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DUALITY FOR VECTORIAL NONCONVEX OPTIMIZATION BY CONVEXIFICATION AND APPLICATIONS

BY

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Introduction. Many papers concerning (strong) vectorial optimization have appeared in the last few years. In the convex case, the main attempt was to generalize the results from the scalar one as, f.i., the Fenchel-Rockafellar duality theorem ([1], [10], [23], [30]), the formulae for conjugate operators and ε-subdifferentials ([10], [13], [14], [15], [23], [24]) and theorems of Kuhn-Tucker type ([10], [30]). The non-convex case was also considered and for this case Fenchel type theorems ([8] for the scalar case, [20]) were established and the Lagrangeans asociated with such problems were studied ([7], [16]). We elaborated ourselves a paper ([28]) in which we extended the Fenchel-Rockafellar duality theory for (strong) vectorial programming. The objective functions were non-convex, the imposed conditions assured the existence of their convex hulls and the validity of duality results for the relaxed problems. Thus it was reobtained all the formulae for the calculus of conjugate operators and subdifferentials established in [13], [14], some of them in more general conditions, in an unified and simpler manner. Meantime some papers have appeared with similar results (see [3], [4], [24]), so that our mentioned paper could not be published in the initial form. So, the main ideas are taken from [28], but we insist mainly on the non-convex case. The paper is divided in four sections. The first section presents the preliminary results and notations needed in the sequel. The second one is devoted to duality theory. The main result is Theorem 2.3 which states the existence of the convex hull of the considered function and gives a duality result. Theorem 2.3 will help to obtain many duality results and formulae for calculating conjugate operators and ε-subdifferentials. In fact, as mentioned in [11], as soon as a duality result is valid, it is easy to establish such formulae. We also apply Theorem 2.3 to obtain the same kind of results for functions of the form $g\circ f$, where f, g are convex and g is increasing, getting the corresponding formulae from [14], [15] in more general conditions. In the same manner there are obtained a „Farkas lemma" and a „sandwich" theorem. It is also given a simple proof of [17, Th. 1] concerning the separation of sets in product spaces. In the third section there are given conditions which assure that the primal problem and the relaxed primal problem have the same value, or an optimal solution of the primal problem is also optimal for the relaxed one. As applications of these conditions and the results of Section 2, we reobtain the results concerning the non-convex programming in [8], [16], [20], some of them in more general conditions. We also give a Kuhn-Tucker theorem for convex programming. Finally, in Section 4 it is underlined the continuous case.

The main results of this section are Theorem 44., which generalizes [18, Th. 3.14.20] from the scalar case, and Theorem 4.5, generalizing [30, Th. 6].

1. **Preliminary Definitions and Results.** Let E, F be real linear spaces. If $P \subset E$ is a convex cone (i.e. $x+y \in P$, $\lambda x \in P$ for all $x, y \in P$, $\lambda \in R^+$) we write $x \leqslant_P y$ (or simply $x \leqslant y$) for $y-x \in P$. $L(E, F)$ denotes the space of linear operators from E into F; if $P \subset E$ and $Q \subset F$ are convex cones, $L^+(E, F)=\{T \in L(E, F) \mid T(P) \subset Q\}$. If E, F are real topological linear spaces (t.l.s.), $B(E, F)=\{T \in L(E, F) \mid T$ is continuous$\}$, while $B^+(E, F)=B(E, F) \cap L^+(E, F)$ when $P \subset E$, $Q \subset F$ are given convex cones. For $T \in L(E, F)$, $N(T)$, $R(T)$ denote the *kernel* and the *range* of T, respectively. The projection of $E \times F$ on E is denoted by P_E. If E is a t.l.s. $\mathcal{V}(E)$ denotes the class of symmetrical neighborhoods of the origin of E. Let $\emptyset \neq A \subset E$; ${}^lA$ denotes the *affine manifold spanned* by A, while ${}^LA$ denotes the *closed affine manifold spanned* by A when E is a t.l.s. ${}^iA$ denotes the *algebraical relative interior* of A, that is ${}^iA=\{x \in A \mid \forall y \in {}^lA\ \exists \lambda>0\ \forall \alpha \in(0, \lambda): x+\alpha(y-x) \in A\}$, while $\operatorname{ri} A$ denotes the *topological relative interior* when E is a t.l.s., i.e. $\operatorname{ri} A=\{x \in A \mid \exists U \in \mathcal{V}(E):(x+U) \cap {}^LA \subset A\}$. If ${}^lA=E$, ${}^iA$ is denoted by A^i, and, if E is a t.l.s. and ${}^LA=E$, $\operatorname{ri} A$ is denoted by $\operatorname{int} A$. $\operatorname{co} A$ and $\operatorname{cone} A$ denote the *convex hull* and the *conic hull* of A, respectively. So $\operatorname{cone} A=\bigcup_{\lambda \geqslant 0} \lambda A$. Note that if A is convex (convex cone) then $0 \in {}^iA$ iff $\forall x \in A$, $\exists \lambda>0$, such that $-\lambda x \in A$ (A is a linear subspace) or equivalently $\operatorname{cone} A$ is a linear subspace. If $A \subset E$ is a nonempty (convex) set and $x \in A$, we denote by $C(A, x)$ and $H(A)$ the cones $\bigcup_{\lambda \geqslant 0} \lambda(A-x)$ and $\bigcup_{\lambda \geqslant 0} \lambda(A \times\{1\})$, respectively.

Proposition 1.1. *Let E, F be real linear spaces, $A, B \subset E$, $C \subset F$ and $T \in L(E, F)$. Then*

(*i*) $\operatorname{co}(A \times C)=\operatorname{co} A \times \operatorname{co} C$.

(*ii*) $T(\operatorname{co} A)=\operatorname{co} T(A)$; $T(\operatorname{cone} A)=\operatorname{cone} T(A)$.

(*iii*) $\operatorname{co}(A+B)=\operatorname{co} A+\operatorname{co} B$.

(*iv*) $\operatorname{cone} A$ *is convex iff* $\operatorname{co} A \subset \operatorname{cone} A$.

Proof. (*iv*) can be found in [17]. So it is sufficient to show that $\operatorname{co} A \times \operatorname{co} C \subset \operatorname{co}(A \times C)$. Let $x=\sum_{i=1}^n \lambda_i x_i$, $y=\sum_{j=1}^m \mu_j y_j$, $\lambda_i, \mu_j \geqslant 0$, $\sum_{i=1}^n \lambda_i=\sum_{j=1}^m \mu_j=1$, $x_i \in A$, $y_j \in C$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$. Then $(x, y)=\sum_{i=1}^n \sum_{j=1}^m \lambda_i \mu_j(x_i, y_j) \in \operatorname{co}(A \times C)$.

Definition 1.1. *The nonempty convex subsets $A_1, \ldots, A_n$ of E are in general position if*

$$0 \in {}^i\Big(\bigcap_{j=1}^{k-1} A_j-A_k\Big)\ \forall k,\ 2 \leqslant k \leqslant n. \tag{1.1}$$

Proposition 1.2. [29]. *Let $A_1, \ldots, A_n \subset E$ be nonempty convex sets. The following statements are equivalent:*

(*i*) *$A_1, \ldots, A_n$ are in general position,*

(*ii*) *$\bigcap_{j=1}^n A_j \neq \emptyset$ and for all (one) $x \in \bigcap_{i=1}^n A_i$, $C(A_1, x), \ldots, C(A_n, x)$ are in general position,*

(*iii*) *$H(A_1), \ldots, H(A_n)$ are in general position,*

(*iv*) $(0, \ldots, 0) \in {}^i\{(x-x_1, \ldots, x-x_n) \mid x \in E,\ x_j \in A_j,\ 1 \leqslant j \leqslant n\}$.

Remark 1.1. Kutateladze [13] has introduced the notion of convex cones in general position asking that (1.1) be satisfied for a rearrangement of the sets. From Proposition 1.2 we see this notion does not depend on the order of the sets.

Throughout this paper Z is an ordered linear space with positive cone Z^+, having the *least upper bound property* (l.u.b.p.), i.e. every nonempty subset B which has an upper bound (lower bound) has a least upper bound (greatest lower bound) called a *supremum* (*infimum*) of B. Such an element will be denoted $\sup B$ ($\inf B$), although it is generally not unique. To avoid complication of writing, we shall suppose $Z^+ \cap -Z^+ = \{0\}$, so that $\sup B$ and $\inf B$ are unique when they exist. As usually, max is an attained supremum. Z is a linear lattice when $\inf\{x, y\} = x \wedge y$ and $\sup\{x, y\} = x \vee y$ exist for any $x, y \in Z$. $A \subset Z$ is said to be *lineally closed* if each line meets the set in a closed subset of the line.

Proposition 1.3. *Let Z be an ordered linear space with the* l.u.b.p. *Then*

(*i*) *Z^+ is lineally closed,*

(*ii*) *if $\{\lambda x \mid \lambda \in R^+\}$ is upper (lower) bounded then $x \leqslant 0$ ($x \geqslant 0$).*

Proof. (*i*) is proved in [21], while (*ii*) follows from (*i*) applying [12, Th. 1.3.4].

Let $f : D \subset E \to F$ be an operator, where F is ordered by the convex cone Q. We say that f is *convex* if D is convex and $f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y)$ for all $x, y \in D$, $\lambda \in (0, 1)$; f is a *sublinear operator* if *D is a convex cone and* $f(\lambda x) = \lambda f(x)$, $f(x+y) \leqslant f(x) + f(y)$ for all $x, y \in D$, $\lambda \in R^+$. If E is also ordered by the convex cone P, f is *increasing* if $D - P \subset D$ and $f(x) \leqslant f(y)$ for $x, y \in D$ with $x \leqslant y$. Every time when we write $f(x) \leqslant z$, we mean $x \in D$. Sometimes the domain of definition of f will be denoted by $D(f)$.

Let now X be a linear space, Z as above and $A \subset X \times Z$; we say that A is of *epigraph type* if $(x, z) \in A$, $z' \geqslant z$ imply $(x, z') \in A$. Note that if $A \subset X \times Z$ then $E(A) = A + \{0\} \times Z^+$ is the smallest set of epigraph type containing A. From Proposition 1.1 (*i*) we have that $E(\operatorname{co} A) = \operatorname{co} E(A)$.

Propsiotion 1.4. (*i*) *$f : D \subset X \to Z$ is convex iff* $\operatorname{epi} f = \{(x, z) \in X \times Z \mid f(x) \leqslant z\}$ *is convex.*

(*ii*) *Let $A \subset X \times Z$ be such that* $\inf\{z \mid (x, z) \in A\} = \varphi_A(x)$ *exists for all $x \in P_X(A)$. If $E(A)$ is convex then $\varphi_A : P_X(A) \to Z$ is convex. Moreover, if $\varphi : P_X(A) \to Z$, $A \subset \operatorname{epi} \varphi$ then $\varphi \leqslant \varphi_A$.*

The proof can be made in the same way as in the scalar case.

Proposition 1.5. (*i*) *Let $A \subset X \times Z$ be such that $E(A)$ is convex. Suppose that* $\inf\{z \mid (x_0, z) \in A\}$ *exists for some $x_0 \in {}^iP_X(A)$. Then* $\inf\{z \mid (x, z) \in A\}$ *exists for any $x \in P_X(A)$.*

(*ii*) *Let $f : D \subset X \to Z$. If* $\inf\{z \mid (x_0, z) \in \operatorname{co}(\operatorname{epi} f)\}$ *exists for some $x_0 \in {}^i\operatorname{co} D$, then* $\inf\{z \mid (x, z) \in \operatorname{co}(\operatorname{epi} f)\}$ *exists for every $x \in \operatorname{co} D$.*

The proof of (*i*) is as in the case $Z = R$. For (*ii*) apply (*i*) for $A = \operatorname{co}(\operatorname{epi} f)$.

When $\inf\{z \mid (x, z) \in \operatorname{co}(\operatorname{epi} f)\} = \varphi_{\operatorname{co}(\operatorname{epi} f)}(x)$ exists for all $x \in \operatorname{co} D$, the operator $\operatorname{co} f : \operatorname{co} D \to Z$, $\operatorname{co} f(x) = \varphi_{\operatorname{co}(\operatorname{epi} f)}(x)$ is called the *convex hull* of f.

Very important for the sequel is the following result.

Theorem 1.1 [31]. *Let X, Z be as above and $f : D \subset X \to Z$ a convex operator. If $x_0 \in {}^iD$ then there exists $T \in L(X, Z)$ such that $Tx - Tx_0 \leqslant f(x) - f(x_0)$ for all $x \in D$.*

Definition 1.2. *Let* $f: D \subset X \to Z$ *and* $x_0 \in D$, $\varepsilon \in Z^+$. *The ε-subdifferential of f at* x_0 *is the set*

$$\partial_\varepsilon f(x_0) = \{T \in L(X, Z) \mid Tx - Tx_0 \leqslant f(x) - f(x_0) + \varepsilon \quad \forall x \in D\}.$$

The *subdifferential* of f at $x_0 \in D$ is the set $\partial f(x_0) = \partial_0 f(x_0)$.

Remark 1.2. If $f: D \subset X \to Z$ is a sublinear operator then $\partial_\varepsilon f(0) = \partial f(0)$ and $\partial_\varepsilon f(x) = \{T \in \partial f(0) \mid Tx \geqslant f(x) - \varepsilon\}$ for every $x \in D$ and $\varepsilon \geqslant 0$.

We shall also deal with problems of the form

$$\text{(P)} \qquad \inf\{f(x) \mid x \in C\}.$$

x_0 is an *optimal solution* (*ε-solution*) for (P) if $x_0 \in C$ and $f(x_0) = \inf P = \inf\{f(x) \mid x \in C \subset D\}$ $(f(x_0) \leqslant \inf P + \varepsilon)$.

2. **Duality Theory.** In the following, X, Y are real linear spaces and Z is as above. Let $A \subset Y \times Z$ be a nonempty set. According to Stoer and Witzgall [22, Def. 4.65], we introduce the *conjugate* A^c of the set A by $A^c = \{(T, z') \in L(Y, Z) \times Z \mid z + z' \geqslant Ty \quad \forall (y, z) \in A\}$.

Proposition 2.1. (*i*) A^c *is a convex set of epigraph type,*

(*ii*) $A^c = (\mathrm{co}\, A)^c = (E(A))^c$;

(*iii*) *If* $A^c \neq \emptyset$ *then* $\varphi_{\mathrm{co}\, A}(y) = \inf\{z \mid (y, z) \in \mathrm{co}\, A\}$ *exists for all* $y \in {}^iP_Y(\mathrm{co}\, A)$ *and*

$$\varphi_{\mathrm{co}\, A}(y) \geqslant \sup\{Ty - z' \mid (T, z') \in A^c\}. \tag{2.1}$$

The proof is immediate.

Theorem 2.1 *Let* $A \subset Y \times Z$ *and* $y_0 \in {}^iP_Y(\mathrm{co}\, A)$. *If* $\inf\{z \mid (y_0, z) \in \mathrm{co}\, A\}$ *exists then* $A^c \neq \emptyset$ *and* $\inf\{z \mid (y_0, z) \in \mathrm{co}\, A\} = \max\{Ty_0 - z' \mid (T, z') \in A^c\}$.

Proof. From Prop. 1.5 (*i*) and Prop. 1.4 (*ii*), we have that $\varphi_{\mathrm{co}\, A}$ exists and is a convex operator. Since $y_0 \in {}^iD(\varphi_{\mathrm{co}\, A})$ we can apply Th. 1.1 to get $T_0 \in L(Y, Z)$ so that $T_0 y - T_0 y_0 \leqslant \varphi_{\mathrm{co}\, A}(y) - \varphi_{\mathrm{co}\, A}(y_0)$ $\forall y \in D(\varphi_{\mathrm{co}\, A})$. Therefore, $z + T_0 y_0 - \varphi_{\mathrm{co}\, A}(y_0) \geqslant T_0 y$ $\forall (y, z) \in \mathrm{epi}\, \varphi_{\mathrm{co}\, A} \supset A$, so that $(T_0, T_0 y_0 - \varphi_{\mathrm{co}\, A}(y_0)) \in A^c \neq \emptyset$. Hence

$$\inf\{z \mid (y_0, z) \in \mathrm{co}\, A\} = \varphi_{\mathrm{co}\, A}(y_0) \geqslant \sup\{Ty_0 - z' \mid (T, z') \in A^c\} \geqslant$$
$$\geqslant T_0 y_0 - T_0 y_0 + + \varphi_{\mathrm{co}\, A}(y_0) = \varphi_{\mathrm{co}\, A}(y_0),$$

which ends the proof.

Corollary 2.1 [17]. *Let* $B, C \subset Y \times Z$ *be such that a)* $0 \in {}^iP_Y(\mathrm{co}(B - C))$ *and b)* $(0, z) \in \mathrm{co}(B - C) \Rightarrow z \geqslant 0$. *Then there exists* $T \in L(Y, Z)$ *and* $z_0 \in Z$ *such that* $Ty_1 - z_1 \leqslant z_0 \leqslant Ty_2 - z_2$ $\forall (y_1, z_1) \in B$, $(y_2, z_2) \in C$.

Proof. Let $A = B - C$. We are in position to apply Th. 2.1 with $y_0 = 0$, so that there exists $(T, z') \in A^c$ such that $0 \leqslant \inf\{z \mid (0, z) \in \mathrm{co}(B - C)\} = -z'$. But $(T, z') \in A^c$ is the same with

$$z_1 - z_2 + z' \geqslant Ty_1 - Ty_2 \quad \forall (y_1, z_1) \in B,\ (y_2, z_2) \in C \Leftrightarrow$$
$$\Leftrightarrow Ty_1 - z_1 \leqslant Ty_2 - z_2 + z' \leqslant Ty_2 - z_2 \quad \forall (y_1, z_1) \in B,\ (y_2, z_2) \in C.$$

The proof is complete taking $z_0 = \sup\{Ty_1 - z_1 \mid (y_1, z_1) \in B\}$.

In fact, in [17] condition *b*) is given in the equivalent form $(0, z) \in \mathrm{co}(\mathrm{cone}(B - C)) \Rightarrow z \geqslant 0$, while condition *a*) is replaced by the condition *a'*) $\forall y \in {}^iP_Y(\mathrm{co}(\mathrm{cone}(B - C)))$ $\exists z \in Z$ such that $(y, z) \in \mathrm{co}(\mathrm{cone}(B - C))$, i.e. ${}^iP_Y(\mathrm{co}(\mathrm{cone}(B - C))) \subset P_Y(\mathrm{co}(\mathrm{cone}(B - C))) \Leftrightarrow P_Y(\mathrm{co}(\mathrm{cone}(B - C)))$ is a linear subspace. But $P_Y(\mathrm{co}(\mathrm{cone}(B - C))) = \mathrm{cone}\, P_Y(\mathrm{co}(B - C))$, so that *a*) and *a'*) are equivalent. It is clear that from Corollary 2.1 one can state Th. 2.1, so that the above results are equivalent.

Definition 2.1. *Let $f : F \subset X \to Z$. The conjugate operator of f is $f^c : \tilde{F} \to Z$,* $f^c(T) = \sup\{Tx - f(x) \mid x \in F\}$, where $\tilde{F} = \{T \in L(X, Z) \mid \sup\{Tx - f(x) \mid x \in F\}$ exists$\}$.

Theorem 2.2. *Let $f : F \subset X \mapsto Z$. Then*

(*i*) $\operatorname{epi} f^c = (\operatorname{epi} f)^c$ *and f^c is a convex operator.*

(*ii*) *If $\tilde{F} \neq \varnothing$ then* $\operatorname{co} f$ *exists and* $f^c = (\operatorname{co} f)^c$.

(*iii*) $T \in \partial_\varepsilon f(x_0) \Leftrightarrow T \in \tilde{F}$ *and* $f^c(T) + f(x_0) \leqslant Tx_0 + \varepsilon$,

$$T \in \partial f(x_0) \Leftrightarrow T \in \tilde{F} \text{ and } f^c(T) + f(x_0) = Tx_0.$$

The proof is immediate.

Let now $\Phi : D \subset X \times Y \to Z$ be an operator. Consider the problem:

$$\text{(P)} \qquad \inf \Phi(x, 0),$$

called the *primal problem* (PP), and, when $\operatorname{co} \Phi$ exists, *the relaxed primal problem* (RPP):

$$(\overline{\text{P}}) \qquad \inf \operatorname{co} \Phi(x, 0).$$

To the operator Φ, we associate the set $A = \{(y, z) \in Y \times Z \mid \exists x \in X : \Phi(x, y) \leqslant z\}$. It is clear that the following relations hold:

$$(2.2) \qquad A = P_{Y \times Z}(\operatorname{epi} \Phi), \ \operatorname{co} A = P_{Y \times Z}(\operatorname{co}(\operatorname{epi} \Phi)),$$

$$P_Y(A) = P_Y(D) = P_Y(\operatorname{epi} \Phi), \ P_Y(\operatorname{co} A) = P_Y(\operatorname{co} D) = P_Y(\operatorname{co}(\operatorname{epi} \Phi))$$

Remark 2.1. A is a set of epigraph type. If Φ is convex, so is A.

Lemma 2.1. *Let $\Phi : D \subset X \times Y \to Z$. Then:*

(*i*) $\inf\{\Phi(x, 0) \mid (x, 0) \in D\} = \inf\{z \mid (0, z) \in A\}$.

(*ii*) *If* $\operatorname{co} \Phi$ *exists then* $\inf\{\operatorname{co} \Phi(x, 0) \mid (x, 0) \in \operatorname{co} D\} = \inf\{z \mid (0, z) \in \operatorname{co} A\}$.

Proof. It is clear that for a function $f : F \subset X \times Y \to Z$, we have

$$\inf_{x \in P_Y(F)} \inf_{y \in F_x} f(x, y) = \inf_{y \in P_Y(F)} \inf_{x \in F_Y} f(x, y) = \inf_{(x,y) \in F} f(x, y)$$

as soon as one of the above quantities exists, where $F_x = \{y \mid (x, y) \in F\}$ for $x \in P_X(F)$. Let us take $F = \{(x, z) \mid (x, 0, z) \in \operatorname{epi} \Phi\}$ and $f : F \to Z$, $f(x, z) = z$. Therefore $\inf\limits_{x \in P_X(F)} \inf\limits_{z \in F_x} z = \inf\limits_{z \in P_Z(F)} \inf\limits_{x \in F_z} z = \inf\limits_{z \in P_Z(F)} z$. But $\inf\limits_{z \in F_X} z = \inf\{z \mid (x, 0, z) \in \operatorname{epi} \Phi\} = \Phi(x, 0)$, so that $\inf\limits_{x \in P_X(F)} \inf\limits_{z \in F_x} z = \inf\{\Phi(x, 0) \mid (x, 0) \in D\}$, while $\inf\{z \mid z \in P_Z(F)\} = \inf\{z \mid (0, z) \in A\}$, since $z \in P_Z(F) \Leftrightarrow \exists x \in X : (x, 0, z) \in \operatorname{epi} \Phi \Leftrightarrow (0, z) \in A$, so that (*i*) holds. To obtain (*ii*) replace Φ by $\operatorname{co} \Phi$ in (*ii*).

Let us determine now A^c

$$(T, z') \in A^c \Leftrightarrow \forall (y, z) \in A : z + z' \geqslant Ty \Leftrightarrow \forall (x, y) \in D : \Phi(x, y) + z' \geqslant Ty \Leftrightarrow$$

$$\Leftrightarrow \forall (x, y) \in D : z' \geqslant Ox + Ty - \Phi(x, y) \Leftrightarrow (0, T) \in \tilde{D} \text{ and } z' \geqslant \Phi^c(0, T).$$

Therefore $A^c = \{(T, z') \mid (0, T) \in \tilde{D}, \ z' \geqslant \Phi^c(0, T)\}$.

Taking into account the above relation, Lemma 2.1. and (2.1) it is natural to consider that the *dual problem* (DP) of (P) or $(\overline{\text{P}})$ is

$$\text{(D)} \qquad \sup[-\Phi^c(0, T)].$$

If $A^c \neq \varnothing$ then from the definitions of (P), $(\overline{\text{P}})$ and (D) we have $\inf P \geqslant \inf \overline{P} \geqslant \sup D$.

Theorem 2.3. *If* $0\in {}^{i}\mathrm{co}\,P_Y(D)$ *and* $\inf\{z\,|\,(x,0,z)\in \mathrm{co}(\mathrm{epi}\ \Phi)\}$ *exists, then* $\mathrm{co}\,\Phi$ *exists and*

(2.3) $$\inf\{\mathrm{co}\,\Phi(x,0)\,|\,(x,0)\in \mathrm{co}\,D\}=\max\{-\Phi^c(0,T)\,|\,(0,T)\in \tilde{D}\}.$$

Moreover, x_0 *is an optimal solution for* (P) *and* $\min P=\max D$ *iff* $\exists T\in L(Y,Z)$ *such that* $(0,T)\in \partial\Phi(x_0,0)$.

Proof. In our conditions, by (2.2) and Lemma 2.1 we have $0\in {}^{i}P_Y(\mathrm{co}\,A)={}^{i}\mathrm{co}\,P_Y(D)$ and $\inf\{z\,|\,(0,z)\in \mathrm{co}\,A\}=\inf\{z\,|\,(x,0,z)\in \mathrm{co}(\mathrm{epi}\,\Phi)\}$ exists. Hence we can apply Th. 2.1 to obtain that $\inf\{z\,|\,(0,z)\in \mathrm{co}\ A\}=\max\{-z\,|\,(T,z)\in A^c\}$. Therefore $A^c\neq\emptyset$, so that $\mathrm{epi}\,\Phi^c\neq\emptyset$. Hence $\mathrm{co}\,\Phi$ exists and $\max\{-z\,|\,(T,z)\in A^c\}=\max\{-\Phi^c(0,T)\,|\,(0,T)\in\tilde{D}\}$. The rest of the proof follows immediately.

Theorem 2.4. *Let* $f:F\subset Y\to Z$ *and* $S\in L(X,Y)$ *be such that* $0\in {}^{i}(R(S)-\mathrm{co}\,F)$.

(*i*) *If* $\inf\{z\,|\,(Sx,z)\in \mathrm{co}(\mathrm{epi}\,f)\}$ *exists, then* $\mathrm{co}\,f$ *exists and*

(2.4) $$\inf\{\mathrm{co}\,f(Sx)\,|\,Sx\in \mathrm{co}\,F\}=\max\{-f^c(T)\,|\,T\in\tilde{F},\ T\circ S=0\}.$$

x_0 *is an optimal solution of the corresponding primal problem and* $\min P=\max D$ *iff* $\exists T\in \partial f(Sx_0)$ *such that* $T\circ S=0$. *Moreover, if f is convex and* $\varphi(x)=f(Sx)$ *then*

(*ii*) $D(\varphi^c)=\{T\circ S\,|\,T\in\tilde{F}\}$, $\varphi^c(U)=\min\{f^c(T)\,|\,T\in\tilde{F},\ T\circ S=U\}$,

(*iii*) $\partial_\varepsilon\varphi(x)=\{T\circ S\,|\,T\in\partial_\varepsilon f(Sx)\}\quad \forall x\in D(\varphi)=S^{-1}(F),\ \varepsilon\geq 0.$

Proof. Let $\Phi:D\to Z$, $\Phi(x,y)=f(Sx-y)$, where $D=\{(x,y)\in X\times Y\,|\,Sx-y\in F\}=\{(x,Sx-y)\,|\,x\in X,\ y\in F\}$. Therefore $\mathrm{co}\,P_Y(D)=R(S)-\mathrm{co}\,F$, so that $0\in {}^{i}\mathrm{co}\,P_Y(D)$. To apply the preceding theorem we must calculate $\mathrm{epi}\,\Phi$.

$$\mathrm{epi}\,\Phi=\{(x,y,z)\in X\times Y\times Z\,|\,\Phi(x,y)\leq z\}=\{(x,y,z)\,|\,f(Sx-y)\leq z\}=$$
$$=\{(x,Sx-y,z)\,|\,x\in X,\ (y,z)\in \mathrm{epi}\,f\}=V(X\times \mathrm{epi}\,f),$$

where $V:X\times Y\times Z\to X\times Y\times Z$, $V(x,y,z)=(x,Sx-y,z)$. V is a linear operator, so that $\mathrm{co}(\mathrm{epi}\,\Phi)=V(X\times \mathrm{co}(\mathrm{epi}\,f))$. Therefore $\inf\{z\,|\,(x,0,z)\in \mathrm{co}(\mathrm{epi}\,\Phi)\}=\inf\{z\,|\,x\in X,\ (Sx,z)\in \mathrm{co}(\mathrm{epi}\,f)\}$ exists, so that formula (2.3) is true. We have

$$\begin{aligned}\Phi^c(T_1,T_2)&=\sup\{T_1x+T_2y-\Phi(x,y)\,|\,(x,y)\in D\}\\&=\sup\{T_1x+T_2y-f(Sx-y)\,|\,x\in X,\ y\in Y,\ Sx-y\in F\}\\&=\sup\{T_1x+T_2(Sx-y)-f(y)\,|\,x\in X,\ y\in F\}\\&=\sup\{(T_1+T_2\circ S)x\,|\,x\in X\}+\sup\{(-T_2)y-f(y)\,|\,y\in F\}.\end{aligned}$$

Therefore

(2.5) $$\tilde{D}=\{(T\circ S,T)\,|\,T\in\tilde{F}\}\ \text{and}\ \Phi^c(T\circ S,T)=f^c(T).$$

Hence, since $\tilde{D}\neq\emptyset$, $\tilde{F}\neq\emptyset$. so that $\mathrm{co}f$ exists. On the other hand, taking into account (2.5), (2.3) becomes (2.4) in our case. The rest of part (*i*) is obvious.

(*ii*) Suppose f is convex and take $U\in D(\varphi^c)$. Consider $\bar{\Phi}:D\to Z$, $\bar{\Phi}(x,y)=\Phi(x,y)-Ux$, where Φ is defined at (*i*). We have that $0\in {}^{i}P_Y(D)$ and $\inf\{\bar{\Phi}(x,0)\,|\,(x,0)\in D\}=-\varphi^c(U)$. Therefore

(2.6) $$-\varphi^c(U)=\max\{-\bar{\Phi}^c(0,T)\,|\,(0,T)\in D(\bar{\Phi}^c)\}.$$

But $\bar{\Phi}^c(0, T) = \sup\{Ty - \Phi(x, y) + Ux \mid (x, y) \in D\} = \Phi^c(U, T)$. From (2.5) and (2.6), (*ii*) follows.

(*iii*) It is obvious that $\{T\circ S \mid T \in \partial_\varepsilon f(Sx_0)\} \subset \partial_\varepsilon \varphi(x_0)$ for $x_0 \in S^{-1}(F)$. Let $U \in \partial_\varepsilon \varphi(x_0) \subset D(\varphi^c)$; then there exists $T \in \tilde{F}$ such that $U = T\circ S$ and $\varphi^c(U) = f^c(T)$. Thus $f(Sx_0) + f^c(T) = \varphi(x_0) + \varphi^c(U) \leqslant Ux_0 + \varepsilon = T(Sx_0) + \varepsilon$. Therefore $T \in \partial_\varepsilon f(Sx_0)$, and the proof is complete.

From Theorem 2.4 we obtain the following important cases.

Theorem 2.5. *Let* $f : F \subset X \times Y \to Z$, $S \in L(X, Y)$ *and suppose that* $0 \in {}^i\{Sx - y \mid (x, y) \in \mathrm{co}\, F\}$.

(*i*) *If* $\inf\{z \mid (x, Sx, z) \in \mathrm{co}(\mathrm{epi}\, f)\}$ *exists then* $\mathrm{co}\, f$ *exists and*

$$\inf\{\mathrm{co}\, f(x, Sx) \mid (x, Sx) \in \mathrm{co}\, F\} = \max\{-f^c(T\circ S, -T) \mid (T\circ S, -T) \in \tilde{F}\}$$

x_0 *is an optimal solution of the corresponding primal problem and* $\min P = \max D$ *iff* $\exists T \in L(Y, Z)$ *such that* $(T\circ S, -T) \in \partial f(x_0, Sx_0)$. *Moreover, if* f *is convex and* $\varphi(x) = f(x, Sx)$ *then*

(*ii*) $D(\varphi^c) = \{U \mid U = T_1 + T_2\circ S,\ (T_1, T_2) \in \tilde{F}\}$, $\varphi^c(U) = \min\{f^c(T_1, T_2) \mid (T_1, T_2) \in \tilde{F},\ T_1 + T_2\circ S = U\}$.

(*iii*) $\partial_\varepsilon \varphi(x) = \{T_1 + T_2\circ S \mid (T_1, T_2) \in \partial_\varepsilon f(x, Sx)\}$ $\forall x \in D(\varphi) = \{x \mid (x, Sx) \in F\}$, $\varepsilon \geqslant 0$.

Proof. This case can be reduced to the one in Theorem 2.4 considering the operator $\tilde{S} \in L(X, X \times Y)$, $\tilde{S}x = (x, Sx)$, and noting that $0 \in {}^i\{Sx - y \mid (x, y) \in \mathrm{co}\, F\} \Leftrightarrow (0, 0) \in {}^i(R(\tilde{S}) - \mathrm{co}\, F)$.

Theorem 2.6. *Let* X, Y_k, $1 \leqslant k \leqslant n$ *be linear spaces*, $f_k : F_k \subset Y_k \to Z$ *and* $S_k \in L(X, Y_k)$, $1 \leqslant k \leqslant n$. *Suppose that* $(0, \ldots, 0) \in {}^i\{(S_1x - y_1, \ldots, S_nx - y_n) \mid x \in X,\ y_k \in \mathrm{co}\, F_k,\ 1 \leqslant k \leqslant n\}$.

(*i*) *If* $\inf\{z_1 + \ldots + z_n \mid (S_kx, z_k) \in \mathrm{co}(\mathrm{epi}\, f_k),\ 1 \leqslant k \leqslant n\}$ *exists, then* $\mathrm{co}\, f_k$ *exists for all* k, $1 \leqslant k \leqslant n$ *and*

$$\inf\left\{\sum_{k=1}^{n} \mathrm{co}\, f_k(S_kx) \mid S_kx \in \mathrm{co}\, F_k,\ 1 \leqslant k \leqslant n\right\} =$$

$$= \max\left\{-\sum_{k=1}^{n} f_k^c(T_k) \mid T_k \in \tilde{F}_k, \sum_{k=1}^{n} T_k\circ S_k = 0\right\}.$$

x_0 *is an optimal solution of the corresponding primal problem and* $\min P = \max D$ *iff* $\forall k$, $1 \leqslant k \leqslant n$ $\exists T_k \in \partial f_k(S_kx_0)$ *such that* $\sum_{k=1}^{n} T_k \circ S_k = 0$. *Moreover, if* f_k, $1 \leqslant k \leqslant n$ *are convex operators and* $\varphi(x) = \sum_{k=1}^{n} f_k(S_kx)$, *then*

(*ii*) $$D(\varphi^c) = \left\{U \mid U = \sum_{k=1}^{n} T_k\circ S_k,\ T_k \in \tilde{F}_k,\ 1 \leqslant k \leqslant n\right\},$$

$$\varphi^c(U) = \min\left\{\sum_{k=1}^{n} f_k^c(T_k) \mid T_k \in \tilde{F}_k,\ 1 \leqslant k \leqslant n, \sum_{k=1}^{n} T_k\circ S_k = U\right\}.$$

(*iii*) $$\partial_\varepsilon \varphi(x) = \left\{\sum_{k=1}^{n} T_k\circ S_k \mid T_k \in \partial_{\varepsilon_k} f_k(S_kx),\ \varepsilon_k \geqslant 0,\ 1 \leqslant k \leqslant n, \sum_{k=1}^{n} \varepsilon_k = \varepsilon\right\}$$

$\forall x \in D(\varphi)$.

Proof. Take $S: X \to Y = Y_1 \times \ldots \times Y_n$, $Sx = (S_1 x, \ldots, S_n x)$ and $f: F = F_1 \times \ldots \times F_n \subset Y \to Z$, $f(y_1, \ldots, y_n) = f_1(y_1) + \ldots + f_n(y_n)$ in Theorem 2.4. Note that $\operatorname{co} f$ exists exactly when $\operatorname{co} f_k$ exists for all $1 \leqslant k \leqslant n$ and then

$$\operatorname{co} f(y_1, \ldots, y_n) = \sum_{k=1}^{n} \operatorname{co} f_k(y_k),$$

$$D(f^c) = D(f_1^c) \times \ldots \times D(f_n^c),\ f^c(T_1, \ldots, T_n) = \sum_{k=1}^{n} f_k^c(T_k), \tag{2.7}$$

$$\partial_\varepsilon f(y_1, \ldots, y_n) = \bigcup \{\partial_{\varepsilon_1} f_1(y_1) \times \ldots \times \partial_{\varepsilon_n} f_n(y_n) \mid \varepsilon_k \geqslant 0,\ \varepsilon_1 + \ldots + \varepsilon_n = \varepsilon\}. \tag{2.8}$$

Corollary 2.2. *Let* $f_k : F_k \subset X \to Z$, $1 \leqslant k \leqslant n$. *Suppose that* $\operatorname{co} F_k$, $1 \leqslant k \leqslant n$, *are in general position.*

(*i*) *If* $\inf\{z_1 + \ldots + z_n \mid x \in X,\ (x, z_k) \in \operatorname{co}(\operatorname{epi} f_k),\ 1 \leqslant k \leqslant n\}$ *exists, then* $\operatorname{co} f_k$ *exists for all* $1 \leqslant k \leqslant n$ *and*

$$\inf\left\{\sum_{k=1}^{n} \operatorname{co} f_k(x) \mid x \in \bigcap_{k=1}^{n} \operatorname{co} F_k\right\} = \max\left\{-\sum_{k=1}^{n} f_k^c(T_k) \mid T_k \in \tilde{F}_k,\ \sum_{k=1}^{n} T_k = 0\right\}.$$

x_0 *is an optimal solution for* (P) *and* $\min P = \max D$ *iff* $\forall k$, $1 \leqslant k \leqslant n$ $\exists T_k \in \partial f_k(x_0)$ *such that* $\sum_{k=1}^{n} T_k = 0$. *Moreover, if* f_k *are convex and* $\varphi(x) = \sum_{k=1}^{n} f_k(x)$, *then*

(*ii*) $D(\varphi^c) = \left\{U \mid U = \sum_{k=1}^{n} T_k,\ T_k \in \tilde{F}_k\right\}$, $\varphi^c(U) = \min\left\{\sum_{k=1}^{n} f_k^c(T_k) \mid T_k \in \tilde{F}_k,\ \sum_{k=1}^{n} T_k = U\right\}$.

(*iii*) $\partial_\varepsilon \varphi(x) = \bigcup \left\{\sum_{k=1}^{n} \partial_{\varepsilon_k} f_k(x) \mid \varepsilon_k \geqslant 0,\ 1 \leqslant k \leqslant n,\ \varepsilon_1 + \ldots + \varepsilon_n = \varepsilon\right\}$,

$\forall x \in D(\varphi) = \bigcap_{k=1}^{n} F_k$.

Proof. Apply Theorem 2.6 for $S_k =$ *identity of* X, $1 \leqslant k \leqslant n$, and take into account Proposition 1.2.

Theorem 2.7. *Let* $f: F \subset X \to Z$ *and* $S \in L(Y, Z)$. *Suppose that* $0 \in {}^i\operatorname{co} S(F)$. *If* $\inf\{z \mid x \in X,\ (x, z) \in \operatorname{co}(\operatorname{epi} f),\ Sx = 0\}$ *exists, then* $\operatorname{co} f$ *exists and*

$$\inf\{\operatorname{co} f(x) \mid Sx = 0\} = \max\{-f^c(T \circ S) \mid T \in L(Y, Z),\ T \circ S \in \tilde{F}\}. \tag{2.9}$$

Moreover, x_0 *is an optimal solution for the corresponding primal problem and* $\min P = \max D$ *iff*

$$Sx_0 = 0 \ \textit{and}\ \exists T \in L(Y, Z) \ \textit{such that}\ T \circ S \in \partial f(x_0).$$

Proof. Consider $\Phi : D \to Z$, $\Phi(x, y) = f(x)$, where $D = \{(x, y) \in F \times Y \mid Sx = y\} = \{(x, Sx) \mid x \in F\}$. Therefore $P_Y(D) = S(F)$, so that $0 \in {}^i\operatorname{co} P_Y(D)$. To apply Theorem 2.3 we must calculate $\operatorname{epi} \Phi$.

$$\operatorname{epi} \Phi = \{(x, y, z) \mid Sx = y,\ f(x) \leqslant z\} = \{(x, Sx, z) \mid (x, z) \in \operatorname{epi} f\}.$$

so that $\mathrm{co}(\mathrm{epi}\,\Phi)=\{(x, Sx, z)\,|\,(x, z)\in \mathrm{co}(\mathrm{epi}\, f)\}$. Therefore $\inf\{z\,|\,(x, 0, z)\in \mathrm{co}(\mathrm{epi}\,\Phi)\}=\inf\{z\,|\,(x, z)\in \mathrm{co}(\mathrm{epi}\, f),\ Sx=0\}$ exists. But $\Phi^c(0, T)=\sup\{Ty-\Phi(x, y)\,|\,(x, y)\in D\}=\sup\{Ty-f(x)\,|\,Sx=y\}=f^c\,(T\circ S)$. Hence $\mathrm{co}\, f$ exists and (2.9) follows from (2.3). The rest of the theorem follows immediately.

Corollary 2.3. *Let* $f: F\subset X\to Z$ *be a convex operator and* $S\in L(X, Y)$. *Suppose that* $\varphi(y)=\inf\{f(x)\,|\,Sx=y\}$ *exists for every* $y\in S(F)$ *(this is the case when* $\varphi(y_0)$ *exists for some* $y_0\in {}^{i}S(F)$*). Then*

(*i*) $D(\varphi^c)=\{U\,|\,U\circ S\in \tilde{F}\}$, $\varphi^c(U)=f^c(U\circ S)$,

(*ii*) *for every attained* $\varphi(y)$, e.g. $\varphi(y)=f(x)$, $Sx=y$, *we have*

$$\partial_\varepsilon\varphi(y)=\{U\,|\,U\circ S\in\partial_\varepsilon f(x)\}.$$

Proof. (*i*) It is just clear that φ is convex. Let $U\in D(\varphi^c)$.

$$-\varphi^c(U)=\inf\{\varphi(y)-Uy\,|\,y\in S(F)\}=\inf\{f(x)-Uy\,|\,Sx-y=0,\ \ x\in F,\ \ y\in Y\}.$$

Let us take $\bar{f}: F\times Y\to Z$, $\bar{f}(x, y)=f(x)-Uy$ and $\bar{S}: X\times Y\to Y$, $\bar{S}(x, y)=Sx-y$. Since $0\in {}^{i}\bar{S}(F\times Y)={}^{i}(S(F)-Y)=Y$ we can apply the preceding theorem, so that $-\varphi^c(U)=\max\{-\bar{f}^c(T\circ\bar{S})\,|\,T\circ S\in D(\bar{f}^c)\}$. But for $T\in L(Y, Z)$, $T\circ\bar{S}=(T\circ S, -T)$ and $\bar{f}^c(T\circ\bar{S})=f^c(T\circ S, -T)=\sup{}^{*}\{T\circ Sx-Ty-f(x)+Uy\,|\,x\in F,\ y\in Y\}=\sup\{T\circ Sx-f(x)\,|\,x\in F\}+\sup\{(U-T)y\,|\,y\in Y\}$, so that $T\circ\bar{S}\in D(\bar{f}^c)$ iff $T\circ S\in\tilde{F}$ and $U=T$. This shows that $D(\varphi^c)=\{U\,|\,U\circ S\in\tilde{F}\}$ and $\varphi^c(U)=f^c(U\circ S)$.

(*ii*) Let $y=Sx$ such that $\varphi(y)=f(x)$. Then for $U\in\partial_\varepsilon\varphi(y)$, $U\in D(\varphi^c)$, so that $U\circ S\in\tilde{F}$ and $f(x)+f^c(U\circ S)=\varphi(y)+\varphi^c(U)\leqslant Uy+\varepsilon=U\circ Sx+\varepsilon$, which shows that $U\circ S\in\partial_\varepsilon f(x)$, i.e. $\partial_\varepsilon\varphi(y)\subset\{U\,|\,U\circ S\in\partial_\varepsilon f(x)\}$. The converse inclusion is obvious.

Corollary 2.4. *Let* $f_k: F_k\subset X\to Z$, $1\leqslant k\leqslant n$, *be convex operators. Suppose* $\varphi(x)=\inf\{f_1(x_1)+\ldots+f_n(x_n)\,|\,x_k\in F_k,\ 1\leqslant k\leqslant n,\ \ x_1+\ldots+x_n=x\}$ *exists for every* $x\in F_1+\ldots+F_n=F$ *(this is the case when* $\varphi(x_0)$ *exists for some* $x_0\in {}^{i}F$*). Then*

(*i*) $D(\varphi^c)=\bigcap\limits_{k=1}^{n}\tilde{F}_k$, $\varphi^c(U)=\sum\limits_{k=1}^{n}f_k^c(U)$,

(*ii*) *for every attained* $\varphi(x)$, e.g. $\varphi(x)=f_1(x_1)+\ldots+f_n(x_n)$, $x_1+\ldots+x_n=x$, $\partial_\varepsilon\varphi(x)=\bigcup\{\bigcap\limits_{k=1}^{n}\partial_{\varepsilon_k}f_k(x_k)\,|\,\varepsilon_k\geqslant 0,\ \varepsilon_1+\ldots+\varepsilon_n=\varepsilon\}$.

Proof. Take in Corollary 2.3 $Y_k=X$, $S: X^n\to X$, $S(x_1, \ldots, x_n)=x_1+\ldots+x_n$ and $f: F_1\times\ldots\times F_n\to Z$, $f(x_1, \ldots, x_n)=f_1(x_1)+\ldots+f_n(x_n)$ and use (2.7) and (2.8).

In the sequel, suppose that Y is ordered by the convex cone Q, $f: F\to Y$ is convex and $g: G\to Z$ is convex and increasing (see Section 1). Note that $D(g^c)\subset L^+(Y, Z)$, so that $\partial_\varepsilon g(y)\subset L^+(Y, Z)$. It is simply enough to show that $\varphi: H\to Z$, $\varphi(x)=g(f(x))$, where $H=\{x\in F\,|\,f(x)\in G\}$, is a convex operator.

Theorem 2.8. *Let* f, g *be as above and* $0\in {}^{i}(G-f(F))$.

(*i*) *If* $\inf\{\varphi(x)\,|\,x\in H\}$ *exists then*

(2.10) $\inf\{\varphi(x) \mid x \in H\} = \max\{-g^c(T) - (T \circ f)^c(0) \mid T \in \tilde{G},\ 0 \in D((T \circ f)^c)\}$.

Moreover x_0 *is an optimal solution iff*

$$\exists T \in \partial g(f(x_0)) \text{ such that } 0 \in \partial(T \circ f)(x_0).$$

(*ii*) $D(\varphi^c) = \bigcup\{D((T \circ f)^c) \mid T \in \tilde{G}\}$,

(2.11) $\varphi^c(U) = \min\{g^c(T) + (T \circ f)^c(U) \mid U \in D((T \circ f)^c),\ T \in \tilde{G}\}$.

(*iii*) $\partial_\varepsilon \varphi(x) = \bigcup\{\partial_{\varepsilon_2}(T \circ f)(x) \mid T \in \partial_{\varepsilon_1} g(f(x)),\ \varepsilon_1, \varepsilon_2 \geqslant 0,\ \varepsilon_1 + \varepsilon_2 = \varepsilon\}$.

Proof. (*i*) Let $\Phi : D \to Z$, $\Phi(x, y) = g(f(x) - y)$, where $D = \{(x, y) \in F \times Y \mid f(x) - y \in G\} = \{(x, f(x) - y) \mid x \in F,\ y \in G\}$. Therefore $0 \in {}^i P_Y(D) = {}^i(f(F) - G)$, so that we can apply Theorem 2.3 to get $\inf\{\varphi(x) \mid x \in H\} = \max\{-\Phi^c(0, T) \mid (0, T) \in \tilde{D}\}$. But

$$\Phi^c(T_1, -T_2) = \sup\{T_1 x - T_2 y - g(f(x) - y) \mid x \in F, f(x) - y \in G\} =$$
$$\sup\{T_1 x - (T_2 \circ f)(x) + T_2 y - g(y) \mid x \in F,\ y \in G = \sup\{T_1 x - (T_2 \circ f)(x) \mid x \in$$
$$\in F\} + \sup\{T_2 y - g(y) \mid y \in G\} = (T_2 \circ f)^c(T_1) + g^c(T_2),$$

as soon as $T_2 \in \tilde{G}$ and $T_1 \in D((T_2 \circ f)^c)$. So (2.10) is verified.

(*ii*) Let $U \in \tilde{H}$ and take $\bar{\Phi} : D \to Z$, $\bar{\Phi}(x, y) = \Phi(x, y) - Ux$, where Φ is defined above. We have

$$-\varphi^c(U) = \inf\{\bar{\Phi}(x, 0) \mid (x, 0) \in D\} = \max\{-\bar{\Phi}^c(0, -T) \mid (0, -T) \in \tilde{D}(\Phi^c)\} =$$
$$= \max\{-\Phi^c(U, -T) \mid (U, -T) \in \tilde{D}\}.$$

So we get immediately (2.11).

(*iii*) Let now $U \in \partial_\varepsilon \varphi(x) \subset \tilde{H}$. There exists $T \in \tilde{G}$ such that $U \in D((T \circ f)^c)$ and $\varphi^c(U) = g^c(T) + (T \circ f)^c(U)$. Therefore

$$g(f(x)) + g^c(T) + (T \circ f)^c(U) = \varphi(x) + \varphi^c(U) \leqslant Ux + \varepsilon,$$

or equivalently,

$$g(f(x)) + g^c(T) - T(f(x)) + (T \circ f)(x) + (T \circ f)^c(U) - Ux \leqslant \varepsilon.$$

Taking $\varepsilon_1 = g(f(x)) + g^c(T) - T(f(x)) \geqslant 0$ and $\varepsilon_2 = \varepsilon - \varepsilon_1$, we have $T \in \partial_{\varepsilon_1} g(f(x))$, $U \in \partial_{\varepsilon_2}(T \circ f)(x)$. Thus we proved an inclusion in (*iii*). The converse is obvious.

Corollary 2.5. *In Theorem 2.8, suppose that g is sublinear. Then*

(*i*) $\inf\{\varphi(x) \mid x \in H\} = \max\{-(T \circ f)^c(0) \mid T \in \partial g(0),\ 0 \in D((T \circ f)^c)\}$,

(*ii*) $D(\varphi^c) = \bigcup\{D((T \circ f)^c) \mid T \in \partial g(0)\}$, $\varphi^c(U) = \min\{(T \circ f)^c(U) \mid T \in \partial g(0),\ U \in D((T \circ f)^c)\}$.

(*iii*) $\partial_\varepsilon \varphi(x) = \bigcup\{\partial_{\varepsilon_2}(T \circ f)(x) \mid T \in \partial g(0),\ T(f(x)) \geqslant g(f(x)) - \varepsilon_1,\ \varepsilon_1, \varepsilon_2 \geqslant 0,\ \varepsilon_1 + \varepsilon_2 = \varepsilon\}$.

Proof. We must only note that $D(g^c) = \partial g(0)$ and take into account Remark 1.2.

Theorem 2.9. *Let* Y *be a linear lattice,* $T\in L^+(Y,Z)$ *and* $f_k: F_k\subset X\to Y$ $1\leqslant k\leqslant n$, *convex operators. Suppose that* $F_1, \dots, F_n$ *are in general position. Take* $\varphi:\bigcap_{k=1}^n F_k\to Z$, $\varphi(x)=T(f_1(x)\vee\dots\vee f_n(x))$.

(*i*) *If* $\inf\left\{T(f_1(x)\vee\dots\vee f_n(x))\mid x\in\bigcap_{k=1}^n F_k\right\}$ *exists, then*

$$\inf\left\{T(f_1(x)\vee\dots\vee f_n(x))\mid x\in\bigcap_{k=1}^n F_k\right\}=\max\left\{-\sum_{k=1}^n (T_k\circ f_k)^c(S_k)\mid T_k\in L^+(Y,Z),\right.$$

$$\left.\sum_{k=1}^n T_k=T,\ S_k\in D((T_k\circ f_k)^c),\ \sum_{k=1}^n S_k=0\right\}.$$

(*ii*) $$D(\varphi^c)=\left\{\sum_{k=1}^n S_k\mid \exists T_k\in L^+(Y,Z),\ \sum_{k=1}^n T_k=T,\ S_k\in D((T_k\circ f_k)^c)\right\},$$

$$\varphi^c(U)=\min\left\{\sum_{k=1}^n (T_k\circ f_k)^c(S_k)\mid T_k\in L^+(Y,Z),\ \sum_{k=1}^n T_k=T,\right.$$

$$\left.S_k\in D((T_k\circ f_k)^c),\ \sum_{k=1}^n S_k=U\right\}.$$

(*iii*) $$\partial_\varepsilon\varphi(x)=\bigcup\left\{\sum_{k=1}^n \partial_{\varepsilon_k}(T_k\circ f_k)(x)\mid T_k\in L^+(Y,Z),\ \sum_{k=1}^n T_k=T,\ \varepsilon_k\geqslant 0,\right.$$

$$\left.\varphi(x)-\sum_{k=1}^n T_k(f_k(x))+\sum_{k=1}^n \varepsilon_k=\varepsilon\right\}.$$

Proof. Take $f:\bigcap_{k=1}^n F_k=F\to Y^n$, $f(x)=(f_1(x),\dots,f_n(x))$ and $g: Y^n\to Z$, $g(y_1,\dots,y_n)=T(y_1\vee\dots\vee y_n)$, Y^n being ordered by Q^n. g is an increasing sublinear operator. Moreover $0\in{}^i(f(F)-Y^n)=Y^n$ and $\partial g(0)=\{(T_1,\dots,T_n)\mid T_k\in L^+(Y,Z),\ \sum_{k=1}^n T_k=T\}$. The assertions follow then from the preceding corollary and Corollary 2.2.

Remark 2.2 The duality results in the preceding propositions are new for vectorial optimization (except the Fenchel duality in Corollary 2.2), as well as those assuring the existence of convex hulls. Also new, for this case, is the method to obtain the formulae for conjugate operators and ε-subdifferentials.

Remark 2.3. The formulae stated in Corollaries 2.3 and 2.4 are new for the vectorial case (for the scalar one see [11], [19]) and are dual to those in Theorem 2.4 and Corollary 2.2.

Remark 2.4. In [14] Theorem 2.8 (*ii*) and (*iii*) are established in the condition $f(F)\cap G^i\neq\varnothing$.

Remark 2.5. From Corollary 2.2 and Remark 3.2 one obtains Theorems 1 and 2 and Lemma 3 from [20], taking into account that $(0, \infty)(F_1-F_2)=X$ implies $0\in(\operatorname{co}F_1-\operatorname{co}F_2)^i$.

Remark 2.6. Corollary 2.2 (i) is nothing else but [1, Th. 6]. Taking into account the disccussion in Section 4, one obtains [8, Thms. 2.3, 2.4] and [30, Thms. 2,3].

An immediate consequence of Corollary 2.2 is the following.

Corollary 2.6. *Let $f: F\subset X\to Z$, $g: G\subset X\to Z$ be such that f and $-g$ are convex and $0\in{}^i(F-G)$. If $f(x)\geqslant g(x)$ for all $x\in F\cap G$ then there exist $T\in L(X, Z)$ and $z\in Z$ such that*

$$g(x)\leqslant Tx-z \text{ for all } x\in G,\ f(x)\geqslant Tx-z \text{ for all } x\in F. \tag{2.12}$$

In the sequel we give a necessary and sufficient condition for the existence of $T\in L(X, Z)$ and $z\in Z$ satisfying (2.12) (compare with Sandwich Theorem 4.3 in [31]).

Theorem 2.10. *Let $f: F\subset X\to Z$, $g: G\subset X\to Z$ be such that f, $-g$ are convex. The following statements are equivalent:*

(*i*) *there exists a convex operator $h: X\to Z$ with $h(0)\leqslant 0$ such that*

$$f(x_1)-g(x_2)\geqslant -h(x_1-x_2) \text{ for all } x_1\in F,\ x_2\in G,$$

(*ii*) *there exist $T\in L(X, Z)$ and $z\in Z$ satisfying* (2.12).

Proof. Taking $h=-T$, it is clear that $(ii)\Rightarrow(i)$. For the converse implication take $\bar h: X\times X\to Z$, $\bar h(x_1, x_2)=h(x_1-x_2)$ and $\bar k: F\times G\to Z$, $\bar k(x_1, x_2)=f(x_1)-g(x_2)$. For $\bar h$ and $-\bar k$ one applies the preceding corollary, and the conclusion follows.

Theorem 2.11. *Let $f: F\subset X\to Z$ be a sublinear operator, $P\subset X$, $Q\subset Y$ convex cones, $S\in L(X, Y)$ and $y_0\in Y$. Suppose that $F-P$ is a linear subspace and $y_0\in{}^i(S(F\cap P)-Q)$. Then*

$$x\geqslant 0,\ Sx\geqslant y_0\Rightarrow f(x)\geqslant z_0 \tag{2.13}$$

iff

$$\exists T_1\in L^+(X, Z),\ T_2\in L^+(Y, Z) \text{ such that } T_1+T_2\circ S\in\partial f(0) \text{ and } T_2y_0\geqslant z_0. \tag{2.14}$$

Proof. It is clear that $(2.14)\Rightarrow(2.13)$. For the converse implication take $\Phi: D\to Z$, $\Phi(x, y)=f(x)$, where $D=\{(x, y)\mid x\in F\cap P,\ Sx\in y_0+y+Q\}$. Φ is convex and $P_Y(D)=S(F\cap P)-Q-y_0$; therefore $0\in{}^iP_Y(D)$. Since $z_0\leqslant\inf\{\Phi(x, 0)\mid(x, 0)\in D\}$, from Theorem 2.3, there exists $T_2\in L(Y, Z)$ such that $(0, T_2)\in\tilde{D}$ and $z_0\leqslant-\Phi^c(0, T_2)$. But

$$\Phi^c(0, T_2)=\sup\{T_2y-\Phi(x, y)\mid(x, y)\in D\}=\sup\{T_2(Sx-y_0-y)-f(x)\mid x\in F\cap P,\ y\in Q\}=\sup\{T_2\circ Sx-f(x)\mid x\in F\cap P\}+\sup\{-T_2y\mid y\in Q\}-T_2y_0.$$

Therefore $T_2\in L^+(Y, Z)$, $T_2\circ Sx\leqslant f(x)$ for all $x\in F\cap P$ and $\Phi^c(0, T_2)=-T_2y_0$. If $I_P: P\to Z$, $I_P(x)=0$, then $T_2\circ S\in\partial(f+I_P)(0)=\partial f(0)+\partial I_P(0)=\partial f(0)-L^+(X, Z)$, which completes the proof.

Remark 2.7. Theorem 2.11 represents a generalization of the Farkas lemma analogous to the one in [26], but in different conditions. Theorem 2.11 provides a new condition in which Problem II in [7] has solution.

3. **Applications.** In this section we emphasize conditions in which $\inf P = \inf \overline{P}$ and an optimal solution for (PP) is also optimal for (RPP), respectively. Using these conditions and the duality results from Section 2, we reobtain the results for nonconvex programming in [16], [20]. We give also a Kuhn-Tucker type theorem.

Let $A \subset Y \times Z$ and suppose that $\inf\{z \mid (0, z) \in A\} = v$ exists. It is obvious that $v = \inf\{z \mid (0, z) \in \mathrm{co}\, A\}$ iff

$$(3.1) \qquad \forall (y_k, z_k) \in A,\ \lambda_k \geqslant 0,\ 1 \leqslant k \leqslant n, \sum_{k=1}^{n} \lambda_k = 1,\ \sum_{k=1}^{n} \lambda_k y_k = 0 \Rightarrow \sum_{k=1}^{n} \lambda_k z_k \geqslant v.$$

A sufficient condition for $v = \inf\{z \mid (0, z) \in \mathrm{co}\, A\}$ is

$$(3.2) \qquad \mathrm{co}\,(A - (0, v)) \subset \mathrm{cone}(A - (0, v)) \Leftrightarrow \mathrm{cone}\,(A - (0, v)) \text{ is convex.}$$

Indeed, if $(y_k, z_k) \in A,\ \lambda_k \geqslant 0,\ 1 \leqslant k \leqslant n, \sum_{k=1}^{n} \lambda_k = 1,\ \sum_{k=1}^{n} \lambda_k y_k = 0$ then $\sum_{k=1}^{n} \lambda_k (y_k, z_k - v) = \left(0, \sum_{k=1}^{n} \lambda_k z_k - v\right) \in \mathrm{co}(A - (0, v))$, so that, by (3.2), there exist $\lambda \geqslant 0$, $(y, z) \in A$ such that $\left(0, \sum_{k=1}^{n} \lambda_k z_k - v\right) = \lambda((y, z) - (0, v)) = \lambda(y, z - v)$. If $\lambda = 0$ then $\sum_{k=1}^{n} \lambda_k z_k = v \geqslant v$; if $\lambda > 0$ then $y = 0$, so that $z \geqslant v$ which implies $\sum_{k=1}^{n} \lambda_k z_k \geqslant v$. Therefore (3.1) holds. Also note that if (3.1) or (3.2) is satisfied for $v \leqslant \inf P$ then $v \leqslant \inf \overline{P}$. We emphasize especially the situation $A = P_{Y \times Z}\,(\mathrm{epi}\,\Phi)$, where $\Phi : D \subset X \times Y \to Z$ (as in Theorem 2.3). In this case conditions (3.1) and (3.2) become

$$(3.3) \qquad \forall (x_k, y_k) \in D,\ \lambda_k \geqslant 0,\ 1 \leqslant k \leqslant n, \sum_{k=1}^{n} \lambda_k = 1,\ \sum_{k=1}^{n} \lambda_k y_k = 0 \Rightarrow \sum_{k=1}^{n} \lambda_k \Phi(x_k, y_k) \geqslant v.$$

$$(3.4) \qquad \begin{cases} \forall (x_k, y_k) \in D,\ \lambda_k \geqslant 0,\ 1 \leqslant k \leqslant n, \sum_{k=1}^{n} \lambda_k = 1\ \exists \lambda \geqslant 0,\ (x, y) \in D : \sum_{k=1}^{n} \lambda_k y_k = y, \\ \sum_{k=1}^{n} \lambda_k \Phi(x_k, y_k) \geqslant \lambda \Phi(x, y) + (1 - \lambda) v, \end{cases}$$

respectively. Let us remark that (3.2)$\Rightarrow$(3.4)$\Rightarrow$(3.3), but the converse implications are generally not true. If x_0 is an optimal solution for (PP), i.e. $v = \Phi(x_0, 0)$ and one of the conditions (3.3) or (3.4) is valid then x_0 is also optimal for (RPP).

We rewrite conditions (3.3) and (3.4) for some situations from Section 2. Thus in the case of Theorem 2.4 the above conditions become respectively:

$$\forall y_k \in F,\ \lambda_k \geqslant 0,\ 1 \leqslant k \leqslant n, \sum_{k=1}^{n} \lambda_k = 1,\ \sum_{k=1}^{n} \lambda_k y_k \in R(S) \Rightarrow \sum_{k=1}^{n} \lambda_k f(y_k) \geqslant v,$$

$$\begin{cases} \forall y_k \in F,\ \lambda_k \geqslant 0,\ 1 \leqslant k \leqslant n,\ \sum_{k=1}^{n} \lambda_k = 1\ \exists \lambda \geqslant 0,\ \ y \in F \text{ with } \sum_{k=1}^{n} \lambda_k y_k - \lambda y \in R(S) \\ \text{and } \sum_{k=1}^{n} \lambda_k f(y_k) \geqslant \lambda f(y) + (1-\lambda) v. \end{cases}$$

For the case of Corollary 2.2 with $n=2$ condition (3.3) becomes

$$(3.5)\quad \begin{cases} \forall x_k \in F_1,\ y_k \in F_2,\ \lambda_k \geqslant 0,\ 1 \leqslant k \leqslant n, \sum_{k=1}^{n} \lambda_k = 1, \sum_{k=1}^{n} \lambda_k x_k = \sum_{k=1}^{n} \lambda_k y_k \Rightarrow \\ \sum_{k=1}^{n} \lambda_k f_1(x_k) + \sum_{k=1}^{n} \lambda_k f_2(y_k) \geqslant v, \end{cases}$$

while in the case of Theorem 2.7 (3.3) and (3.4) become

$$\forall x_k \in F,\ \lambda_k \geqslant 0,\ 1 \leqslant k \leqslant n, \sum_{k=1}^{n} \lambda_k = 1,\ \sum_{k=1}^{n} \lambda_k x_k \in N(S) \Rightarrow \sum_{k=1}^{n} \lambda_k f(x_k) \geqslant v,$$

$$\begin{cases} \forall x_k \in F,\ \lambda_k \geqslant 0,\ 1 \leqslant k \leqslant n, \sum_{k=1}^{n} \lambda_k = 1\ \exists \lambda \geqslant 0, x \in F:\ \sum_{k=1}^{n} \lambda_k x_k - \lambda x \in N(S), \\ \sum_{k=1}^{n} \lambda_k f(x_k) \geqslant \lambda f(x) + (1-\lambda) v. \end{cases}$$

Remark 3.1. The above conditions do not assure the existence of the convex hull of the respective functions.

Remark 3.2. Condition (3.5) is nothing else but condition (CI) from [20] for $f-v$ and $-g$.

Let now $f: F \subset X \to Z$ and $g: G \subset X \to Y$ with Y ordered by the convex cone Q. Let $D = F \cap G$. Consider the problem

$$(\mathrm{P}') \qquad \inf\{f(x) \mid g(x) \leqslant 0\}.$$

To problem (P′) corresponds the *Lagrangean*

$$(3.6) \qquad L: D \times L^+(Y, Z) \to Z,\ L(x, T) = f(x) + Tg(x).$$

The *saddle problem* associated with (P′) is

$$(\mathrm{SP}') \qquad \begin{cases} \textit{Find } (x_0, T_0) \in D \times L^+(Y, Z) \textit{ such that} \\ L(x_0, T) \leqslant L(x_0, T_0) \leqslant L(x, T_0) \quad \forall x \in D,\ T \in L^+(Y, Z). \end{cases}$$

Theorem 3.1. *Suppose that* $0 \in {}^i\mathrm{co}(g(D) + Q)$. *If* x_0 *is an* ε*-solution for* (P′) *and*

$$(3.7) \qquad \mathrm{co}\, B \subset \mathrm{cone}\, B$$

where $B = \{(g(x) + q, f(x) - v + z) \mid x \in D,\ z \geqslant 0\}$, v *being* $\inf \mathrm{P}'$. *Then*

$$(3.8) \quad \exists T \in L^+(Y, Z) \textit{ such that } Tg(x_0) + \varepsilon = \varepsilon_1 \geqslant 0 \textit{ and } 0 \in \partial_{\varepsilon_1}(f + T \circ g)(x_0).$$

Proof. Let $\Phi: D(\Phi) \to Z$, $\Phi(x, y) = f(x)$, where $D(\Phi) = \{(x, y) \mid x \in D,\ g(x) \leqslant y\}$. We have

(3.9) $A = P_{Y\times Z}(\text{epi}\,\Phi) = \{(g(x)+q, f(x)+z) \mid x\in D,\ q\in Q,\ z\geqslant 0\} = B+(0, v)$

and $P_Y(D(\Phi)) = g(D)+Q$. (3.7) and (3.9) show that $\text{co}(A-(0,v))\subset \text{cone}(A-(0,v))$. Therefore the relaxed primal problem has the same value as (P') Since $0\in{}^i\text{co}\,P_Y(D(\Phi))$, by Theorem 2.3, there exists $T\in L(Y,Z)$ such that

$$v = -\Phi^c(0, -T) = -\sup\{-Ty - f(x) \mid g(x)\leqslant y\} =$$
$$= -\sup\{-Tg(x)-f(x) \mid x\in D\} - \sup\{-Tq \mid q\in Q\}.$$

Therefore $T\in L^+(Y,Z)$, $\sup\{-Tq \mid q\in Q\}=0$ and $v=-(f+T\circ g)^c(0)$, so that

$$f(x_0)\leqslant v+\varepsilon = -(f+T\circ g)^c(0)+\varepsilon \Leftrightarrow 0\leqslant f(x_0)+Tg(x_0)+$$
$$+(f+T\circ g)^c(0)\leqslant \varepsilon + Tg(x_0) = \varepsilon_1.$$

Hence $0\in\partial_{\varepsilon_1}(f+T\circ g)(x_0)$ which together with $Tg(x_0)+\varepsilon=\varepsilon_1\geqslant 0$ show that the conclusions of the theorem hold.

Remark 3.3. If (3.8) holds for an admissible x_0 then x_0 is ε-solution for (P') Taking $\varepsilon=0$, and therefore $v=f(x_0)$ in the conditions of Theorem, 3.1, it follows $Tg(x_0)=0$ and (x_0, T) is a saddle point for L given by (3.6) i.e. (x_0, T) is solution for (SP').

Remark 3.4. If x_0 is an optimal solution for (P'), the existence of T such that (x_0, T) be a solution for (SP') is proved in [16] in the conditions: $0\in{}^i(g(D)+Q)$ and $g(D)+Q$ is expansive. In [16] there are also indicated some exemples of nonconvex problems for which (3.7) holds.

Corollary 3.1. *(i) If f and g are convex and $0\in{}^i(g(D)+Q)$, then $x_0\in D$ is an ε-solution of* (P') *iff x_0 is admissible and* (3.8) *holds.*

(ii) If moreover $0\in{}^i(F-G)$ then x_0 is an ε-solution for (P') *iff x_0 is admissible and*

(3.10) $$\exists T\in L^+(Y,Z) \text{ and } \varepsilon_1,\ \varepsilon_2\geqslant 0 \text{ such that } Tg(x_0)+\varepsilon = \varepsilon_1+\varepsilon_2 \text{ and } 0\in\partial_{\varepsilon_1}f(x_0)+\partial_{\varepsilon_2}(T\circ g)(x).$$

Proof. (*i*) was already remarked, while (3.8) and (3.10) are equivalent by Corollary 2.2.

Corollary 3.2. *Let $g: G\subset X\to Y$ be a convex operator and $\bar{g}: \bar{G}\to Z$, $\bar{g}(x)=0$, where $\bar{G}=\{x \mid g(x)\leqslant 0\}$. If $0\in{}^i(g(G)+Q)$ then*

(3.11) $$\partial_\varepsilon \bar{g}(x) = \bigcup\{\partial_{\varepsilon_1}(T\circ g)(x) \mid T\in L^+(Y,Z),\ Tg(x)+\varepsilon=\varepsilon_1\geqslant 0\}.$$

Proof. Let $U\in\partial_\varepsilon\bar{g}(x)$ and take $f=-U$ in Corollary 3.1 (*ii*). Then x is an ε-solution for the appropriate (P'). Therefore there exists $T\in L^+(Y,Z)$ such that $Tg(x)+\varepsilon=\varepsilon_1\geqslant 0$ and $U\in\partial_{\varepsilon_1}(T\circ g)(x)$. The converse inclusion being obviously true, (3.11) holds.

Let consider now the problem

(P'') $$\inf\{f(x) \mid g_k(x)\leqslant 0,\ 1\leqslant k\leqslant n\},$$

where $f: F \subset X \to Z$, $g_k : G_k \subset X \to Y_k$, $1 \leqslant k \leqslant n$ are convex operators, Y_k being ordered by the convex cones Q_k. Let $\bar{G}_k = \{x | g_k(x) \leqslant 0\}$ and $\bar{g}_k : \bar{G}_k \to Z$, $\bar{g}_k(x) = 0$, $1 \leqslant k \leqslant n$.

Theorem 3.2. *Let* f, g_k, $1 \leqslant k \leqslant n$ *be as above. Suppose that*

$$(3.12) \quad \begin{cases} F, G_1, G_2, \ldots, G_n \text{ are in general position and} \\ (0, \ldots, 0) \in {}^i\{(g_1(x) + q_1, \ldots, g_n(x) + q_n) \mid x \in F \cap G_1 \cap \ldots \cap G_n,\ q_k \in \\ \qquad \in Q_k,\ 1 \leqslant k \leqslant n\} \end{cases}$$

or

$$(3.13) \quad \begin{array}{l} F, \bar{G}_1, \ldots, \bar{G}_n \text{ are in general position and} \\ 0 \in {}^i(g_k(G_k) + Q_k),\ 1 \leqslant k \leqslant n. \end{array}$$

Then x_0 *is an* ε*-solution for* (P″) *iff* x_0 *is admissible and*

$$(3.14) \quad \begin{cases} \exists T_k \in L^+(Y_k, Z),\ \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n \geqslant 0 \text{ such that } \sum_{k=1}^n T_k g_k(x_0) + \varepsilon = \sum_{k=1}^n \varepsilon_k \\ \text{and } 0 \in \partial_{\varepsilon_0} f(x_0) + \partial_{\varepsilon_1}(T_1 \circ g_1)(x_0) + \ldots + \partial_{\varepsilon_n}(T_n \circ g_n)(x_0). \end{cases}$$

Proof. If condition (3.12) is satisfied, take $g : \bigcap_{k=1}^n G_k \to Y_1 \times \ldots \times Y_n$, $g(x) = (g_1(x), \ldots, g_n(x))$ and apply Corollary 3.1 (*ii*). Let condition (3.13) be satisfied. x_0 is an ε-solution for (P″) iff $0 \in \partial_\varepsilon(f + \sum_{k=1}^n \bar{g}_k)(x_0)$. Applying now Corollary 2.2 and taking into account (3.11), we obtain that (3.14) holds. It is clear that (3.14) implies x_0 is an ε-solution.

4. **The Continuous Case.** In this section we show that from the preceding algebraic results it is possible to obtain their continuous versions. This means that the linear spaces under consideration are t.l.s. and that in the definitions of conjugate operators and ε-subdifferentials continuous linear operators instead of linear ones are taken. The notations are the same as in the preceding sections and the linear and order structures are keeped (so Z has the l.u.b.p.); in addition all the spaces are separated topological linear spaces. Throughout this section we also assume that Z is *normal*, i.e. there exists a base $\mathcal{W}$ of symmetric neighborhoods of the origin of Z such that $W = (W + Z^+) \cap (W - Z^+)$ $\forall W \in \mathcal{W}$. The operator $f: F \subset X \to Z$ is said to be *continuous* at x_0 if $x_0 \in \operatorname{int} F$ and f is continuous in the usual sense at x_0. Concerning the continuity of convex operators we have the following results whose proofs follow from [3], [4] and can be also found in [28].

Theorem 4.1. *Let* $f: F \subset X \to Z$ *be a convex operator.*

(*i*) f *is continuous at* x_0 *iff*

$$\exists z \in Z \ \ \forall W \in \mathcal{W} \ \ \exists V \in \mathcal{V}(X) \ \ \forall x \in x_0 + V : f(x) \in z + W - Z^+.$$

(*ii*) *If* f *is continuous at* $x_0 \in \operatorname{int} F$ *then* f *is continuous on* $\operatorname{int} F$.

(iii) If

$$\exists z \in Z,\ V \in \mathcal{V}(X)\ \ \forall x \in x_0 + V : f(x) \leqslant z$$

then f is continuous at x_0. *If* $\operatorname{int} Z^+ \neq \varnothing$ *and f is continuous at* x_0 *then the above condition holds.*

(iv) Let $g : G \subset X \to Z$ *be continuous at* $x_0 \in \operatorname{int} G$ *and* $A \subset X \times Z$ *be convex such that* $A \subset \operatorname{epi} g$. *If there exists* $\inf\{z \mid (x_0, z) \in A\}$ *then* $\varphi_A : P_X(A) \to Z$, $\varphi_A(x) = \inf\{z \mid (x, z) \in A\}$ *exists and* φ_A *is continuous on* $\operatorname{int} P_X(A) \supset \operatorname{int}(\operatorname{co} G)$.

(v) Let, $g : G \subset X \to Z$. *If g is continuous at some* $x_0 \in \operatorname{int} G$ *then* $D(g^c) \subset B(X, Z)$ *and therefore* $\partial_\varepsilon g(x) \subset B(X, Z)$ *for every* $x \in G$, $\varepsilon \geqslant 0$.

Theorem 4.2. *Let X be a locally convex space and assume that* $\operatorname{int} Z^+ \neq \varnothing$. *If* $T_0 \in B(X_0, Z)$, *where* $X_0 \subset X$ *is a linear subspace with the induced topology, then there exists* $T \in B(X, Z)$ *such that* $T|_{X_0} = T_0$.

Proof. Since $T_0 \in B(X_0, Z)$, T_0 is convex and continuous at 0. According to Theorem 4.1 (iii), there exist $z \in Z$ and $p : X \to R^+$-a continuous seminorm such that $x \in X_0$, $p(x) \leqslant 1$ imply $T_0 x \leqslant z$. Therefore $z \geqslant 0$ and $T_0 x \leqslant p(x) z$ for $x \in X_0$. Taking $\bar{p} : X \to Z$, $\bar{p}(x) = p(x) z$, $\bar{p}$ is a (continuous) sublinear operator, and therefore we can apply [31, Th. 2.1] to get $T \in L(X, Z)$ such that $T|_{X_0} = T_0$ and $Tx \leqslant \bar{p}(x)$ for all $x \in X$. From Theorem 4.1 (v), we have $T \in B(X, Z)$.

In the following we shall emphasize the conditions which assure, in the convex case, the validity of continuous versions of the results from Sections 2 and 3. For the nonconvex case, we must take into account Theorem 4.1 *(iv)*, or we have to write these conditions for the convex hulls of the operators. So, all the operators are assumed to be convex.

To obtain the continuous version of Th. 2.3 we strengthen the condition $0 \in {}^i P_Y(\operatorname{co} D)$ in such a way that $\varphi_{\operatorname{co} A} = \varphi_A$ be continuous at 0. From Theorem 4.1 such conditions are

$$\exists z \in Z\ \ \forall W \in \mathcal{W}\ \exists V \in \mathcal{V}(Y)\ \forall y \in V\ \exists x \in X : \Phi(x, y) \in z + W - Z^+, \tag{4.1}$$

$$\exists z \in Z,\ V \in \mathcal{V}(Y)\ \ \forall y \in V\ \ \exists x \in X : \Phi(x, y) \leqslant z, \tag{4.2}$$

$$\exists x_0 \in X \text{ such that } \Phi(x_0, .) \text{ is continuous at } 0. \tag{4.3}$$

Let us note that to obtain the formulae for conjugate operators and subdifferentials, we considered also operators $\bar{\Phi}$, $\bar{\Phi}(x, y) = \Phi(x, y) - Ux$. To get that the corresponding φ_A, be continuous at 0 at least for all operators $U \in B(X, Z)$, conditions (4.1) and (4.2) must be rewritten as

$$\begin{aligned} &\exists z \in Z\ \ x \in X\ \ \forall W \in \mathcal{W},\ U \in \mathcal{V}(X)\ \ \exists V \in \mathcal{V}(Y),\ \ \forall y \in \\ &\in V\ \exists x' \in x + U :\ \Phi(x', y) \in z + W - Z^+ \end{aligned} \tag{4.1'}$$

and

$$\exists z \in Z,\ x \in X\ \ \forall U \in \mathcal{V}(X), \exists V \in \mathcal{V}(Y),\ \ \forall y \in V\ \exists x' \in x + U : \Phi(x', y) \leqslant z. \tag{4.2'}$$

Note that if (4.3) holds than it holds also for every $\bar{\Phi}$ with $U \in L(X, Z)$.

It is clear that a necessary condition in order that φ_A be continuous at 0, is $0 \in P_Y(D)^i$. The next result shows that this condition is sufficient to satisfy (4.1′) and therefore that φ_A^- be continuous at 0 for $U \in B(X, Z)$, in enough general situations.

Theorem 4.3. *Let X, Z be Fréchet spaces, Y be a barrelled space and $\Phi : D \subset X \times Y \to Z$ a convex operator with closed epigraph. If $0 \in P_Y(D))^i$ and $\inf\{\Phi(x, 0) \mid (x, 0) \in D\}$ exists, then $\varphi(y) = \inf\{\Phi(x, y) \mid (x, y) \in D\}$ exists for every $y \in P_Y(D)$ and φ is continuous at 0. Moreover condition (4.1′) holds.*

The proof is given in [28] and follows also from [4].

From Theorem 4.3 it follows also that the continuous version of Corollary 2.2 is valid for $n = 2$ if X, Z are Fréchet spaces and $0 \in (F_1 - F_2)^i$.

We remember the form of Φ in Theorems 2.4, 2.5, 2.8, namely $\Phi(x, y) = f(Sx - y)$, $\Phi(x, y) = f(x, Sx - y)$, $\Phi(x, y) = g(f(x) - y)$, respectively, so the conditions (4.1), (4.1′), (4.2), (4.2′) (4.3) can be easily written for each case. We mention that, even in the case $Z = R$, condition (4.3) was used, in most cases, to establish duality results, formulae for subdifferentials or conjugate operators (see [2], [5], [9], [25], [27]). In [19, Th. 19] it is provided a condition, more general than (4.3). We also remark that in the case of Theorem 3.1 condition (4.3) is nothing else but the Slater condition ($\exists x_0 \in F$ such that $-g(x_0) \in \operatorname{int} Q$).

It seems that the most general conditions in which the Fenchel duality theorem in the case $Z = R$ is proved are provided by [18, Th. 3.14.20]. We shall reestablish it in the present context, with a simpler proof.

Theorem 4.4. *Let X be a Fréchet space, $\operatorname{int} Z^+ \neq \emptyset$ and $f_k : F_k \subset X \to Z$, $k = 1, 2$ be convex operators. Assume the following conditions holds : a) $\operatorname{ri} F_1 \cap \operatorname{ri} F_2 \neq \emptyset$, b) f_k is continuous relative to ${}^L F_k$ at some point of $\operatorname{ri} F_k$, $k = 1, 2$, c) ${}^L F_1 + {}^L F_2$ is closed. If $\inf\{f_1(x) + f_2(x) \mid x \in F_1 \cap F_2\}$ exists, then*

$$\inf\{f_1(x) + f_2(x) \mid x \in F_1 \cap F_2\} = \max\{-f_1^c(T) - f_2^c(-T) \mid T \in B(X, Z) \cap \tilde{F}_1 \cap -\tilde{F}_2\}. \tag{4.4}$$

Proof. Without loss of generality we can suppose $0 \in \operatorname{ri} F_1 \cap \operatorname{ri} F_2$, and therefore ${}^L F_1 = X_1$ and ${}^L F_2 = X_2$ are closed linear subspaces. Note that $f^c(T_1) = f^c(T_2)$ for $T_1, T_2 \in L(X, Z)$ with $T_1 - T_2|_{L_{D(f)}} = 0$. On the other hand, taking into account Theorem 4.1 (v) and b), if $T \in \tilde{F}_k = D(f_k^c)$ then $T|_{X_k}$ is continuous. Also note that $T|_{X_1 + X_2}$ is continuous if $T|_{X_k}$, $k = 1, 2$, are so. This follows immediately because X_1, X_2 and $X_1 + X_2$ are Fréchet spaces and $B: X_1 \times X_2 \to X_1 + X_2$, $B(x_1, x_2) = x_1 + x_2$ is a surjective continuous linear operator, and therefore it is open. From Corollary 2.2 we have that $\inf\{f_1(x) + f_2(x) \mid x \in F_1 \cap F_2\} = \max\{-f_1^c(T) - f_2^c(-T) \mid T \in L(X, Z), T \in \tilde{F}_1 \cap - \tilde{F}_2\}$. Let T realize the maximum in the right-hand side. Then $T_0 = T|_{X_1 + X_2}$ is continuous. Applying Theorem 4.2 we get $T' \in B(X, Z)$ such that $T'|_{X_1 + X_2} = T_0$. From the above discussion, $f_1^c(T') = f_1^c(T)$ and $f_2^c(T') = f_2^c(T)$ so that (4.4) holds.

At the end of this section we give the continuous version of Theorem 3.2 which constitutes a generalization of [30, Th. 6]. In this case X, Y_k, Z are s.t.l.s., the cones $Q_k \subset Y_k$, $1 \leqslant k \leqslant n$, are convex and closed and $\operatorname{int} Q_k \neq \emptyset$, and Z^+ is closed.

Theorem 4.5. *Let* $f: F \subset X \to Z$ *and* $g_k : G_k \subset X \to Y_k$, $1 \leqslant k \leqslant n$, *be convex operators such that* g_k *are continuous on* $\operatorname{int} G_k \neq \emptyset$. *Assume there exists* $x_0 \in F$ *such that* $g_k(x_0) \in -\operatorname{int} Q_k$, $1 \leqslant k \leqslant n$. *Then* $\bar{x}$ *is an optimal solution for* (P'') *iff* $\bar{x}$ *is admissible and there exist* $T_k \in B^+(Y_k, Z)$ *with* $T_k g_k(\bar{x}) = 0$, $1 \leqslant k \leqslant n$ *and*

$$0 \in \partial f(\bar{x}) + \sum_{k=1}^{n} \partial (T_k \circ g_k)(\bar{x}). \tag{4.5}$$

Moreover, if g_k *are Gâteaux differentiable at* $\bar{x}$, *then condition* (4.5) (*for* $T_k \in B^+(Y_k, Z)$) *is equivalent with* $\sum_{k=1}^{n} T_k \circ g'_k(\bar{x}) \in -\partial f(\bar{x})$.

Proof. We must only show that for a convex operator $g : G \to Y$

$$\partial g(\bar{x}) = \{g'(\bar{x}) + S \mid S \in L(X, Y),\ R(S) \subset Q \cap -Q\} \tag{4.6}$$

if g is Gâteaux differentiable at $\bar{x}(\in \operatorname{int} G)$. Because g is Gâteaux differentiable at $\bar{x}$, it follows that $\bar{x} \in \operatorname{int} G$. Let $x \in X$ be such that $\bar{x} + x \in G$; there exists $t_0 > 0$ such that $0 \leqslant t \leqslant t_0$ implies $\bar{x} - tx \in G$. For $t \in (0, t_0)$, taking into account the convexity of g, we have that $(g(\bar{x} - tx) - g(\bar{x}))/(-t) \leqslant g(\bar{x} + x) - g(\bar{x})$. Letting $t \downarrow 0$ we obtain $g'(\bar{x})(x) \leqslant g(x + \bar{x}) - g(\bar{x})$ for all $x \in G - \bar{x}$. Hence $g'(\bar{x}) \in \partial g(\bar{x})$ and therefore the inclusion $\supset$ holds in (4.6). Let now $S \in L(X, Y)$, $S \in \partial g(\bar{x})$; we have $Sx \leqslant g(x + \bar{x}) - g(\bar{x})$ for all $x \in G - \bar{x}$. Since $\bar{x} \in \operatorname{int} G$, for $x \in X$ there exists $t_0 > 0$ such that $\bar{x} + tx \in G$ for $t \in (0, t_0)$. For such a t we have $S(tx) \leqslant g(\bar{x} + tx) - g(\bar{x}) \Leftrightarrow Sx \leqslant (g(\bar{x} + tx) - g(\bar{x}))/t$. Taking the limit as $t \downarrow 0$ in the above relation, we obtain $Sx \leqslant g'(\bar{x})x$ for all $x \in X$, and therefore $R(S - g'(\bar{x})) \subset Q \cap -Q$. Thus (4.6) holds. If $T \in B^+(Y, Z)$ then $T \circ g$ is convex and $(T \circ g)'(\bar{x}) = T \circ g'(\bar{x})$. Since $Z^+ \cap -Z^+ = \{0\}$, we have that $\partial(T \circ g)(\bar{x}) = \{T \circ g'(\bar{x})\}$.

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