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PROCEEDINGS OF
12-th Baikal International
Conference

OPTIMIZATION METHODS

AND THEIR APPLICATIONS

June, 24 – July, 1, 2001
Irkutsk, Baikal

Plenary Lectures

Irkutsk
2001

Editors Prof. V. Bulatov, Prof. V. Baturin

Weak sharp minima, well-behaving functions and global error bounds for convex inequalities in Banach spaces

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1 Introduction

The notion of set of weak sharp minima was introduced by Burke and Ferris in [4] as a generalization of the notion of sharp minimum point. They gave several properties in finite dimensional spaces, most of them being equivalent for convex functions. As we shall see, this notion is strongly related to that of well-conditioned functions, notion studied by Lemaire [9], Penot [11] and Cornejo, Jourani and Zălinescu [5].

Auslender and Crouzeix [3] introduced the class of well-behaving asymptotically convex functions. For the study of this class they introduced several functions of a real variable, whose positivity characterizes this class; see also [12] for a more general setting. It turns out that these functions are also related to global error bounds for convex inequalities. There are many papers concerning global error bounds for convex inequalities, the most recent seems to be that of Klatte and Li [8]. Most of the results on the last two topics are established in finite dimension, and many proofs are sufficiently lengthy.

Our aim in this paper is to show that almost all the results still hold in reflexive Banach spaces, many proofs being simpler than those known till now. We also add some new properties or characterizations.

Throughout the paper $(X, \|\cdot\|)$ is a normed vector space, $\Lambda(X)$ is the class of proper convex functions, while $\Gamma(X)$ denotes the set of functions $f \in \Lambda(X)$ which are lower semicontinuous (lsc). For $f \in \Lambda(X)$, $\operatorname{dom} f := \{x \in X \mid f(x) < \infty\}$ and

$$f'(x,u) := \lim_{t \to 0^+} \frac{f(x+tu) - f(x)}{t}$$

denote the domain of f and the directional derivative of f at $x \in \operatorname{dom} f$ in the direction $u \in X$, respectively. We shall also use the notations $[f \le \lambda]$ and $[f < \lambda]$ for the level set $\{x \in X \mid f(x) \le \lambda\}$ and for the strict level set $\{x \in X \mid f(x) < \lambda\}$ of f, respectively; the infimum of f on X is denoted by $\inf f$, while the set where the infimum is attained is denoted by $\operatorname{argmin} f$. When $A \subset X$, ι_A denote the indicator function of A, which is defined by $\iota_A(x) := 0$ if $x \in A$, $\iota_A(x) := \infty$ for $x \in X \setminus A$. As usual, when $f \in \Lambda(X)$,

$$f^* : X^* \to \overline{\mathbb{R}}, \quad f^*(x^*) := \sup\{\langle x, x^* \rangle - f(x) \mid x \in X\},$$

and

$$\partial f(x) := \{x^* \in X^* \mid \langle y - x, x^* \rangle \le f(y) - f(x) \ \forall\, y \in X\}$$

denote the conjugate and the subdifferential of f; the function $f_\infty : X \to \overline{\mathbb{R}}$ defined by

$$f_\infty(u) := \sup\left\{t^{-1}(f(x+tu) - f(x)) \mid t > 0\right\} = \lim_{t \to \infty} t^{-1}(f(x+tu) - f(x)),$$

denotes the asymptotic function of $f \in \Gamma(X)$, where $x \in \operatorname{dom} f$ is an arbitrary element.

Let $C \subset X$ be a nonempty convex set; $P_C(x)$ denotes the set of those $y \in C$ for which $\|x - y\| = d_C(x) := d(x, C) := \inf\{\|x - c\| \mid c \in C\}$. As usual, we set $d(x, \emptyset) := +\infty$. It is well-known (see f.i. [14, Th. 2.10.3]) that $y \in P_C(x)$ iff $F(x - y) \cap N(C, y) \neq \emptyset$, where $F : X \rightrightarrows X^*$ is the duality mapping of X and

$$N(C, x) := \{x^* \in X^* \mid \langle y - x, x^* \rangle \leq 0 \; \forall y \in C\}$$

is the normal cone of C at $x \in C$. From this characterization one obtains easily that

$$y \in P_C(y + tu) \quad \forall y \in C, \; \forall u \in F^{-1}(N(C, y)), \; \forall t \geq 0. \tag{1}$$

Moreover, $P_C(x) \neq \emptyset$ for every $x \in X$ when X is reflexive and C is closed. Recall that

$$\partial d_C(x) = U_{X^*} \cap N(C, x), \quad \forall x \in C, \tag{2}$$

where U_X denotes the closed unit ball of X; S_X denotes the closed unit sphere of X and $B_X := U_X \setminus S_X$. Denoting by $\mathcal{C}(C, x)$ the closed cone generated by $C - x$ for $x \in C$ (which is also convex because C is convex) we also have

$$(d_C)'(x, u) = d(u, \mathcal{C}(C, x)), \quad \forall x \in C, \quad \forall u \in X. \tag{3}$$

2 Weak sharp minima

Let $f \in \Lambda(X)$ and $S \subset X$ be a nonempty set. Burke and Ferris [4] say that S is a *set of weak sharp minima* if there exists $\alpha > 0$ such that

$$f(x) \geq f(\overline{x}) + \alpha \cdot d_S(x), \quad \forall x \in X, \; \forall \overline{x} \in S; \tag{4}$$

when $S = \{\overline{x}\}$ one says that $\overline{x}$ is a *sharp minimum point* of f. It is clear that $S \subset \operatorname{argmin} f \subset \operatorname{cl} S$ if S is a set of weak sharp minima for f. Hence $S = \operatorname{argmin} f$ when S is closed. This is a reason for taking $S = \operatorname{argmin} f$ in the sequel. In this case (4) becomes

$$f(x) \geq \inf f + \alpha \cdot d_S(x) \quad \forall x \in X,$$

i.e. f is ψ_α-conditioned, with $\psi_\alpha(t) = \alpha t$ for $t \geq 0$; see [5] for the definition and properties. Throughout this section we assume that $f \in \Gamma(X)$ and $S := \operatorname{argmin} f \neq \emptyset$.

The next result collects several characterizations for $\operatorname{argmin} f$ to be a set of weak sharp minima.

Theorem 2.1 Let $\alpha > 0$ and consider the following statements:

(i) $\forall x \in X \; : \; f(x) \geq \inf f + \alpha \cdot d_S(x)$;

(ii) $\forall x^* \in \alpha U_{X^*} \; : \; f^*(x^*) = f^*(0) + \iota_S^*(x^*)$;

(iii) $\forall \overline{x} \in S, \; \forall (x, x^*) \in \operatorname{gr} \partial f \; : \; \langle x - \overline{x}, x^* \rangle \geq \alpha \cdot d_S(x)$;

(iv) $d(0, \partial f(X \setminus S)) \geq \alpha$;

(v) $\forall x^* \in \alpha B_{X^*} \; : \; \partial f^*(x^*) \subset S$ [$\partial f^*(x^*)$ being taken for the duality (X, X^*)];

(vi) $\forall x \in X,\ \forall \overline{x} \in P_S(x)\ :\ f'(\overline{x}, x - \overline{x}) \geq \alpha \cdot d_S(x)$;

(vii) $\forall \overline{x} \in S,\ \forall u \in F^{-1}(N(S, \overline{x}))\ :\ f'(\overline{x}, u) \geq \alpha \|u\|$;

(viii) $\forall \overline{x} \in S,\ \forall u \in F^{-1}(N(S, \overline{x})) \cap \operatorname{cone}(\operatorname{dom} f - \overline{x})\ :\ f'(\overline{x}, u) \geq \alpha \|u\|$;

(ix) $\forall \overline{x} \in S,\ \forall u \in X\ :\ f'(\overline{x}, u) \geq \alpha \cdot d(u, \mathcal{C}(S, \overline{x}))$.

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Leftrightarrow$ (v), (i) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Leftrightarrow$ (viii) and (vi) $\Rightarrow$ (ix). If X is a Banach space then (v) $\Rightarrow$ (i). Moreover, if X is a reflexive Banach space then (vii) $\Rightarrow$ (i) and (ix) $\Rightarrow$ (i); hence all nine conditions are equivalent in this case.

Proof. The implications (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (v) and (v) $\Rightarrow$ (i) (the last one for X a Banach space) are established in [5, Th. 5.1]. The implications (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) follow immediately from the implications (ii) $\Rightarrow$ (iv) $\Rightarrow$ (vi) of [5, Th. 5.2] (taking $\psi := \psi_\alpha$ defined above), while (iv) $\Leftrightarrow$ (v) is obvious. The equivalence (vii) $\Leftrightarrow$ (viii) is also obvious because $\operatorname{dom} f'(\overline{x}, \cdot) = \operatorname{cone}(\operatorname{dom} f - \overline{x})$. The equivalence of conditions (i), (vi) and (ix) is obtained by Burke and Ferris [4] for $\dim X < \infty$; we omit the proof in our case.

(vi) $\Rightarrow$ (vii) Let $\overline{x} \in S$ and $u \in F^{-1}(N(S, \overline{x}))$. By (1) we have that $\overline{x} \in P_S(\overline{x} + u)$. From (vi) we get the conclusion.

Assume now that X is a reflexive Banach space.

(vii) $\Rightarrow$ (i) Let $x \in \operatorname{dom} f$ and take $\overline{x} \in P_S(x)$. As noticed in the introduction $F(x - \overline{x}) \cap N(S, \overline{x}) \neq \emptyset$. Hence $x - \overline{x} \in F^{-1}(N(S, \overline{x}))$, and so $f'(\overline{x}, x - \overline{x}) \geq \alpha \|x - \overline{x}\| = \alpha d_S(x)$. The conclusion follows from the inequality $f'(\overline{x}, x - \overline{x}) \leq f(x) - f(\overline{x})$. □

Using the preceding theorem we get the following estimate for the best coefficient α in (4).

Corollary 2.2 Let X be a Banach space and $f \in \Gamma(X)$ be such that $S := \operatorname{argmin} f \neq \emptyset$. Then

$$\inf_{x \in X \setminus S} \frac{f(x) - \inf f}{d(x, S)} = d(0, \partial f(X \setminus S)).$$

Moreover, if X is reflexive, then

$$\begin{aligned} \inf_{x \in X \setminus S} \frac{f(x) - \inf f}{d(x, S)} &= \inf \left\{ f'\left(\overline{x}, \frac{x - \overline{x}}{\|x - \overline{x}\|} \right) \,\middle|\, x \in \operatorname{dom} f \setminus S,\ \overline{x} \in P_S(x) \right\} \\ &= \inf \left\{ f'(\overline{x}, u) \mid \overline{x} \in S,\ u \in S_X \cap F^{-1}(N(S, \overline{x})) \right\} \\ &= \inf \left\{ \frac{f'(\overline{x}, u)}{d(u, \mathcal{C}(S, \overline{x}))} \,\middle|\, \overline{x} \in S,\ u \in X \setminus \mathcal{C}(S, \overline{x}) \right\}. \end{aligned}$$

Proof. It is obvious that anyone of the numbers appearing in the statement of the corollary is nonnegative. By the equivalences (i) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii) $\Leftrightarrow$ (ix) $\Leftrightarrow$ (iv) of the preceding theorem, if one of them is positive then all of them are equal. The conclusion follows. □

Remark 2.1 For every nonempty closed convex set $A \subset X$ we have

$$d(u, \mathcal{C}(A, x)) \geq \|u\|, \quad \forall x \in A,\ \forall u \in F^{-1}(N(A, x)). \tag{5}$$

Indeed, we have that $A = \operatorname{argmin} d_A$ and $d_A(x) \geq \inf d_A + 1 \cdot d_A(x)$ for every $x \in A$. From the implication (i) $\Rightarrow$ (vii) of Theorem 2..1 and the formula (3) one gets (5).

As we shall see later the notion of weak sharp minima is closely related to that of global error bound for (convex) inequality systems.

3 Well-behaved asymptotically convex functions

Auslender and Crouzeix [3] say that $f \in \Gamma(X)$ has *good asymptotic behaviour*, or is *well-behaved asymptotically*, if

$$(d\,(0, \partial f(x_n))\,) \to 0 \;\Rightarrow\; (f(x_n)) \to \inf f.$$

The class of functions $f \in \Gamma(X)$ satisfying the above condition will be denoted by $\mathcal{F}$. As example of well-behaved function is that of a coercive convex function in reflexive Banach spaces.

In the next result we give a characterization of the elements of $\mathcal{F}$.

Proposition 3.1 Let $f \in \Gamma(X)$. Then

$$f \in \mathcal{F} \;\Leftrightarrow\; \overline{r}_f(t) > 0,\; \forall\, t > \inf f,$$

where, for $t \in \mathbb{R}$,

$$\overline{r}_f(t) := \inf\{d\,(0, \partial f(x)) \mid x \in X,\; f(x) \geq t\} = d(0, \partial f(X \setminus [f < t])). \tag{6}$$

Proof. It is obvious that

$$\begin{aligned} f \notin \mathcal{F} &\Leftrightarrow \exists\,(x_n) \subset X \;:\; (d(0, \partial f(x_n))) \to 0,\; (f(x_n)) \not\to \inf f \\ &\Leftrightarrow \exists\,(x_n) \subset X,\; \exists\, t > \inf f \;:\; (d(0, \partial f(x_n))) \to 0 \text{ and } f(x_n) \geq t,\; \forall\, n \in \mathbb{N} \\ &\Leftrightarrow \exists\, t > \inf f \;:\; \overline{r}_f(t) = 0. \end{aligned}$$

Hence the conclusion holds. □

It is obvious that $\overline{r}_f$ is nondecreasing. Moreover, for $t \leq \inf f$ we have that $\overline{r}_f(t) = d\,(0, \operatorname{Im} \partial f)$; in particular, by the Brøndsted-Rockafellar theorem, $\overline{r}_f(\inf f) = d\,(0, \operatorname{dom} f^*)$ when X is a Banach space.

In order to give other expressions for $\overline{r}_f(t)$ we introduce several related quantities.

Proposition 3.2 Let $f \in \Gamma(X)$ and $x \in \operatorname{dom} f$ with $f(x) > \inf f$. Consider the following numbers:

$$\begin{aligned} \gamma_1(x) &:= \sup\left\{ \frac{f(x) - f(y)}{\|x - y\|} \;\middle|\; y \in [f < f(x)] \right\}, \\ \gamma_2(x) &:= \sup\left\{ \frac{f(x) - t}{d(x, [f \leq t])} \;\middle|\; t < f(x) \right\}, \\ \gamma_3(x) &:= \sup\left\{ -f'\left(x, \frac{y - x}{\|y - x\|}\right) \;\middle|\; y \in [f < f(x)] \right\}, \\ \gamma_4(x) &:= \sup\{-f'(x, u) \mid u \in S_X\}, \\ \gamma_5(x) &:= d(0, \partial f(x)), \\ \gamma_6(x) &:= \sup\{-f'(x, -u) \mid u \in S_X \cap N([f \leq f(x)], x)\}. \end{aligned}$$

Then $\gamma_1(x) = \gamma_2(x) = \gamma_3(x) = \gamma_4(x) = \gamma_5(x) \in]0, \infty]$. Moreover, if f is continuous at x and X is a Hilbert space, identified with X^* by Riesz theorem, then $\gamma_1(x) = \gamma_6(x)$.

Before giving the proof we mention a simple property which will be used repeatedly in this section.

Lemma 3.3 Let $f \in \Gamma(X)$, $t \in \mathbb{R}$, and $x, y \in \operatorname{dom} f$ such that $f(y) < t < f(x)$. Then there exists $z \in]x, y[$ such that $f(z) = t$. In particular $\|y - z\| < \|y - x\|$ and $\|x - z\| < \|y - x\|$.

Proof. Of course, the mapping $\varphi : \mathbb{R} \to \overline{\mathbb{R}}$, $\varphi(\lambda) := f(\lambda x + (1 - \lambda)y)$ is a lsc convex function and $[0, 1] \subset \operatorname{dom} \varphi$. By a well-known property of lsc convex function on $\mathbb{R}$, φ is continuous on $[0, 1]$. As $\varphi(0) < t < \varphi(1)$, there exists $\overline{\lambda} \in]0, 1[$ such that $\varphi(\overline{\lambda}) = t$. Taking $z := \overline{\lambda}x + (1 - \overline{\lambda})y$ the conclusion follows. □

Proof of Proposition 3..2. Let us denote, during the proof, the set $[f < f(x)]$ by S. It is obvious that $\gamma_1(x) > 0$, $\gamma_2(x) > 0$ and $\gamma_5(x) > 0$ (because $S \neq \emptyset$). Since $f'(x, y-x) \leq f(y) - f(x) < 0$ for $y \in S$, we have also that $\gamma_3(x) > 0$, and so $\gamma_4(x) > 0$ as, obviously, $\gamma_4(x) \geq \gamma_3(x)$.

Let first $t < f(x)$ be such that $[f \leq t] \neq \emptyset$, and show that

$$\sup_{y \in [f \leq t]} \frac{f(x) - f(y)}{\|x - y\|} = \frac{f(x) - t}{d(x, [f \leq t])}. \tag{7}$$

Consider $(y_n) \subset [f \leq t]$ such that $(\|x - y_n\|) \to d(x, [f \leq t])$. Then

$$\sup_{y \in [f \leq t]} \frac{f(x) - f(y)}{\|x - y\|} \geq \frac{f(x) - f(y_n)}{\|x - y_n\|} \geq \frac{f(x) - t}{\|x - y_n\|} \to \frac{f(x) - t}{d(x, [f \leq t])},$$

which proves the inequality $\geq$ in (7). Conversely, let $y \in [f \leq t]$. Using Lemma 3..3 we get $\lambda \in [0, 1[$ such that $f(y') = t$, where $y' := \lambda x + (1 - \lambda)y$. Of course, $\|x - y'\| = (1 - \lambda)\|x - y\|$ and $t \leq \lambda f(x) + (1 - \lambda)f(y)$, whence $(1 - \lambda)(f(x) - f(y)) \leq f(x) - t$. It follows that

$$\frac{f(x) - f(y)}{\|x - y\|} = \frac{(1 - \lambda)(f(x) - f(y))}{(1 - \lambda)\|x - y\|} \leq \frac{f(x) - t}{\|x - y'\|} \leq \frac{f(x) - t}{d(x, [f \leq t])},$$

which proves the inequality $\leq$ in (7). Hence (7) holds.

Now, by (7), we have

$$\gamma_1(x) = \sup_{t < f(x)} \sup_{y \in [f \leq t]} \frac{f(x) - f(y)}{\|x - y\|} = \sup_{t < f(x)} \frac{f(x) - t}{d(x, [f \leq t])} = \gamma_2(x).$$

(Note that the values of t with $[f \leq t] = \emptyset$ influence neither $\gamma_1(x)$ nor $\gamma_2(x)$.)

Let $y \in S$ and take $u := (y - x)/\|y - x\|$. Since $f'_+(x, y - x) \leq f(y) - f(x)$, we have that $(f(x) - f(y))/\|x - y\| \leq -f'(x, u)$. It follows that $\gamma_1(x) \leq \gamma_3(x)$. On the other hand, we have that $x + tu \in S$ for $t \in]0, \|y - x\|^{-1}]$, and so

$$-f'(x, u) = \sup \left\{ \frac{f(x) - f(x + tu)}{\|x - (x + tu)\|} \;\middle|\; t \in]0, \|y - x\|^{-1}] \right\} \leq \gamma_1(x),$$

which implies that $\gamma_3(x) \leq \gamma_1(x)$. Hence $\gamma_1(x) = \gamma_3(x)$.

It was observed above that $0 < \gamma_3(x) \leq \gamma_4(x)$. Let $u \in S_X$. Suppose first that there exists $\bar{t} > 0$ such that $y := x + \bar{t}u \in S$; then $-f'(x,u) = -f'(x, \frac{y-x}{\|y-x\|}) \leq \gamma_3(x)$. In the contrary case $f(x+tu) \geq f(x)$ for every $t > 0$, and so $f'(x,u) \geq 0$, whence $-f'(x,u) \leq 0 < \gamma_3(x)$. Hence $\gamma_3(x) = \gamma_4(x)$.

Assume that $\gamma \in \mathbb{R}$ is such that $\gamma_4(x) \leq \gamma$ (thus $\gamma > 0$). It follows that $-f'(x,u) \leq \gamma \|u\|$ for all $u \in X$, and so $0 \leq \gamma \|u\| + f'(x,u)$ for $u \in X$. Hence 0 is a minimum point for $\gamma \|\cdot\| + f'(x,\cdot)$, and so $0 \in \partial (\gamma \|\cdot\| + f'(x,\cdot))(0)$. But

$$\partial (\gamma \|\cdot\| + f'(x,\cdot))(0) = \gamma U_{X^*} + \partial f'(x,\cdot)(0) = \gamma U_{X^*} + \partial f(x). \tag{8}$$

Therefore $\gamma U_{X^*} \cap \partial f(x) \neq \emptyset$, which implies that $d(0, \partial f(x)) \leq \gamma$. Conversely, if $\gamma \geq d(0, \partial f(x))$ (hence $\gamma > 0$ because $0 \notin \partial f(x)$), then $\gamma U_{X^*} \cap \partial f(x) \neq \emptyset$ (since the norm of X^* is w^*-lsc and $\partial f(x)$ is w^*-closed), and so $0 \in \gamma U_{X^*} + \partial f(x)$. By (8) we obtain that 0 is a minimum point of $\gamma \|\cdot\| + f'(x,\cdot)$, which shows that $\gamma_4(x) \leq \gamma$. Hence $\gamma_4(x) = \gamma_5(x)$.

Assume now that X is a Hilbert space (identified with its topological dual) and f is continuous at x; then

$$f'(x,u) = \max\{(u \mid x^*) \mid x^* \in \partial f(x)\} \quad \forall\, u \in X, \tag{9}$$

and so, by [13, Prop. 5.4], $K := N([f \leq f(x)], x) = \mathbb{R}_+ \partial f(x)$. It is clear that $\gamma_6(x) \leq \gamma_5(x)$. Let $y \in S$ and take $u := (x-y)/\|x-y\|$; it was observed above that $\gamma := -f'(x,-u) > 0$. Then for $x^* \in \partial f(x)$ we have that $(u \mid x^*) = -(-u \mid x^*) \geq -f'(x,-u) \geq \gamma$. Take $\overline{u} = P_K(u)$. By the well-known characterization of best approximations in Hilbert spaces, we have that $(u - \overline{u} \mid v - \overline{u}) \leq 0$ for every $v \in K$. Replacing v by tv with $t \geq 0$ we get $(u - \overline{u} \mid v) \leq 0 = (u - \overline{u} \mid \overline{u})$ for all $v \in K$. In particular $\gamma \leq (u \mid x^*) \leq (\overline{u} \mid x^*)$ for all $x^* \in \partial f(x)$, and so $\overline{u} \neq 0$. Since P_K is Lipschitz with Lipschitz constant 1 and $0 = P_K(0)$, we have that $\|\overline{u}\| \leq 1$. Taking $u_0 := \overline{u}/\|\overline{u}\|$, we have that $u_0 \in S_X \cap N([f \leq f(x)], x)$ and, by (9), $f'(x,-u) \leq -\gamma$. Hence $\gamma_6(x) \geq \gamma$, whence $\gamma_6(x) \geq \gamma_3(x)$. □

With the aid of the numbers $\gamma_1(x)$–$\gamma_6(x)$ we introduce the functions

$$r_f^i :]\inf f, \infty[\to [0,\infty], \quad r_f^i(t) = \inf\{\gamma_i(x) \mid x \in [f=t]\}, \quad i \in \overline{1,6}.$$

Of course, if $[f=t] = \emptyset$ then $r_f^i(t) = +\infty$.

From the preceding result we obtain immediately

Corollary 3.4 Let $f \in \Gamma(X)$. Then $r_f^1(t) = r_f^2(t) = r_f^3(t) = r_f^4(t) = r_f^5(t)$ for every $t > \inf f$. Moreover, if X is a Hilbert space identified with its dual and f is continuous at every point of $[f=t] \neq \emptyset$ then $r_f^1(t) = r_f^6(t)$.

The numbers $r_f^j(0)$ with $1 \leq j \leq 6$ and $0 > \inf f$ are introduced by Klatte and Li in [8] for f finite valued and $X = \mathbb{R}^n$. In [8] some of these quantities are expressed directly using the functions f_i when $f := \max\{f_1, \ldots, f_n\}$; they follow easily from the formulae (23) and (24) mentioned below.

In the sequel we shall denote by r_f anyone of the functions r_f^1–r_f^5 defined above. From the preceding proposition we obtain that

$$\overline{r}_f(t) = \inf\{r_f(s) \mid s \geq t\}, \quad \forall\, t > \inf f. \tag{10}$$

Consider now the function

$$l_f : [\inf f, \infty[\to [0, \infty], \quad l_f(t) := \inf \left\{ \frac{f(x) - t}{d(x, [f \le t])} \;\middle|\; x \in \operatorname{dom} f \setminus [\, f \le t] \right\}. \quad (11)$$

As usual, we take $\inf \emptyset = +\infty$. Hence $l_f(\inf f) = 0$ if $\inf f$ is not attained; if $\inf f$ is attained then $\operatorname{argmin} f$ is a set of weak sharp minima if, and only if, $l_f(\inf f) > 0$. Even for $t > \inf f$ the number l_f is related to weak sharp minima. Indeed, for $t > \inf f$, taking $h : X \to \overline{\mathbb{R}}$ with $h(x) := \max\{f(x) - t, 0\}$ we have that $\inf h = 0$ and $\operatorname{argmin} h = [f \le t]$. Then $[f \le t]$ is a set of weak sharp minima for h if, and only if, $l_f(t) > 0$. This remark suggests having formulae for $l_f(t)$ similar to those in Corollary 2..2.

Proposition 3.5 Let X be a Banach space and $f \in \Gamma(X)$. Assume that $t \in [\inf f, \infty[$ is such that $[f \le t] \ne \emptyset$. Then

$$l_f(t) = d(0, \partial f(X \setminus [f \le t])). \quad (12)$$

Assume now that X is reflexive. Then

$$\begin{aligned} l_f(t) &= \inf \left\{ f'\left(y, \frac{x - y}{\|x - y\|}\right) \;\middle|\; x \in \operatorname{dom} f \setminus [f \le t],\ y \in P_{[f \le t]}(x) \right\} & (13)\\ &= \inf \left\{ f'(y, u) \mid y \in [f = t],\ u \in S_X \cap F^{-1}\left(N([f \le t], y)\right\} \right. & (14)\\ &= \inf \left\{ \frac{f'(y, u)}{d(u, \mathcal{C}([f \le t], y))} \;\middle|\; y \in [f = t],\ u \in X \setminus \mathcal{C}([f \le t], y) \right\}. & (15) \end{aligned}$$

Moreover, if $t > \inf f$ then

$$l_f(t) = k_f(t) := \inf \left\{ f'\left(y, \frac{u}{\|u\|}\right) \;\middle|\; y \in [f = t],\ u \in F^{-1}(\partial f(y)) \right\}. \quad (16)$$

Proof. If $t = \inf f$ the conclusion is given by Corollary 2..2. Assume that $t > \inf f$ and denote by $l_1(t)$, $l_2(t)$, $l_3(t)$, and $l_4(t)$ the quantities in the right-hand side of (12)–(15), respectively. Consider $h : X \to \overline{\mathbb{R}}$ with $h(x) := \max\{f(x) - t, 0\}$; we have that $\inf h = 0$ and $\operatorname{argmin} h = [f \le t]$. The equality of $l_f(t)$ and $l_1(t)$ follows directly from Corollary 2..2 because $U := X \setminus [f \le t]$ is open and $f|_U = h|_U + t$, and so $\partial f(x) = \partial h(x)$ for every $x \in U$. Since, obviously, $l_f(t) = l_h(0)$, we apply again Corollary 2..2 for h. Hence, in order to obtain (13)–(15), it is sufficient to observe the following:

a) Let $x \in \operatorname{dom} f \setminus [f \le t]$ and $y \in P_{[f \le t]}(x)$. Then, by Lemma 3..3, we have that $f(y) = t$. Since $y + s(x - y) \in \operatorname{dom} f \setminus [f \le t]$ for every $s \in]0, 1]$, we obtain that $f'(y, x - y) = h'(y, x - y)$.

b) Let $y \in [f = t]$ and $u \in S_X \cap F^{-1}\left(N([f \le t], y)\right.$. Then $y \in P_{[f \le t]}(y + su)$ for every $s > 0$. It follows that either $y + su \notin \operatorname{dom} f$ for every $s > 0$, and so $f'(y, u) = h'(y, u) = \infty$, or $y + s_0 u \in \operatorname{dom} f$ for some $s_0 > 0$; in this case $f(y) = t$ and again $f'(y, x - y) = h'(y, x - y)$.

c) Let $y \in [f = t]$ and $u \in X \setminus \mathcal{C}([f \le t], y)$. Because $u \notin \mathcal{C}([f \le t], y)$, we have that $y + su \notin [f \le t]$ for every $s > 0$. Hence $f(y + su) = h(y + su) + t$ for all $s > 0$. If $y + su \notin \operatorname{dom} f$ for $s > 0$ then $f'(y, u) = h'(y, u) = \infty$. Assume that

$y + s_0 u \in \operatorname{dom} f$ for some $s_0 > 0$; as above we obtain that $f(y) = t$ and so, once again, $f'(y, x-y) = h'(y, x-y)$.

Let us prove now (16). Because $\partial f(y) \subset N([f \le t], y)$ for every $y \in [f = t]$, we have that $k_f(t) \ge l_3(t) = l_f(t)$. Let now $x \in \operatorname{dom} f \setminus [f \le t]$ and take $y \in P_{[f \le t]}(x)$; of course, $y \ne x$. By Lemma 3..3 we have that $y \in [f = t]$. Since y is a solution of the the problem

min $\frac{1}{2}\|z-x\|^2$ subject to $f(z) - t \le 0$,

and the Slater condition holds, we have that $0 \in \partial(\frac{1}{2}\|\cdot - x\|^2 + \lambda(f-t))$ for some $\lambda \ge 0$, i.e. y is a minimum point of $\frac{1}{2}\|\cdot - x\|^2 + \lambda(f-t)$. Hence $F(x-y) \cap \partial(\lambda f)(y) \ne \emptyset$. Assuming that $\lambda = 0$, we obtain that $\|y-x\| \le \|z-x\|$ for every $z \in \operatorname{dom} f$. Taking $z := x$ we obtain the contradiction $\|y-x\| \le 0$, i.e. $y = x$. Therefore $\lambda > 0$. Then $u := \lambda^{-1}(x-y) \in F^{-1}(\partial f(y))$. It follows that $l_f(t) = l_2(t) \ge k_f(t)$. The proof is complete. □

Relation (12) shows that l_f is nondecreasing on $[\inf f, \infty[$ when X is a Banach space. In fact l_f has this property in any normed vector space.

Proposition 3.6 Let $f \in \Gamma(X)$ and $\inf f \le t_1 < t_2 < \infty$. Then

$$l_f(t_1) \le l_f(t_2) \quad \text{and} \quad l_f(t_1) \le r_f(t_2).$$

In particular l_f is nondecreasing on $[\inf f, \infty[$. Moreover, if X is a reflexive Banach space then $r_f(t) \le l_f(t)$ for every $t \in]\inf f, \infty[$, and so r_f is nondecreasing on $]\inf f, \infty[$; hence $r_f = \overline{r}_f$ in this case.

Proof. Let first $\overline{x} \in \operatorname{dom} f$ and take $t_1, t_2 \in \mathbb{R}$ such that $t_1 < t_2 < f(\overline{x})$; then

$$\frac{f(\overline{x}) - t_1}{d(\overline{x}, [f \le t_1])} \le \frac{f(\overline{x}) - t_2}{d(\overline{x}, [f \le t_2])}. \tag{17}$$

It is obvious that (17) holds if $[f \le t_1] = \emptyset$. Assume that $[f \le t_1] \ne \emptyset$ and take $x \in [f \le t_1]$. Let $\overline{\lambda} := (f(\overline{x}) - t_2)/(f(\overline{x}) - t_1) \in]0,1[$. Then

$$f\left(\overline{\lambda} x + (1-\overline{\lambda})\overline{x}\right) \le \overline{\lambda} f(x) + (1-\overline{\lambda}) f(\overline{x}) \le \overline{\lambda} t_1 + (1-\overline{\lambda}) f(\overline{x}) = t_2.$$

Hence $\overline{\lambda} x + (1-\overline{\lambda})\overline{x} \in [f \le t_2]$, and so $d(\overline{x}, [f \le t_2]) \le \overline{\lambda}\,\|\overline{x} - x\|$. Taking the infimum for $x \in [f \le t_1]$ we get (17). The inequality (17) is obvious if $\overline{x} \notin \operatorname{dom} f$ and $[f \le s_1] \ne \emptyset$.

Using (17) we obtain that

$$\begin{aligned} \inf_{x \in \operatorname{dom} f \setminus [f \le t_1]} \frac{f(x) - t_1}{d(x, [f \le t_1])} &\le \inf_{x \in \operatorname{dom} f \setminus [f \le t_2]} \frac{f(x) - t_1}{d(x, [f \le t_1])} \\ &\le \inf_{x \in \operatorname{dom} f \setminus [f \le t_2]} \frac{f(x) - t_2}{d(x, [f \le t_2])}, \end{aligned}$$

and

$$\inf_{x \in \operatorname{dom} f \setminus [f \le t_1]} \frac{f(x) - t_1}{d(x, [f \le t_1])} \le \inf_{x \in [f = t_2]} \frac{f(x) - t_1}{d(x, [f \le t_1])} \le \inf_{x \in [f = t_2]} \sup_{t < t_2} \frac{f(x) - t}{d(x, [f \le t])}.$$

The first inequality shows that $l_f(t_1) \le l_f(t_2)$, while the second one shows that $l_f(t_1) \le r_f(t_2)$.

Assume now that X is reflexive and take $t \in]\inf f, \infty[$. If there is no $x \in \operatorname{dom} f$ with $f(x) > t$ then $l_f(t) = \infty \ge r_f(t)$. Let $x \in \operatorname{dom} f$ with $f(x) > t$. Because X is reflexive and $[f \le t]$ is closed and convex, there exists $y \in P_{[f\le t]}(x)$. Using Lemma 3..3 we obtain that $f(y) = t$. Let $z \in [f < t]$. Taking $\mu := \|x-y\| / (\|x-y\| + \|y-z\|) \in]0,1[$, we have that $f((1-\mu)x + \mu z) \ge t$. In the contrary case $u := (1-\mu)x + \mu z \in [f < t]$. Then, using again Lemma 3..3, there exists $\lambda \in]0,1[$ with $f(\lambda u + (1-\lambda)x) = t$. It follows that

$$\|x-y\| \le \lambda \|u-x\| < \|u-x\| = \mu \|x-z\| = \frac{\|x-y\| \cdot \|x-z\|}{\|x-y\| + \|y-z\|},$$

whence the contradiction $\|x-y\| + \|y-z\| < \|x-z\|$. Therefore $t \le f((1-\mu)x + \mu z) \le (1-\mu)f(x) + \mu f(z)$, whence $\|x-y\| (t - f(z)) \le \|y-z\| (f(x) - t)$. Hence

$$\frac{t - f(z)}{\|y-z\|} \le \frac{f(x)-t}{\|x-y\|} = \frac{f(x)-t}{d(x,[f\le t])}.$$

Since $z \in [f < t]$ is arbitrary, we get $r_f(t) = r_f^1(t) \le l_f(t)$. □

Corollary 3.7 Let X be a reflexive Banach space, $f \in \Gamma(X)$ and $t > \inf f$. If f is Gâteaux differentiable on $[f = t] \subset \operatorname{int}(\operatorname{dom} f)$ then

$$l_f(t) = r_f(t) = \inf\{\|\nabla f(x)\| \mid x \in [f = t]\}. \tag{18}$$

Proof. If $[f = t] = \emptyset$ then, obviously, $r_f(t) = l_f(t) = \infty$, and so (18) holds. Let $[f = t]$ be nonempty and take $x_0 \in [f < t]$ and $x \in [f = t]$. Since $x \in \operatorname{int}(\operatorname{dom} f)$ we obtain that $]x_0, x] \subset \operatorname{int}(\operatorname{dom} f)$. As f is continuous on $\operatorname{int}(\operatorname{dom} f)$, it follows that $]x_0, x[\subset \operatorname{int}[f \le t]$. Because $x \in [f = t]$, $x \notin \operatorname{int}[f \le t]$; hence, by a separation theorem applied for x and $[f \le t]$, we get $u^* \in S_{X^*} \cap N([f \le t], x)$. Let $u \in F^{-1}(u^*)$. Taking into account relation (14) we get $\|\nabla f(x)\| \ge \langle u, \nabla f(x)\rangle = f'(x, u)$. It follows that

$$\begin{aligned} r_f(t) &= r_f^5(t) = \inf\{\|\nabla f(x)\| \mid x \in [f = t]\} \\ &\ge \inf\left\{f'(x,u) \mid x \in [f = t],\ u \in S_X \cap F^{-1}(N([f \le t], x))\right\} = l_f(t). \end{aligned}$$

Using Proposition 3..6 we obtain that (18) holds. □

We conclude this section with a characterization of the elements of $\mathcal{F}$ which follows easily from Proposition 3..6 and relation (10).

Theorem 3.8 Let X be a reflexive Banach space and $f \in \Gamma(X)$. Then

$$f \in \mathcal{F} \;\Leftrightarrow\; r_f(t) > 0 \;\forall t \in]\inf f, \infty[\;\Leftrightarrow\; l_f(t) > 0 \;\forall t \in]\inf f, \infty[.$$

The numbers $l_f(t)$, $k_f(t)$ and $r_f^5(t)$ were introduced by Auslender and Crouzeix in [3]. Propositions 3..1, 3..6, relation (16) of Proposition 3..5 and Theorem 3..8 were stated and proved in [3] for $X = \mathbb{R}^n$ under the supplementary condition that $\operatorname{dom} f$ is relatively open; in [1] Auslender, Cominetti and Crouzeix remove this supplementary condition on $\operatorname{dom} f$, simplify several proofs and mention that the above mentioned results hold in reflexive Banach spaces.

4 Global error bound for convex inequality systems

Let $g \in \Lambda(X)$; consider $C = [g \leq 0]$ the solution set of the inequality

$$g(x) \leq 0, \quad x \in X; \tag{19}$$

One says that the global error bound holds for (19) if C is nonempty and there exists $\alpha > 0$ such that

$$d_C(x) \leq \alpha \cdot [g(x)]_+ \quad \forall\, x \in X, \tag{20}$$

where $\gamma_+ := \max(\gamma, 0)$ for $\gamma \in \overline{\mathbb{R}}$.

Taking $h = \max(g, 0)$ we have that $\inf h = 0$ (under our assumption that $C \neq \emptyset$) and $C = \operatorname{argmin} h$. Hence the global error bound for C holds if, and only if, C is a set of weak sharp minima for h. On the other hand (20) is equivalent to $l_g(0) \geq \alpha^{-1}$, and so the global error bound for C holds if, and only if, $[g \leq 0] \neq \emptyset$ and $l_g(0) > 0$.

Taking into account the above discussion, Proposition 3..5 can be used to obtain the best estimate for the constant α in (20). The formula (14) for $l_g(0)$ is established by Lewis and Pang [10, Cor. 1] when $X = \mathbb{R}^n$.

In the next result we collect several sufficient conditions for the existence of $\alpha > 0$ verifying (20).

Theorem 4.1 Let $g \in \Gamma(X)$ be such that $C := [g \leq 0] \neq \emptyset$. Consider the following conditions:

(i) there exists $u \in X$ such that $g_\infty(u) < 0$;

(ii) there exists $\eta > 0$ such that $\sup_{x \in [g \leq 0]} d(x, [g \leq -\eta]) < \infty$;

(iii) there exists $\eta > 0$ such that $\sup_{x \in [g = 0]} d(x, [g \leq -\eta]) < \infty$;

(iv)
$$\inf_{\eta > 0} \sup_{x \in [g=0]} \inf_{y \in [g \leq -\eta]} \frac{\|x - y\|}{-g(y)} = \inf_{\eta > 0} \sup_{x \in [g=0]} \frac{d(x, [g \leq -\eta])}{\eta} < \infty; \tag{21}$$

(v) $0 > \inf g$ and $r_g(0) > 0$;

(vi) $l_g(0) > 0$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv) $\Rightarrow$ (v) and (ii) $\Rightarrow$ (vi). If g is not constant on $\operatorname{dom} g$ and $[g \leq 0] \subset \operatorname{int}(\operatorname{dom} g)$ then (iii) $\Rightarrow$ (ii). If X is a reflexive Banach space, then (v) $\Rightarrow$ (vi); moreover, if g is not constant, $[g = 0] \subset \operatorname{int}(\operatorname{dom} g)$ and g is Gâteaux differentiable on $[g = 0]$ then (vi) $\Rightarrow$ (v) and $l_g(0) = r_g(0)$.

Proof. First note that the equality in (21) follows from (7) when $[g = 0] \neq \emptyset$. When $[g = 0] = \emptyset$, both sides of the equality are $-\infty$.

The implications (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv) are obvious.

(i) $\Rightarrow$ (ii) Let us take $u \in S_X$ such that $g_\infty(u) < 0$ and $\eta > 0$. Consider $x \in [g \leq 0]$. Taking $\gamma := -\eta / g_\infty(u) > 0$, we have that

$$g(x + \gamma u) \leq g(x) + \gamma g_\infty(u) \leq \gamma g_\infty(u) = -\eta,$$

and so $d(x,[g\le -\eta])\le \|x+\gamma u-x\|=\gamma<\infty$.

(iii) $\Rightarrow$ (ii) Assume that g is not constant on $\operatorname{dom} g$ and $[g\le 0]\subset \operatorname{int}(\operatorname{dom} g)$. We shall show that

$$\sup\{d(x,[g\le -\eta])\mid x\in[g=0]\}=\sup\{d(x,[g\le -\eta])\mid x\in[g\le 0]\}>0 \tag{22}$$

for any $\eta>0$.

If $[g<0]=\emptyset$ then $[g\le 0]=[g=0]$ and (22) is obvious. Assume that $g(x_0)<0$ and take $x\in\operatorname{dom} g$ with $g(x)\ne g(x_0)$; we may assume that $g(x_0)<g(x)$. If $g(x)\ge 0$, by Lemma 3..3, there exists $y\in\,]x_0,x]$ such that $g(y)=0$. Assume that $g(x)<0$. Since $t^{-1}\left(g(x_0+t(x-x_0))-g(x_0)\right)\ge g(x)-g(x_0)>0$ for $t\ge 1$, there exists $t>1$ such that $g(x_0+t(x-x_0))>0$. The set $\Lambda:=\{t\ge 1\mid g(x_0+t(x-x_0))\le 0\}$ is a closed bounded interval; take $\bar t:=\max\Lambda$. Since $x_0+\bar t(x-x_0)\in[g\le 0]\subset\operatorname{int}(\operatorname{dom} g)$, there exists $t>\bar t$ such that $x_0+t(x-x_0)\in\operatorname{dom} g$. It follows that $g(x_0+t(x-x_0))>0$, and so necessarily $x_0+\bar t(x-x_0)\in[g=0]$. Hence there exists $y\in[g=0]$ such that $x\in\,]x_0,y[$.

Let show (22) for $\eta>0$. This relation is obvious if $[g\le -\eta]=\emptyset$. Assume that $[g\le -\eta]\ne\emptyset$ and take $x\in[g<0]\setminus[g\le -\eta]$. Consider $x_0\in[g\le -\eta]$. As above, there exist $y\in[g=0]$ and $\lambda\in\,]0,1[$ such that $x=\lambda y+(1-\lambda)x_0$. Take now $z\in[g\le -\eta]$; then $\lambda z+(1-\lambda)x_0\in[g\le -\eta]$, and so

$$d(x,[g\le -\eta])\le \|x-\lambda z-(1-\lambda)x_0\|=\lambda\,\|y-z\|\le\|y-z\|,$$

whence $d(x,[g\le -\eta])\le d(y,[g\le -\eta])$. It follows that $\sup_{x\in[g=0]}d(x,[g\le -\eta])\ge \sup_{x\in[g\le 0]}d(x,[g\le -\eta])$. As the converse inequality is obvious, (22) holds. Hence (iii) $\Rightarrow$ (ii) in our case.

(iv) $\Rightarrow$ (v) If $0=\inf g$ then $C=[g=0]\ne\emptyset$ and $[g\le -\eta]=\emptyset$ for any $\eta>0$. Hence $d(x,[g\le -\eta])=\infty$ for $x\in[g=0]$ and $\eta>0$, whence the contradiction $\inf_{\eta>0}\sup_{x\in[g=0]}\eta^{-1}d(x,[g\le -\eta])=\infty$. Hence $0>\inf g$. The conclusion then follows from the relation $r_g(0)=r_g^2(0)$ in Corollary 3..4 and the obvious inequality

$$\inf_{\eta>0}\sup_{x\in[g=0]}\frac{d(x,[g\le -\eta])}{\eta}\ge \sup_{x\in[g=0]}\inf_{\eta>0}\frac{d(x,[g\le -\eta])}{\eta}=\frac{1}{r_g^2(0)}.$$

(ii) $\Rightarrow$ (vi) Because $\sup_{x\in[g\le 0]}d(x,[g\le -\eta])<\infty$, we have that $[g\le -\eta]\ne\emptyset$. Consider $0<\mu<\eta$. Since $[g\le -\mu]\subset[g\le -\eta]$, we obtain that $\sup_{x\in[g\le -\mu]}d(x,[g\le -\eta])<\infty$. From the implication (ii) $\Rightarrow$ (v) with 0 replaced by $-\mu$ we obtain that $r_g(-\mu)>0$. Using Proposition 3..6 we obtain that $l_g(0)\ge r_g(-\mu)>0$.

Assume now that X is a reflexive Banach space. Then the implication (v) $\Rightarrow$ (vi) is true because, by Proposition 3..6, $l_g(0)\ge r_g(0)$.

(vi) $\Rightarrow$ (v) Assume that g is not constant, $l_g(0)>0$ and g is Gâteaux differentiable on $[g=0]\subset\operatorname{int}(\operatorname{dom} g)$. It is sufficient to show that $0>\inf g$; then the conclusion follows from Corollary 3..7. If $[g=0]=\emptyset$ then $0>\inf g$ (because $C\ne\emptyset$). Assume that $[g=0]\ne\emptyset$; if $[g<0]=\emptyset$ then every $\overline{x}\in C$ is a minimum point of g, and so $0\in\partial g(\overline{x})=\{\nabla g(\overline{x})\}$. Assuming that g is 0 on $\operatorname{dom} g$ then $[g=0]=\operatorname{dom} g=X$ ($\operatorname{dom} g$ being closed and open), a contradiction. It follows that $\operatorname{dom} g\setminus[g\le 0]\ne\emptyset$, and so, by relation (13) in Proposition 3..5, $l_g(0)=0$. Hence $[g<0]\ne\emptyset$. □

When $g:=\max_{i\in\overline{1,m}}g_i$ with $g_i\in\Lambda(X)$ for $1\le i\le m$, several conditions of the preceding theorem can be formulated in terms of the functions g_i. Let us denote the set $\{j \in \overline{1,m} \mid g_j(x) = g(x)\}$ by $I(x)$ for $x \in \operatorname{dom} g$. Assume that $g_i \in \Lambda(X)$ is continuous on $\operatorname{int}(\operatorname{dom} g_i)$ and $\emptyset \neq [g \leq 0] \subset \operatorname{int}(\operatorname{dom} g_i)$ for every $i \in \overline{1,m}$. Then, by [14, Cor. 2.7.4], we have that

$$\partial g(x) = \operatorname{co}\left(\bigcup\nolimits_{i \in I(x)} \partial g_i(x)\right) \quad \forall\, x \in [g \leq 0], \tag{23}$$

and so, because g and g_i are continuous at x,

$$g'(x,u) = \max_{i \in I(x)} \max\{\langle u, x^*\rangle \mid x^* \in \partial g_i(x)\} \quad \forall\, x \in [g \leq 0],\ \forall\, u \in X. \tag{24}$$

The condition $r_g(0) > 0$ in the preceding theorem may be described in several ways, taking into account that $r_g = r_g^i$ for $1 \leq i \leq 5$. One of them (corresponding to r_g^5) is

$$0 \notin \operatorname{cl}\left(\bigcup\nolimits_{x \in [g=0]} \operatorname{co}\big(\bigcup\nolimits_{i \in I(x)} \partial g_i(x)\big)\right),$$

another one (corresponding to r_g^4) being

$$\inf_{x \in [g=0]} \sup_{u \in S_X} \min_{i \in I(x)} \min_{x^* \in \partial g_i(x)} \langle u, x^*\rangle > 0.$$

The implications (i) $\Rightarrow$ (vi) and (ii) $\Rightarrow$ (vi) (for $g = f + \iota_A$ with f finite and continuous and A closed and convex) of Theorem 4..1 are obtained by Deng in [6] and [7], respectively, while the other ones can be found in [8] (for finite valued functions on $\mathbb{R}^n$). Lewis and Pang [10, Cor. 2] obtained, when $X = \mathbb{R}^n$ and $[g = 0] \subset {}^i(\operatorname{dom} g)$, that (i) $\Rightarrow r_g^5(0) > 0 \Rightarrow$ (vi) and that $\inf g < 0 \wedge g \in \mathcal{F} \Rightarrow r_g^5(0) > 0$.

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