

# Contents

<b>1</b>	<b>Introduction</b> .....	1
1.1	Motivating Examples .....	1
1.2	Book Structure .....	4
1.3	Useful Notation .....	9
<b>2</b>	<b>Order Relations and Ordering Cones</b> .....	11
2.1	Order Relations .....	11
2.2	Cone Properties Related to the Topology and the Order .....	17
2.3	Convexity Notions for Sets and Set-Valued Maps.....	22
2.4	Solution Concepts in Vector Optimization .....	28
2.5	Vector Optimization Problems with Variable Ordering Structure .....	43
2.6	Solution Concepts in Set-Valued Optimization.....	45
2.6.1	Solution Concepts Based on Vector Approach .....	45
2.6.2	Solution Concepts Based on Set Approach .....	48
2.6.3	Solution Concepts Based on Lattice Structure.....	55
2.6.4	The Embedding Approach by Kuroiwa .....	65
2.6.5	Solution Concepts with Respect to Abstract Preference Relations .....	67
2.6.6	Set-Valued Optimization Problems with Variable Ordering Structure .....	70
2.6.7	Approximate Solutions of Set-Valued Optimization Problems .....	73
2.7	Relationships Between Solution Concepts .....	74
<b>3</b>	<b>Continuity and Differentiability</b> .....	77
3.1	Continuity Notions for Set-Valued Maps .....	77
3.2	Continuity Properties of Set-Valued Maps Under Convexity Assumptions .....	90
3.3	Lipschitz Properties for Single-Valued and Set-Valued Maps ....	96
3.4	Clarke's Normal Cone and Subdifferential .....	102

3.5	Limiting Cones and Generalized Differentiability .....	103
3.6	Approximate Cones and Generalized Differentiability .....	107
<b>4</b>	<b>Tangent Cones and Tangent Sets</b> .....	<b>109</b>
4.1	First-Order Tangent Cones .....	110
4.1.1	The Radial Tangent Cone and the Feasible Tangent Cone .....	110
4.1.2	The Contingent Cone and the Interiorly Contingent Cone .....	112
4.1.3	The Adjacent Cone and the Interiorly Adjacent Cone .....	120
4.2	Modified First-Order Tangent Cones .....	123
4.2.1	The Modified Radial and the Modified Feasible Tangent Cones .....	124
4.2.2	The Modified Contingent and the Modified Interiorly Contingent Cones .....	124
4.2.3	The Modified Adjacent and the Modified Interiorly Adjacent Cones .....	126
4.3	Miscellaneous Properties of First-Order Tangent Cones .....	129
4.4	First-Order Tangent Cones on Convex Sets .....	132
4.4.1	Connections Among First-Order Tangent Cones on Convex Sets .....	132
4.4.2	Properties of First-Order Tangent Cones on Convex Sets .....	137
4.5	First-Order Local Cone Approximation .....	143
4.6	Convex Subcones of the Contingent Cone .....	147
4.7	First-Order Inversion Theorems and Intersection Formulas .....	156
4.8	Expressions of the Contingent Cone on Some Constraint Sets .....	161
4.9	Second-Order Tangent Sets .....	169
4.9.1	Second-Order Radial Tangent Set and Second-Order Feasible Tangent Set .....	170
4.9.2	Second-Order Contingent Set and Second-Order Interiorly Contingent Set .....	170
4.9.3	Second-Order Adjacent Set and Second-Order Interiorly Adjacent Set .....	173
4.10	Generalized Second-Order Tangent Sets .....	175
4.11	Second-Order Asymptotic Tangent Cones .....	181
4.11.1	Second-Order Asymptotic Feasible Tangent Cone and Second-Order Asymptotic Radial Tangent Cone .....	182
4.11.2	Second-Order Asymptotic Contingent Cone and Second-Order Asymptotic Interiorly Contingent Cone .....	183

4.11.3	Second-Order Asymptotic Adjacent Cone and Second-Order Asymptotic Interiorly Adjacent Cone .....	185
4.12	Miscellaneous Properties of Second-Order Tangent Sets and Second-Order Asymptotic Tangent Cones .....	187
4.13	Second-Order Inversion Theorems .....	192
4.14	Expressions of the Second-Order Contingent Set on Specific Constraints .....	197
4.15	Miscellaneous Second-Order Tangent Cones .....	202
4.15.1	Second-Order Tangent Cones of Ledzewicz and Schaettler .....	202
4.15.2	Projective Tangent Cones of Second-Order .....	204
4.15.3	Second-Order Tangent Cone of N. Pavel .....	206
4.15.4	Connections Among the Second-Order Tangent Cones .....	207
4.16	Second-Order Local Approximation .....	207
4.17	Higher-Order Tangent Cones and Tangent Sets .....	210
<b>5</b>	<b>Nonconvex Separation Theorems</b> .....	213
5.1	Separating Functions and Examples .....	213
5.2	Nonlinear Separation .....	217
5.2.1	Construction of Scalarizing Functionals .....	217
5.2.2	Properties of Scalarization Functions .....	219
5.2.3	Continuity Properties .....	224
5.2.4	Lipschitz Properties .....	225
5.2.5	The Formula for the Conjugate and Subdifferential of $\varphi_A$ for $A$ Convex .....	231
5.3	Scalarizing Functionals by Hiriart-Urruty and Zaffaroni .....	232
5.4	Characterization of Solutions of Set-Valued Optimization Problems by Means of Nonlinear Scalarizing Functionals .....	236
5.4.1	An Extension of the Functional $\varphi_A$ .....	236
5.4.2	Characterization of Solutions of Set-Valued Optimization Problems with Lower Set Less Order Relation $\preceq_C^l$ by Scalarization .....	240
5.5	The Extremal Principle .....	244
<b>6</b>	<b>Hahn-Banach Type Theorems</b> .....	249
6.1	The Hahn–Banach–Kantorovich Theorem .....	250
6.2	Classical Separation Theorems for Convex Sets .....	258
6.3	The Core Convex Topology .....	261
6.4	Yang’s Generalization of the Hahn–Banach Theorem .....	264
6.5	A Sufficient Condition for the Convexity of $\mathbb{R}_+ A$ .....	271

<b>7</b>	<b>Conjugates and Subdifferentials</b> .....	275
7.1	The Strong Conjugate and Subdifferential .....	275
7.2	The Weak Subdifferential .....	288
7.3	Subdifferentials Corresponding to Henig Proper Efficiency .....	296
7.4	Exact Formulas for the Subdifferential of the Sum and the Composition .....	298
<b>8</b>	<b>Duality</b> .....	307
8.1	Duality Assertions for Set-Valued Problems Based on Vector Approach .....	308
8.1.1	Conjugate Duality for Set-Valued Problems Based on Vector Approach .....	308
8.1.2	Lagrange Duality for Set-Valued Optimization Problems Based on Vector Approach .....	313
8.2	Duality Assertions for Set-Valued Problems Based on Set Approach .....	317
8.3	Duality Assertions for Set-Valued Problems Based on Lattice Structure .....	322
8.3.1	Conjugate Duality for $\mathcal{F}$ -Valued Problems .....	323
8.3.2	Lagrange Duality for $\mathcal{I}$ -Valued Problems .....	326
8.4	Comparison of Different Approaches to Duality in Set-Valued Optimization .....	338
8.4.1	Lagrange Duality .....	339
8.4.2	Subdifferentials and Stability .....	341
8.4.3	Duality Statements with Operators as Dual Variables ...	345
<b>9</b>	<b>Existence Results for Minimal Points</b> .....	349
9.1	Preliminary Notions and Results Concerning Transitive Relations .....	349
9.2	Existence of Minimal Elements with Respect to Transitive Relations .....	352
9.3	Existence of Minimal Points with Respect to Cones .....	355
9.4	Types of Convex Cones and Compactness with Respect to Cones .....	360
9.5	Existence of Optimal Solutions for Vector and Set Optimization Problems .....	362
<b>10</b>	<b>Ekeland Variational Principle</b> .....	369
10.1	Preliminary Notions and Results .....	369
10.2	Minimal Points in Product Spaces .....	373
10.3	Minimal Points in Product Spaces of Isac–Tammer’s Type .....	381
10.4	Ekeland’s Variational Principles of Ha’s Type .....	384
10.5	Ekeland’s Variational Principle for Bi-Set-Valued Maps .....	390
10.6	EVP Type Results .....	391
10.7	Error Bounds .....	394

<b>11</b>	<b>Derivatives and Epiderivatives of Set-Valued Maps</b> .....	399
11.1	Contingent Derivatives of Set-Valued Maps .....	400
11.1.1	Miscellaneous Graphical Derivatives of Set-valued Maps .....	407
11.1.2	Convexity Characterization Using Contingent Derivatives .....	414
11.1.3	Proto-Differentiability, Semi-Differentiability, and Related Concepts .....	416
11.1.4	Weak Contingent Derivatives of Set-Valued Maps .....	422
11.1.5	A Lyusternik-Type Theorem Using Contingent Derivatives .....	426
11.2	Calculus Rules for Derivatives of Set-Valued Maps .....	428
11.2.1	Calculus Rules by a Direct Approach .....	429
11.2.2	Derivative Rules by Using Calculus of Tangent Cones .....	432
11.3	Contingently $C$ -Absorbing Maps .....	437
11.4	Epiderivatives of Set-Valued Maps .....	445
11.4.1	Contingent Epiderivatives of Set-Valued Maps with Images in $\mathbb{R}$ .....	446
11.4.2	Contingent Epiderivatives in General Spaces .....	452
11.4.3	Existence Theorems for Contingent Epiderivatives .....	457
11.4.4	Variational Characterization of the Contingent Epiderivatives .....	464
11.5	Generalized Contingent Epiderivatives of Set-Valued Maps .....	470
11.5.1	Existence Theorems for Generalized Contingent Epiderivatives .....	474
11.5.2	Characterizations of Generalized Contingent Epiderivatives .....	478
11.6	Calculus Rules for Contingent Epiderivatives .....	482
11.7	Second-Order Derivatives of Set-Valued Maps .....	488
11.8	Calculus Rules for Second-Order Contingent Derivatives .....	500
11.9	Second-Order Epiderivatives of Set-Valued Maps .....	504
<b>12</b>	<b>Optimality Conditions in Set-Valued Optimization</b> .....	509
12.1	First-Order Optimality Conditions by the Direct Approach .....	512
12.2	First-Order Optimality Conditions by the Dubovitskii-Milyutin Approach .....	522
12.2.1	Necessary Optimality Conditions by the Dubovitskii-Milyutin Approach .....	523
12.2.2	Inverse Images and Subgradients of Set-Valued Maps .....	527
12.2.3	Separation Theorems and the Dubovitskii-Milyutin Lemma .....	534

- 12.2.4 Lagrange Multiplier Rules  
by the Dubovitskii-Milyutin Approach ..... 537
- 12.3 Sufficient Optimality Conditions in Set-Valued Optimization .... 542
  - 12.3.1 Sufficient Optimality Conditions Under  
Convexity and Quasi-Convexity ..... 542
  - 12.3.2 Sufficient Optimality Conditions Under  
Paraconvexity ..... 545
  - 12.3.3 Sufficient Optimality Conditions Under  
Semidifferentiability ..... 549
- 12.4 Second-Order Optimality Conditions in Set-Valued  
Optimization ..... 549
  - 12.4.1 Second-Order Optimality Conditions  
by the Dubovitskii-Milyutin Approach ..... 550
  - 12.4.2 Second-Order Optimality Conditions  
by the Direct Approach ..... 554
- 12.5 Generalized Dubovitskii-Milyutin Approach  
in Set-Valued Optimization..... 557
  - 12.5.1 A Separation Theorem for Multiple Closed  
and Open Cones ..... 559
  - 12.5.2 First-Order Generalized  
Dubovitskii-Milyutin Approach ..... 562
  - 12.5.3 Second-Order Generalized  
Dubovitskii-Milyutin Approach ..... 567
- 12.6 Set-Valued Optimization Problems with a Variable  
Order Structure..... 568
- 12.7 Optimality Conditions for Q-Minimizers  
in Set-Valued Optimization..... 572
  - 12.7.1 Optimality Conditions for Q-Minimizers  
Using Radial Derivatives..... 572
  - 12.7.2 Optimality Conditions for Q-Minimizers  
Using Coderivatives ..... 574
- 12.8 Lagrange Multiplier Rules Based on Limiting Subdifferential ... 578
- 12.9 Necessary Conditions for Approximate Solutions  
of Set-Valued Optimization Problems ..... 591
- 12.10 Necessary and Sufficient Conditions for Solution  
Concepts Based on Set Approach ..... 594
- 12.11 Necessary Conditions for Solution Concepts  
with Respect to a General Preference Relation ..... 598
- 12.12 KKT-Points and Corresponding Stability Results ..... 600
- 13 Sensitivity Analysis in Set-Valued Optimization  
and Vector Variational Inequalities..... 605**
  - 13.1 First Order Sensitivity Analysis in Set-Valued Optimization .... 606
  - 13.2 Second Order Sensitivity Analysis in Set-Valued  
Optimization ..... 613

- 13.3 Sensitivity Analysis in Set-Valued Optimization  
Using Coderivatives..... 623
- 13.4 Sensitivity Analysis for Vector Variational Inequalities ..... 634
- 14 Numerical Methods for Solving Set-Valued  
Optimization Problems ..... 645**
  - 14.1 A Newton Method for Set-Valued Maps ..... 645
  - 14.2 An Algorithm to Solve Polyhedral Convex Set-Valued  
Optimization Problems ..... 651
    - 14.2.1 Formulation of the Polyhedral Convex  
Set-Valued Optimization Problem ..... 653
    - 14.2.2 An Algorithm for Solving Polyhedral  
Convex Set-Valued Optimization Problems..... 655
    - 14.2.3 Properties of the Algorithm ..... 658
- 15 Applications ..... 663**
  - 15.1 Set-Valued Approaches to Duality in Vector Optimization ..... 663
    - 15.1.1 Fenchel Duality for Vector Optimization  
Problems Using Corresponding Results  
for  $\mathcal{F}$ -Valued Problems ..... 667
    - 15.1.2 Lagrange Duality for Vector Optimization  
Problems Based on Results for  $\mathcal{S}$ -Valued Problems .... 670
    - 15.1.3 Duality Assertions for Linear Vector  
Optimization Based on Lattice Approach ..... 677
    - 15.1.4 Further Set-Valued Approaches to Duality  
in Linear Vector Optimization ..... 682
  - 15.2 Applications in Mathematical Finance..... 696
  - 15.3 Set-Valued Optimization in Welfare Economics ..... 701
  - 15.4 Robustness for Vector-Valued Optimization Problems..... 706
    - 15.4.1  $\preceq_C^u$ -Robustness ..... 710
    - 15.4.2  $\preceq_C^l$ -Robustness ..... 720
    - 15.4.3  $\preceq_C^s$ -Robustness ..... 722
    - 15.4.4 Algorithms for Solving Special Classes  
of Set-Valued Optimization Problems..... 724
- Appendix ..... 727**
- References..... 733**
- Index ..... 759**



<http://www.springer.com/978-3-642-54264-0>

Set-valued Optimization

An Introduction with Applications

Khan, A.A.; Tammer, C.; Zălinescu, C.

2015, XXII, 765 p. 29 illus., Hardcover

ISBN: 978-3-642-54264-0