{x}, \bar{y}) \preceq_H (x_0, y_0)$ *and* $(x, y) \in \operatorname{gph}\Gamma$, $(x, y) \preceq_H (\bar{x}, \bar{y})$ *imply* $x = \bar{x}$.

Proof. In order to apply Theorem 4 for $A := \operatorname{gph}\Gamma$ we have only to show that A verifies condition (H2). For this consider the sequence $((x_n, y_n))_{n \geq 1} \subset A$ such that (y_n) is $\leq_K$–decreasing and $x_n \to \overline{x}$. As in the proof of the preceding corollary, $\overline{x} \in L(y_n)$ for every $n \in \mathbb{N}$. Because $\Gamma(\overline{x}) \subset K + \operatorname{Min}\Gamma(\overline{x})$, for every $n \in \mathbb{N}$ there exists $y'_n \in \operatorname{Min}\Gamma(\overline{x})$ such that $y'_n \leq y_n$. Because $\operatorname{Min}\Gamma(\overline{x})$ is compact, (y'_n) has a subnet $(y'_{\psi(i)})_{i \in I}$ converging to some $\overline{y} \in \operatorname{Min}\Gamma(\overline{x})$; here $\psi : (I, \succeq) \to \mathbb{N}$ is such that for every n there exists $i_n \in I$ with $\psi(i) \geq n$ for $i \succeq i_n$. Hence $y'_{\psi(i)} \leq y_{\psi(i)} \leq y_n$ for $i \succeq i_n$, whence $\overline{y} \leq y_n$ because K is closed. Therefore, (H2) holds. By Theorem 4, for $(x_0, y_0) \in \operatorname{gph}\Gamma$, there exists $(x, y) \in A$ such that $(x, y) \preceq_H (x_0, y_0)$ and $(x', y') \in \operatorname{gph}\Gamma$, $(x', y') \preceq_H (x, y)$ imply $x' = x$. Set $\bar{x} := x$ and take $\bar{y} \in \operatorname{Min}\Gamma(\bar{x})$ such that $\bar{y} \leq_K y$. As in the proof of Corollary 5 we find that $(\bar{x}, \bar{y})$ is the desired element. The proof is complete. □

3 Ekeland's variational principles of Isac–Tammer's type

Besides $F : X \times X \rightrightarrows K$ considered in the preceding section we consider also $F' : Y \times Y \rightrightarrows K$ satisfying conditions (F1) and F(2), that is, $0 \in F'(y, y)$ for all $y \in Y$ and $F'(y_1, y_2) + F(y_2, y_3) \subset F(y_1, y_3) + K$ for all $y_1, y_2, y_3 \in Y$. Then $\Phi : Z \times Z \rightrightarrows K$ with $Z := X \times Y$, defined by $\Phi((x_1, y_1), (x_2, y_2)) := F(x_1, x_2) + F'(y_1, y_2)$, satisfies conditions (F1) and (F2), too. As in Section 1 we obtain that the relation $\preccurlyeq_{F,F'}$ defined by

$$(x_1, y_1) \preccurlyeq_{F,F'} (x_2, y_2) \Longleftrightarrow y_2 \in y_1 + F(x_1, x_2) + F'(y_1 y_2) + K$$

is reflexive and transitive. Moreover, for $x, x_1, x_2 \in X$ and $y_1, y_2 \in Y$ we have

$$(x_1, y_1) \preccurlyeq_{F,F'} (x_2, y_2) \Longrightarrow (x_1, y_1) \preccurlyeq_F (x_2, y_2) \Longrightarrow y_1 \leq_K y_2,$$
$$(x, y_1) \preccurlyeq_{F,F'} (x, y_2) \Longleftrightarrow y_1 \leq_K y_2.$$

As in the preceding section, for F satisfying (F1)–(F3), F' satisfying (F1), (F2) and z^* from (F3) we define the partial order $\preceq_{F,F',z^*}$ by

$$(x_1, y_1) \preceq_{F,F',z^*} (x_2, y_2) \iff \begin{cases} (x_1, y_1) = (x_2, y_2) \text{ or} \\ (x_1, y_1) \preceq_{F,F'} (x_2, y_2) \text{ and } z^*(y_1) < z^*(y_2). \end{cases}$$

Theorem 7 *Assume that (X, d) is a complete metric space, Y is a real topological vector space and $K \subset Y$ is a proper convex cone. Let $F : X \times X \rightrightarrows Y$ satisfy conditions (F1)–(F3), let $F' : Y \times Y \rightrightarrows K$ satisfy (F1) and (F2), and let $A \subset X \times Y$ satisfy the condition*

(H1b) *for every $\preccurlyeq_{F,F'}$–decreasing sequence $((x_n, y_n)) \subset A$ with $x_n \to x \in X$ there exists $y \in Y$ such that $(x, y) \in A$ and $(x, y) \preccurlyeq_{F,F'} (x_n, y_n)$ for every $n \in \mathbb{N}$.*

Suppose that z^* (from (F3)) is bounded from below on $\Pr_Y(A)$. Then for every $(x_0, y_0) \in A$ there exists a minimal element $(\bar{x}, \bar{y})$ of A with respect to $\preceq_{F,F',z^*}$ such that $(\bar{x}, \bar{y}) \preceq_{F,F',z^*} (x_0, y_0)$.

Proof. It is easy to verify that $\preceq_{F,F',z^*}$ is reflexive, transitive and antisymmetric. To get the conclusion one follows the lines of the proof of Theorem 1. □

Clearly, taking $F' = 0$ in Theorem 7 we get Theorem 1. As mentioned after the proof of Theorem 1, this extends significantly [10, Th. 3.10.7], keeping practically the same proof. We ask ourselves if [10, Th. 3.10.15] could be extended to this framework, taking into account that the boundedness condition on A in [10, Th. 3.10.15] is much less restrictive. In [10, Th. 3.10.15] we used a functional φ_{C,k^0} (defined in (12) below) in order to prove the minimal point theorem. Because an element k^0 does not impose itself naturally, and we need a stronger condition on the functional φ_{C,k^0} even if $k^0 \in K \subset \operatorname{int} C$, we consider an abstract K–monotone functional φ to which we impose some conditions φ_{C,k^0} has already.

Theorem 8 *Assume that (X, d) is a complete metric space, Y is a real topological vector space and $K \subset Y$ is a proper convex cone. Let $F : X \times X \rightrightarrows Y$ satisfy conditions (F1)–(F3), let $F' : Y \times Y \rightrightarrows K$ satisfy (F1) and (F2), and let $A \subset X \times Y$ satisfy the condition (H1b). Assume that there exists a functional $\varphi : Y \to \overline{\mathbb{R}}$ such that*

(F4) $\quad (x_1, y_1) \preceq_{F,F'} (x_2, y_2) \Longrightarrow \varphi(y_1) + d(x_1, x_2) \leq \varphi(y_2).$

If φ is bounded below on $\Pr_Y(A)$ then for every point $(x_0, y_0) \in A$ with $\varphi(y_0) \in \mathbb{R}$, there exists $(\bar{x}, \bar{y}) \in A$ such that $(\bar{x}, \bar{y}) \preceq_{F,F'} (x_0, y_0)$, and $(x', y') \in A$, $(x', y') \preceq_{F,F'} (\bar{x}, \bar{y})$ imply $x' = \bar{x}$. Moreover, if φ is strictly K–monotone on $\Pr_Y(A)$, that is, $y_1, y_2 \in \Pr_Y(A)$, $0 \neq y_2 - y_1 \in K \Longrightarrow \varphi(y_1) < \varphi(y_2)$, then $(\bar{x}, \bar{y})$ is a minimal point of A with respect to $\preceq_{F,F'}$.

Proof. First note that from (F4) we have that φ is K–monotone, that is, $y_1 \leq_K y_2 \Longrightarrow \varphi(y_1) \leq \varphi(y_2)$. Let us construct a sequence $((x_n, y_n))_{n \geq 0} \subset A$ as follows: Having $(x_n, y_n) \in A$, we take $(x_{n+1}, y_{n+1}) \in A$, $(x_{n+1}, y_{n+1}) \preceq_{F,F'} (x_n, y_n)$, such that

$$\varphi(y_{n+1}) \leq \inf\{\varphi(y) \mid (x, y) \in A,\ (x, y) \preceq_{F,F'} (x_n, y_n)\} + 1/(n + 1). \tag{11}$$

Of course, the sequence $((x_n, y_n))$ is $\preceq_{F,F'}$–decreasing, and so (y_n) $(\subset \mathrm{Pr}_Y(A))$ is K–decreasing. It follows that the sequence $(\varphi(y_n))$ is non-increasing and bounded from below, hence convergent in $\mathbb{R}$. Because $(x_{n+p}, y_{n+p}) \preceq_{F,F'} (x_n, y_n) \preceq_{F,F'} (x_{n-1}, y_{n-1})$, using (F4) and (11) we get

$$d(x_{n+p}, x_n) \le \varphi(y_n) - \varphi(y_{n+p}) \le 1/n \quad \forall\, n, p \in \mathbb{N}^*.$$

It follows that (x_n) is a Cauchy sequence in the complete metric space (X, d), and so (x_n) is convergent to some $\bar{x} \in X$.

By (H1b) there exists $\overline{y} \in Y$ such that $(\overline{x}, \overline{y}) \in A$ and $(\bar{x}, \bar{y}) \preceq_{F,F'} (x_n, y_n)$ for every $n \in \mathbb{N}$. Let us show that $(\overline{x}, \overline{y})$ is the desired element. Indeed, $(\bar{x}, \bar{y}) \preceq_{F,F'} (x_0, y_0)$. Suppose that $(x', y') \in A$ is such that $(x', y') \preceq_{F,F'} (\bar{x}, \bar{y})$ $(\preceq_{F,F'} (x_n, y_n)$ for every $n \in \mathbb{N})$. Thus $\varphi(y') + d(x', \overline{x}) \le \varphi(\overline{y})$ by (F4), whence

$$d(x', \bar{x}) \le \varphi(\bar{y}) - \varphi(y') \le \varphi(y_n) - \varphi(y') \le 1/n \quad \forall\, n \ge 1.$$

It follows that $d(x', \bar{x}) = \varphi(\bar{y}) - \varphi(y') = 0$. Hence $x' = \overline{x}$.

Assuming that φ is strictly K–monotone, because $y' \le_K \overline{y}$ and $\varphi(\bar{y}) - \varphi(y') = 0$, we have necessarily $y' = \overline{y}$. Hence $(\bar{x}, \bar{y})$ is a minimal point with respect to $\preceq_{F,F'}$. □

Note that if $C \subset Y$ is a proper convex cone such that $K \setminus \{0\} \subset \operatorname{int} C$ and $k^0 \in K \setminus \{0\}$, the functional $\varphi_{C,k^0} : Y \to \mathbb{R}$ defined by

$$\varphi_{C,k^0}(y) := \inf \left\{ t \in \mathbb{R} \mid y \in tk^0 - C \right\} \tag{12}$$

is a strictly K–monotone continuous sublinear functional (see [10, Th. 2.3.1]). Moreover, if the condition

(B) $\quad \mathrm{Pr}_Y(A) \cap (\tilde{y} - \operatorname{int} C) = \emptyset$ for some $\tilde{y} \in Y$

holds, then $\varphi := \varphi_{C,k^0}$ is bounded from below on $\mathrm{Pr}_Y(A)$. Indeed, by [10, Th. 2.3.1 (f)], we have that $\varphi(y) + \varphi(-\widetilde{y}) \ge \varphi(y - \widetilde{y}) \ge 0$ for $y \in \mathrm{Pr}_Y(A)$, whence $\varphi(y) \ge -\varphi(-\widetilde{y})$ for $y \in \mathrm{Pr}_Y(A)$.

Another example for a function φ is that defined by

$$\varphi(y) := \varphi_{K,k^0}(y - \widehat{y}), \tag{13}$$

where K is a proper convex cone, $k^0 \in K \setminus \{0\}$, and $\widehat{y} \in Y$ is such that

(B1) $\quad y_0 - \widehat{y} \in \mathbb{R}k^0 - K, \quad \mathrm{Pr}_Y(A) \cap (\widehat{y} - K) = \emptyset.$

Then φ is K–monotone, $\varphi(y_0) < \infty$ and $\varphi(y) \ge 0$ for every $y \in \mathrm{Pr}_Y(A)$.

For both of these functions in (12) and (13) we have to impose condition (F4) in order to be used in Theorem 8.

Remark 5 Using the function $\varphi = \varphi_{K,k^0}(\cdot - \widehat{y})$ (defined by (13)) in Theorem 8 we can derive [14, Th. 4.2] taking $F(x_1, x_2) := \{d(x_1, x_2)k^0\}$ and $F'(y_1, y_2) := \{\varepsilon \left\| y_1 - y_2 \right\| k^0\}$ when Y is a Banach space; note that, at its turn, [14, Th. 4.2] extends [16, Th. 8].

4 Ekeland's variational principles of Ha's type

The previous EVP type results correspond to Pareto optimality. Ha [13] established an EVP type result which corresponds to Kuroiwa optimality. The next result is an extension of this type of result. For its proof we use [18, Th. 3.1] or [14, Th. 2.2].

Theorem 9 *Assume that (X, d) is a complete metric space, Y is a real topological vector space and $K \subset Y$ is a proper convex cone. Let $F : X \times X \rightrightarrows Y$ satisfy conditions (F1)–(F3) and $\Gamma : X \rightrightarrows Y$ be such that z^* (from (F3)) is bounded below on $\Gamma(X)$. If $\{x \in X \mid \Gamma(u) \subset \Gamma(x) + F(x, u) + K\}$ is closed for every $u \in X$, then for every $x_0 \in \operatorname{dom}\Gamma$ there exists $\overline{x} \in X$ such that $\Gamma(x_0) \subset \Gamma(\overline{x}) + F(\overline{x}, x_0) + K$, and $\Gamma(\overline{x}) \subset \Gamma(x) + F(x, \overline{x}) + K$ implies $x = \overline{x}$.*

Proof. Let us consider the relation $\preccurlyeq$ on X defined by $x' \preccurlyeq x$ if $\Gamma(x) \subset \Gamma(x') + F(x', x) + K$. By our hypotheses we have that $S(x) := \{x' \in X \mid x' \preccurlyeq x\}$ is closed for every $x \in X$. Note that for $x \in X \setminus \operatorname{dom}\Gamma$ we have that $S(x) = X$, while for $x \in \operatorname{dom}\Gamma$ we have that $S(x) \subset \operatorname{dom}\Gamma$. The relation $\preccurlyeq$ is reflexive and transitive. The reflexivity of $\preccurlyeq$ is obvious. Let $x' \preccurlyeq x$ and $x'' \preccurlyeq x'$. Then $\Gamma(x) \subset \Gamma(x') + F(x', x) + K$ and $\Gamma(x') \subset \Gamma(x'') + F(x'', x') + K$. Using (F2) we get

$$\Gamma(x) \subset \Gamma(x'') + F(x'', x') + K + F(x', x) + K \subset \Gamma(x'') + F(x'', x) + K,$$

that is, $x'' \preccurlyeq x$. Consider

$$\varphi : X \to \overline{\mathbb{R}}, \quad \varphi(x) := \inf z^*(\Gamma(x)),$$

with the usual convention $\inf \emptyset := +\infty$. Clearly, $\varphi(x) \geq m := \inf z^*(\Gamma(X)) > -\infty$. Moreover, if $x' \preccurlyeq x \in \operatorname{dom}\Gamma$ then $z^*(\Gamma(x)) \subset z^*(\Gamma(x')) + z^*(F(x', x)) + z^*(K)$, whence $\varphi(x) \geq \varphi(x') + \inf z^*(F(x', x)) \geq \varphi(x')$.

Fix $x_0 \in \operatorname{dom}\Gamma$. The conclusion of the theorem asserts that there exists $\overline{x} \in X$ such that $\overline{x} \in S(x_0)$ and $S(\overline{x}) = \{\overline{x}\}$. To get this conclusion we apply [14, Th. 2.2] or [18, Th. 3.1]. Because (X, d) is complete and $S(x)$ is closed for every $x \in X$, we may (and we do) assume that $\operatorname{dom}\Gamma = X$ (otherwise we replace X by $S(x_0)$). In order to apply [14, Th. 2.2] we have to show that $d(x_n, x_{n+1}) \to 0$ provided $(x_n)_{n\geq 1} \subset X$ is $\preccurlyeq$–decreasing. In the contrary case there exist $\delta > 0$ and $(n_p)_{p\geq 1} \subset \mathbb{N}^*$ an increasing sequence such that $d(x_{n_p}, x_{n_{p+1}}) \geq \delta$ for every $p \geq 1$. Then, as seen above, $\varphi(x_n) \geq \varphi(x_{n+1}) + \inf z^*(F(x_{n+1}, x_n))$, and so

$$\varphi(x_{n_1}) \geq \varphi\left(x_{n_{p+1}}\right) + \sum_{l=n_1}^{n_p} \inf z^*(F(x_{l+1}, x_l)) \geq m + p \cdot \eta(\delta)$$

with $\eta(\delta) > 0$ from (F3). Letting $p \to \infty$ we get a contradiction. Hence $d(x_n, x_{n+1}) \to 0$. The conclusion follows. □

Note that instead of assuming $S(u)$ to be closed for every $u \in X$ it is sufficient to have that $S(u)$ is $\preccurlyeq$–lower closed, that is, for every $\preccurlyeq$–decreasing sequence $(x_n) \subset S(u)$ with $x_n \to x$ we have that $x \in S(u)$. Moreover, instead of using [14, Th. 2.2] it is possible to give a slightly longer direct proof similar to that of Theorem 1 (and using φ instead of z^* in the construction of (x_n)).

Remark 6 Taking Y to be a separated locally convex space, $K \subset Y$ a pointed closed convex cone and $F(x, x') := \{d(x, x')k^0\}$ with $k^0 \in K \setminus \{0\}$, we can deduce [13, Th. 3.1]. For this assume that $\Gamma(X)$ is quasi bounded, $\Gamma(x)+K$ is closed for every $x\in X$ and Γ is level-closed (or K-l.s.c. with the terminology from [13]). Since clearly z^* is bounded from below on $\operatorname{Im}\Gamma$, in order to apply the preceding theorem we need to have that $S(u)$ is closed for every $u\in X$; this is done in [13, Lem. 3.2]. Below we provide another proof for the closedness of $S(u)$.

First, if $x\notin L(b)$ then there exists $\delta>0$ such that $B(x,\delta)\cap L(b+\delta k^0)=\emptyset$. Indeed, because $x\notin L(b)$ we have that $b\notin\Gamma(x)+K$, and so $b+\delta' k^0\notin\Gamma(x)+K$, that is, $x\notin L(b+\delta' k^0)$, for some $\delta'>0$ (since $\Gamma(x)+K$ is closed). Because $L(b+\delta' k^0)$ is closed, there exists $\delta\in(0,\delta']$ such that $B(x,\delta)\cap L(b+\delta' k^0)=\emptyset$, and so $B(x,\delta)\cap L(b+\delta k^0)=\emptyset$.

Fix $u\in X$ and take $x\in X\setminus S(u)$, that is, $\Gamma(u)\not\subset\Gamma(x)+d(x,u)k^0+K$. Then there exists $y\in\Gamma(u)$ with $b:=y-d(x,u)k^0\notin\Gamma(x)+K$. By the argument above there exists $\delta'>0$ such that $B(x,\delta')\cap L(b+\delta' k^0)=\emptyset$, that is, $y-d(x,u)k^0+\delta' k^0\notin\Gamma(x')+K$ for every $x'\in B(x,\delta')$. Taking $\delta\in(0,\delta']$ sufficiently small we have that $d(x',u)\geq d(x,u)-\delta'$ for $x'\in B(x,\delta)$, and so $y\notin\Gamma(x')+d(x',u)k^0+K$ for every $x'\in B(x,\delta)$, that is, $B(x,\delta)\cap S(u)=\emptyset$.

If we assume that $\Gamma(x_0)\not\subset\Gamma(x)+k^0+K$ for every $x\in X$, then $\overline{x}$ provided by the preceding theorem satisfies $d(\overline{x},x_0)<1$. Indeed, in the contary case, because $\Gamma(x_0)\subset\Gamma(\overline{x})+d(\overline{x},x_0)k^0+K$ and $d(\overline{x},x_0)k^0+K\subset k^0+K$, we get the contradiction $\Gamma(x_0)\subset\Gamma(\overline{x})+k^0+K$. Replacing k^0 by εk^0 and d by $\lambda^{-1}d$ for some $\varepsilon,\lambda>0$ we obtain exactly the statement of [13, Th. 3.1].

In the case in which Y is just a topological vector space we have the following version of the preceding theorem under conditions similar to those in Theorem 4.

Theorem 10 *Assume that (X,d) is a complete metric space, Y is a real topological vector space and $K\subset Y$ is a proper closed convex cone. Let $H\subset K$ be nonempty cs-complete bounded set with $0\notin\operatorname{cl}(H+K)$, and $\Gamma:X\rightrightarrows Y$. If $\{x\in X\mid\Gamma(u)\subset\Gamma(x)+d(x,u)H+K\}$ is closed for every $u\in X$ and $\Gamma(X)$ is quasi bounded, then for every $x_0\in\operatorname{dom}\Gamma$ there exists $\overline{x}\in X$ such that $\Gamma(x_0)\subset\Gamma(\overline{x})+d(\overline{x},x_0)H+K$ and $\Gamma(\overline{x})\subset\Gamma(x)+d(x,\overline{x})H+K$ implies $x=\overline{x}$.*

Proof. Let $B\subset Y$ be a bounded set such that $\Gamma(X)\subset B+K$.

Consider $F(x,x'):=d(x,x')H$ for $x,x'\in X$. As seen before, F satisfies conditions (F1) and (F2), and so the relation $\preccurlyeq$ defined in the proof of Theorem 9 is reflexive and transitive; moreover, by our hypotheses, $S(x):=\{x'\in X\mid x'\preccurlyeq x\}$ is closed for every $x\in X$. As in the proof of Theorem 9 we may (and do) assume that $X=\operatorname{dom}\Gamma$ and it is sufficient to show that $d(x_n,x_{n+1})\to 0$ provided $(x_n)_{n\geq1}\subset X$ is $\preccurlyeq$–decreasing. In the contrary case there exist $\delta>0$ and $(n_p)_{p\geq1}\subset\mathbb{N}^*$ an increasing sequence such that $d(x_{n_p},x_{n_{p+1}})\geq\delta$ for every $p\geq1$.

Fixing $y_1\in\Gamma(x_1)$, inductively we find the sequences $(y_n)_{n\geq0}\subset Y$, $(h_n)_{n\geq0}\subset H$ and $(k_n)_{n\geq0}\subset K$ such that $y_n=y_{n+1}+d(x_n,x_{n+1})h_n+k_n$ for every $n\geq1$. Using the convexity of H, and the facts that $H\subset K$ and $\Gamma(X)\subset B+K$, for $p\in\mathbb{N}$ we get $h'_p\in H$, $b_p\in B$ and $k'_p,k''_p\in K$ such that

$$y_1=y_{n_{p+1}}+\sum_{l=1}^{n_p}d(x_l,x_{l+1})h_l+\sum_{l=0}^{p}k_l=b_p+\delta(h_{n_1}+\ldots+h_{n_p})+k'_p=b_p+p\delta h'_p+k''_p.$$

It follows that $(p\delta)^{-1}(y_0-b_p)\in H+K$ for every $p\geq1$. Since (b_p) is bounded we obtain the contradiction $0\in\operatorname{cl}(H+K)$. The conclusion follows. □

Again, instead of assuming that $S(u)$ is closed for every $u \in X$, it is sufficient to assume that $S(u)$ is $\preccurlyeq$-lower closed for $u \in X$. A slightly longer direct proof, similar to that of Theorem 4, is possible. Also Theorem 10 covers [13, Th. 3.1].

5 Ekeland's variational principle for bi-multifunctions

In [6] Bianchi, Kassay and Pini obtained an EVP type result for vector functions of two variables; previously such results were obtained by Isac [15] and Li et al. [17]. The next result extends [6, Th. 1] in two directions: d is replaced by F satisfying (F1)–(F3) and instead of a single-valued function $f : X \times X \to Y$ we take a multi-valued one. For its proof we use again [18, Th. 3.1] or [14, Th. 2.2].

Theorem 11 *Assume that (X, d) is a complete metric space, Y is a real topological vector space and $K \subset Y$ is a proper convex cone. Let $F : X \times X \rightrightarrows Y$ satisfy conditions (F1)–(F3). Assume that $G : X \times X \rightrightarrows Y$ has the properties:*

(i) $0 \in G(x, x)$ *for every* $x \in X$,

(ii) $G(x_1, x_2) + G(x_2, x_3) \subset G(x_1, x_3) + K$ *for all* $x_1, x_2, x_3 \in X$,

(iii) z^* *(from (F3)) is bounded below on the set* $\operatorname{Im} G(x, \cdot)$ *for every* $x \in X$.

If

(iv) $\{x' \in X \mid [G(x, x') + F(x, x')] \cap (-K) \neq \emptyset\}$ *is closed for every* $x \in X$,

then for every $x_0 \in X$ there exists $\overline{x} \in X$ such that $[G(x_0, \overline{x}) + F(x_0, \overline{x})] \cap (-K) \neq \emptyset$ and $[G(\overline{x}, x) + F(\overline{x}, x)] \cap (-K) \neq \emptyset$ implies $x = \overline{x}$.

Proof. Let us consider the relation $\preccurlyeq$ on X defined by

$$x \preccurlyeq x' \iff \left[G(x', x) + F(x', x)\right] \cap (-K) \neq \emptyset.$$

Then $\preccurlyeq$ is reflexive and transitive. The reflexivity is immediate from (i) and (F1). Assume that $x \preccurlyeq x'$ and $x' \preccurlyeq x''$. Then $-k \in G(x', x) + F(x', x)$ and $-k' \in G(x'', x') + F(x'', x')$ with $k, k' \in K$. Hence, by (ii) and (F2),

$$-k - k' \in G(x', x) + F(x', x) + G(x'', x') + F(x'', x') \subset G(x'', x) + K + F(x'', x) + K,$$

whence $[G(x'', x) + F(x'', x)] \cap (-K) \neq \emptyset$, that is, $x \preccurlyeq x''$.

Setting $S(x) := \{x' \in X \mid x' \preccurlyeq x\}$, by (iv) we have that $S(x)$ is closed for every $x \in X$. We have to show that for $(x_n)_{n \geq 1} \subset X$ a $\preccurlyeq$–decreasing sequence one has $d(x_n, x_{n+1}) \to 0$. In the contrary case there exist an increasing sequence $(n_l)_{l \geq 1} \subset \mathbb{N}$ and $\delta > 0$ such that $d(x_{n_l}, x_{n_l+1}) \geq \delta$ for every $l \geq 1$. Because (x_n) is $\preccurlyeq$–decreasing, we have that $-k_n \in G(x_n, x_{n+1}) + F(x_n, x_{n+1})$ for some $k_n \in K$ and every $n \geq 1$. Then

$$-k_1 - \ldots - k_n \in G(x_1, x_{n+1}) + F(x_1, x_2) + \ldots + F(x_n, x_{n+1}) + K,$$

and so

$$\inf z^* \left(\operatorname{Im} G(x_1, \cdot)\right) + \inf z^* \left(F(x_1, x_2)\right) + \ldots + \inf z^* \left(F(x_n, x_{n+1})\right) \leq 0 \quad \forall n \geq 1.$$

Since $\inf z^* \left(F(x_n, x_{n+1})\right) \geq 0$ for every $n \geq 1$ and $\inf z^* \left(F(x_{n_l}, x_{n_l+1})\right) \geq \eta(\delta) > 0$ for every $l \geq 1$, taking $n := n_p$ with $p \geq 1$, we obtain that $p\eta(\delta) \leq -\inf z^* \left(\operatorname{Im} G(x_1, \cdot)\right)$ for every $p \geq 1$. This yields the contradiction $\eta(\delta) \leq 0$. Hence $d(x_n, x_{n+1}) \to 0$. Applying [14, Th. 2.2] we get some $\overline{x} \in S(x_0)$ with $S(\overline{x}) = \{\overline{x}\}$, that is, our conclusion holds. □

Remark 7 If $x_0 \in X$ is given a priori, we may replace condition (iii) by the fact that z^* (from (F3)) is bounded below on the set $\operatorname{Im} G(x_0, \cdot)$.

Indeed, $X_0 := S(x_0)$ is closed by (iv), and so (X_0, d) is complete. If $x \in X_0$ then $-k \in G(x_0, x) + F(x_0, x) \subset G(x_0, x) + K$ for some $k \in K$, and so $-k' \in G(x_0, x)$ for some $k' \in K$. It follows that $-k' + G(x, u) \subset G(x_0, x) + G(x, u) \subset G(x_0, u) + K$, whence $G(x, u) \subset G(x_0, u) + K$ for every $u \in X$. Hence condition (iii) is verified on X_0, and so the conclusion of the theorem holds for x_0.

Remark 8 For $F(x, x') := \{d(x, x')k^0\}$ with $k^0 \in K \setminus \{0\}$ and G single-valued, using Theorem 11 and the preceding remark one obtains [15, Th. 8] and [17, Th. 3]; in [15] K is normal and closed, while in [17] $k^0 \in \operatorname{int} K$.

Note that condition (iv) in the preceding theorem holds when G is compact-valued, $G(u, \cdot)$ is level-closed, K is closed and $F(x, x') := \{d(x, x')k^0\}$ for some $k^0 \in K$. Indeed, assume that $-k_n \in G(u, x_n) + d(x_n, u)k^0$ for every $n \geq 1$, where $k_n \in K$. Take $\varepsilon > 0$. Then there exists $n_\varepsilon \geq 1$ such that $d(x_n, u) \geq d(x, u) - \varepsilon =: \gamma_\varepsilon$ for every $n \geq n_\varepsilon$. Then for such n we have that $G(u, x_n) \cap \left(-\gamma_\varepsilon k^0 - K\right) \neq \emptyset$, whence $G(u, x) \cap \left(-\gamma_\varepsilon k^0 - K\right) \neq \emptyset$. Hence there exists $y_\varepsilon \in G(u, x)$ such that $y_\varepsilon + \gamma_\varepsilon k^0 \in -K$. Since $G(u, x)$ is compact, $(y_\varepsilon)_{\varepsilon > 0}$ has a subnet converging to $y \in G(x, u)$. Since $\lim_{\varepsilon \to 0} \gamma_\varepsilon = d(x, u)$ and K is closed, we obtain that $y + d(x, u)k^0 \in -K$.

If Y is a separated locally convex space then we may assume that G is weakly compact-valued instead of being compact-valued.

When G is single-valued and $F(x, x') := \{d(x, x')k^0\}$ with $k^0 \in K$, where K is closed and $z^*(k^0) = 1$, the preceding theorem reduces to [6, Th. 1].

6 EVP type results

The framework is the same as in the previous sections. We want to apply the preceding results to obtain vectorial EVPs. To envisage functions defined on subsets of X we add to Y an element ∞ not belonging to the space Y, obtaining thus the space $Y^\bullet := Y \cup \{\infty\}$. We consider that $y \leq_K \infty$ for all $y \in Y$. Consider now the function $f : X \to Y^\bullet$. As usual, the domain of f is $\operatorname{dom} f = \{x \in X \mid f(x) \neq \infty\}$; the epigraph of f is $\operatorname{epi} f = \{(x, y) \in X \times Y \mid f(x) \leq_K y\}$; the graph of f is $\operatorname{gph} f = \{(x, f(x)) \mid x \in \operatorname{epi} f\}$. Of course, f is proper if $\operatorname{dom} f \neq \emptyset$. For $y^* \in K^+$ we set $(y^* \circ f)(x) := +\infty$ for $x \in X \setminus \operatorname{dom} f$.

Theorem 12 *Assume that (X, d) is a complete metric space, Y is a real topological vector space and $K \subset Y$ is a proper convex cone. Let F satisfy the conditions (F1)–(F3) and let $f : X \to Y^\bullet$ be proper. Assume that $z^* \circ f$ (with z^* from (F3)) is bounded from below and the next condition holds:*

(H3) *for every sequence $(x_n) \subset \operatorname{dom} f$ with $x_n \to x \in X$ and $f(x_n) \in f(x_{n+1}) + F(x_{n+1}, x_n) + K$ for every $n \in \mathbb{N}$ one has $f(x_n) \in f(x) + F(x, x_n) + K$ for every $n \in \mathbb{N}$.*

Then for every $x_0 \in \operatorname{dom} f$ there exists $\overline{x} \in \operatorname{dom} f$ such that

$$f(x_0) \in f(\overline{x}) + F(\overline{x}, x_0) + K$$

and

$$\forall x \in \operatorname{dom} f \ : \ f(\overline{x}) \in f(x) + F(\overline{x}, x) + K \Rightarrow x = \overline{x}.$$

Proof. Consider $A := \operatorname{gph} f := \{(x, f(x)) \mid x \in \operatorname{dom} f\}$. Condition (H3) says nothing than (H1) is verified. Applying Theorem 1 we get the conclusion. □

As for Theorem 1, in the above theorem we may assume that z^* is bounded from below on the set

$$B_0 := \{f(x) \mid x \in \operatorname{dom} f,\ f(x_0) \in f(x) + F(x, x_0) + K\}.$$

The preceding theorem is very close to [12, Th. 3.8] for $\gamma = 1$, which is stated for F and K satisfying conditions (a), (b), (c), (A1), (A2) and $f : S \to Y$ (with $S \subset X$ a nonempty closed set) satisfying the conditions

(A3) Let us denote $A_x^{\gamma F} := \{z \in X \mid (f(z) + \gamma F(z, x)) \cap (f(x) - K) \neq \emptyset\}$ for $x \in S$. For each $x \in S$ and $(z_n) \subset A_x^{\gamma F}$, $z_n \to z$ such that $f(z_n) \leq f(z_m)$ for $n > m$, it follows that $z \in A_x^{\gamma F}$.

(A4) The set $(f(S) - f(x_0)) \cap (-D_F)$ is K-bounded.

Because S is closed one may assume that $S = X$ and $\operatorname{dom} f = X$. Observe that (A4) implies that $y^*(B_0)$ is bounded from below for every $y^* \in K^+$, and so $z^*(B_0)$ is bounded from below. Let us prove that (A3) implies (H3) (for $\gamma = 1$). Consider $(x_n) \subset X = \operatorname{dom} f$ with $x_n \to x \in X$ and $f(x_n) \in f(x_{n+1}) + F(x_{n+1}, x_n) + K$ for every $n \in \mathbb{N}$. Clearly, for a fixed $\overline{n} \in \mathbb{N}$ we have that $(x_n)_{n \geq \overline{n}} \subset A_{x_{\overline{n}}}^{1F}$ and $f(x_n) \leq f(x_m)$ for $n \geq m \geq \overline{n}$. By (A3) we have that $x \in A_{x_{\overline{n}}}^{1F}$, that is, $f(x_{\overline{n}}) \in f(x) + F(x, x_{\overline{n}}) + K$. Hence (H3) holds.
In the case in which $F(x, x') = d(x, x')H$ for some $H \subset K$ the condition (H3) becomes

(H4) for every sequence $(x_n) \subset \operatorname{dom} f$ with $x_n \to x \in X$ and $f(x_n) \in f(x_{n+1}) + d(x_{n+1}, x_n)H + K$ for every $n \in \mathbb{N}$ one has $f(x_n) \in f(x) + d(x, x_n)H + K$ for every $n \in \mathbb{N}$.

In the case $H := \{k^0\}$ condition (H4) is nothing else than condition (E1) in [14].

Using Theorem 12 and Proposition 3 we have the following variant of the preceding result.

Theorem 13 *Assume that (X, d) is a complete metric space, Y is a real topological vector space and $K \subset Y$ is a proper closed convex cone. Let $f : X \to Y^\bullet$ be a proper function and $H \subset K$ be a nonempty cs-complete bounded set with $0 \notin \operatorname{cl}(H + K)$. Assume that $f(\operatorname{dom} f)$ is quasi bounded. If*

(H5) *for every sequence $(x_n) \subset \operatorname{dom} f$ such that $x_n \to x \in X$ and $(f(x_n))$ is $\leq_K$–decreasing one has $f(x) \leq_K f(x_n)$ for every $n \in \mathbb{N}$*

holds, then for every $x_0 \in \operatorname{dom} f$ there exists $\overline{x} \in \operatorname{dom} f$ such that

$$(f(x_0) - K) \cap (f(\overline{x}) + d(\overline{x}, x_0)H) \neq \emptyset$$

and

$$(f(\overline{x}) - K) \cap (f(x) + d(\overline{x}, x)H) = \emptyset \quad \forall x \in \operatorname{dom} f \setminus \{\overline{x}\}.$$

Proof. Since condition (H4) is exactly condition (H1) for $A := \operatorname{gph} f$ and $F = F_H$, in order to have the conclusion of the theorem it is sufficient to show that (H2) is verified for this situation; then just use Proposition 3 and Theorem 12.
Let $((x_n, y_n)) \subset \operatorname{gph} f$ be such that $x_n \to x \in X$ and (y_n) is $\leq_K$–decreasing. Hence $y_n = f(x_n)$ for every n. By (H5) we have that $y := f(x) \leq_K f(x_n) = y_n$ for every $n \in \mathbb{N}$ and, of course, $(x, f(x)) \in \operatorname{gph} f$. The proof is complete. □

Remark 9 Taking H to be complete, convex and bounded, then H is cs-complete. In this case we obtain the main result in [5], that is, [5, Th. 4.1].

Note that the closed convex subsets as well as the open convex subsets of a separated locally convex space are cs-closed; moreover, all the convex subsets of finite dimensional normed spaces are cs-closed (hence cs-complete).

Remark 10 Taking $H := \{k^0\}$ in the preceding theorem one obtains practically [10, Cor. 3.10.6]; there K is assumed to be closed in the direction k^0, the present condition (H5) being condition (H4) in [10, Cor. 3.10.6].

Remark 11 Similar results can be stated using Theorems 7 and 8. When specializing to $F(x_1, x_2) = \left\{d(x_1, x_2)k^0\right\}$ and $F'(y_1, y_2) = \left\{\varepsilon \left\|y_1 - y_2\right\| k^0\right\}$ one recovers [14, Cor. 3.1] and [14, Th. 4.2].

7 Other vectorial EVP type results in the literature

In [1] one states the following result in the case in which (X, d) is a complete metric space, Y is a Banach space, $C \subset Y$ is a a closed convex cone and $k^0 \in \operatorname{int} C$:

"**Theorem 3.1** ([1]). Let $f : X \to Y$ be a vector-valued function. For every $\varepsilon > 0$ there is an initial point $x_0 \in X$ such that $f(X) \cap (f(x_0) - \varepsilon k^0 - \operatorname{int} C) = \emptyset$ and f satisfies

(H) $\{x' \in X \mid f(x') + d(x, x')k^0 \leq_C f(x)\}$ is closed for every $x \in X$.

Then there exists $\overline{x} \in X$ such that (i) $f(\overline{x}) \leq_{\operatorname{int} C} f(x_0)$, (ii) $d(x_0, \overline{x}) \leq 1$, (iii) $f(x) + \varepsilon d(x, \overline{x})k^0 \not\leq_C f(\overline{x})$ for all $x \neq \overline{x}$."

With the present condition (i) the result is false; just take $Y := \mathbb{R}$, $C := \mathbb{R}_+$ and $f : X \to \mathbb{R}$ any (lower semi-) continuous function having $x_0 \in \mathbb{R}$ as a global minimum point. If (i) is replaced by the condition (i') $f(\overline{x}) \leq_C f(x_0)$ the result is well known, because besides (ii) and (iii) the usual conclusion is $f(\overline{x}) + d(\overline{x}, x_0)k^0 \leq_C f(x_0)$. Another possibility is to replace (i) by (i") $f(\overline{x}) \leq_{C^0} f(x_0)$ with $C^0 := \{0\} \cup \operatorname{int} C$. However, this conclusion is immediate from $f(\overline{x}) + d(\overline{x}, x_0)k^0 \leq_C f(x_0)$ because $k^0 \in \operatorname{int} C$.

In the case $\operatorname{int} C = \emptyset$ in [1, Th. 3.2] one takes $k^0 \in C \setminus \{0\}$ and one assumes: $f(X) \subset \cup_{t \in \mathbb{R}}\{tk^0\}$, condition (H) in [1, Th. 3.1] (above), and that "for every $\varepsilon > 0$ there is an initial point $x_0 \in X$ such that $f(X) \cap (f(x_0) - \varepsilon k^0 - C) = \emptyset$", the conclusions being (i') mentioned above and (ii), (iii) from [1, Th. 3.1] (above). Of course, in this situation $f(x) = \varphi(x) \cdot k^0$ for some function $\varphi : X \to \mathbb{R}$. The condition $\forall \varepsilon > 0, \exists x_0 \in X : f(X) \cap (f(x_0) - \varepsilon k^0 - C) = \emptyset$ means that $-k^0 \notin C$, $\inf \varphi > -\infty$ and $\forall \varepsilon > 0, \exists x_0 \in X : \varphi(x_0) < \inf \varphi + \varepsilon$ (which clearly holds because $\inf \varphi \in \mathbb{R}$). Because $-k^0 \notin C$, condition (H) becomes $\{x' \in X \mid \varphi(x') + d(x, x') \leq \varphi(x)\}$ is closed for all $x \in X$. So, [1, Th. 3.2] is essentially a scalar version of the EVP.

Note that for the proof of [10, Th. 3.10.17] it is sufficient to take $(I, \succcurlyeq)$ totally ordered in conditions (H5) and (H6) (in its proof such an index set is taken). With this observation, the first part of [7, Th. 4.1], with the corrections from [8], that is the conclusions (IV) and (V), can be deduced from [10, Th. 3.10.17]. Indeed, by [10, Th. 3.10.17] one finds $(\overline{v}, \overline{y}) \in \operatorname{gph} F$ such that $\overline{y} + \varepsilon r(\overline{v}, v_0) \leq_K y_0$ and $(v, y) \in \operatorname{gph} F$, $y + \varepsilon r(v, \overline{v}) \leq_K \overline{y}$ imply $y = \overline{y}$. Because $F(\overline{v}) \subset E(F(\overline{v})) + K$, there exists $y^* \in F(\overline{v})$ such that $y^* \leq_K \overline{y}$. Taking $v^* := \overline{v}$, (v^*, y^*) verifies the conditions (IV) and (V).

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