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On Zero Duality Gap and the Farkas Lemma for Conic Programming

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Recently S. A. Clark published an interesting duality result in linear conic programming dealing with a convex cone that is not closed in which the usual (algebraic) dual problem is replaced by a topological dual with the aim of having zero duality gap under certain usual hypotheses met in mathematical finance. We present some examples to show that an extra condition is needed to reach a conclusion; this supplementary condition is also provided. We also give counterexamples for three results on hedging prices and simple proofs for two known solvability results (see Propositions 4.1 and 4.2).

Key words: conic programming; counterexample; duality gap; Farkas lemma; hedging prices
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1. Introduction. It is well known that the Farkas lemma outside the polyhedral case can be given in an asymptotic way when closedness conditions are not imposed; see (11) and Jeyakumar [10]. Attempts were made to obtain the existence of solutions of the problem $Ax = b$, $x \geq 0$, without using closedness conditions.

A similar problem was considered for the case of duality in linear conic programming. This is not a surprise because such duality results can be obtained by applying a variant of the Farkas lemma. Recently, Clark [6] obtained a duality theorem for linear conic programming problems without using closedness conditions but in which the algebraic dual is replaced by the so-called topological dual. In this note; we provide a reformulation of the primal and (topological) dual problems using just a closed linear subspace and its orthogonal; we provide sufficient conditions for having zero duality gap, conditions of the type used by Clark; and we show that the result of Clark is not valid even in finite-dimensional spaces. We also show that the nonasymptotic versions of the Farkas lemma obtained by Lasserre [11] and Clark [7] can be deduced easily from the asymptotic versions of this result.

As in Clark [6], we consider (X, Y) a dual pair of real linear spaces with respect to a bilinear form $\langle\cdot,\cdot\rangle$ on $X \times Y$ which separates points. It is known that X and Y become separated locally convex spaces, the topology on X being $\sigma(X, Y)$ determined by the seminorms $x \longmapsto |\langle x, y\rangle|$ for $y \in Y$. Then the topological dual of X is (identified with) Y. Similarly, Y is endowed with the topology $\sigma(Y, X)$ and its topological dual is (identified with) X. On X one considers a convex cone (not necessarily closed or pointed) K; this means that $\lambda x \in K$ and $x + x' \in K$ for all $\lambda \in \mathbb{R}_+ := [0, \infty)$ and $x, x' \in K$; hence $0 \in K$. The cone K determines as usual a (not necessarily antisymmetric) partial order on X denoted by $\geq$; so, for $x, x' \in X$ we set $x' \geq x$, or equivalently $x \leq x'$, if $x' - x \in K$. For a nonempty set $C \subset X$ we set $C^\circ := \{y \in Y \mid \langle x, y\rangle \geq 0 \ \forall x \in C\}$, $C^\# := \{y \in Y \mid \langle x, y\rangle > 0 \ \forall x \in C\backslash\{0\}\}$, and $C^\perp := \{y \in Y \mid \langle x, y\rangle = 0 \ \forall x \in C\}$; when C is a linear subspace, we have that $C^\circ = C^\perp$. In the case $C \subset Y$, C°, $C^\#$, and $C^\perp$ are subsets of X and are defined similarly. Of course K° is a closed convex cone which induces a partial order on Y denoted by $\geq$ and $(K^\circ)^\circ = \operatorname{cl} K$, where $\operatorname{cl} C$ or $\overline{C}$ means the closure of $C \subset X$. For $C, D \subset X$, $x \in X$, $\Gamma \subset \mathbb{R}$, and $\gamma \in \mathbb{R}$, we set

$$C + D := \{x + x' \mid x \in C,\ x' \in D\}, \qquad x + D := \{x\} + D,$$

$$\Gamma C := \{\gamma x \mid \gamma \in \Gamma,\ x \in C\}, \qquad \gamma C := \{\gamma\}C.$$

We also denote by $\operatorname{int} C$, $\operatorname{span} C$, $\overline{\operatorname{span}} C$, $\operatorname{core} C$, and $\operatorname{icr} C$ the interior, linear hull, closed linear hull, the algebraic interior (or core), and the relative algebraic interior (or intrinsic core) of C. One considers also another dual pair (Z, W) whose bilinear form (which separates points) is denoted also by $\langle\cdot,\cdot\rangle$, and a continuous linear operator A: $Y \to W$. Then the adjoint operator A^* of A is the linear operator A^*: $Z \to X$ defined by $\langle A^* z, y\rangle = \langle z, Ay\rangle$ for $z \in Z$ and $y \in Y$. It is well known that $(\operatorname{Im} A^*)^\perp = \ker A$ and $(\ker A)^\perp = \operatorname{cl}(\operatorname{Im} A^*)$. The *primal problem* is then the problem

$$\sup\langle c, y\rangle \quad \text{s.t. } Ay = b,\ y \geq 0, \tag{1}$$

and its *algebraic dual problem* is

$$\inf\langle z, b\rangle \quad \text{s.t. } A^* z \geq c, \tag{2}$$

where $b \in W$ and $c \in X$ are fixed elements. Assuming that the equation $Ay = b$ has a solution $\bar{y}$, and setting $M := \operatorname{Im} A^*$, the problems (1) and (2) become

$$\sup\langle c, y\rangle \quad \text{s.t. } y \in \bar{y} + M^\perp,\ y \geq 0,$$

and

$$\inf\langle x, \bar{y}\rangle \quad \text{s.t. } x \in M,\ x \geq c,$$

respectively. In order to make the duality gap smaller between these two problems in Clark [6], one replaces M by $\operatorname{cl} M$ in the last problem, obtaining so the so-called *topological dual problem*. So, setting $L := \operatorname{cl} M = \operatorname{cl}(\operatorname{Im} A^*)$ and $\bar{x} := c$, the primal problem becomes

$$\sup\langle \bar{x}, y\rangle \quad \text{s.t. } y \in \bar{y} + L^\perp,\ y \geq 0, \tag{3}$$

while the topological dual problem becomes

$$\inf\langle x, \bar{y}\rangle \quad \text{s.t.} \quad x \in L, \quad x \geq \bar{x}. \tag{4}$$

Replacing x by $x' := x - \bar{x}$ in (4), this problem becomes

$$\inf(\langle x', \bar{y}\rangle + \langle \bar{x}, \bar{y}\rangle) \quad \text{s.t.} \quad x' \in (-\bar{x}) + L, \quad x' \geq 0,$$

which shows that the (sole) difference between problems (3) and (4) is that in (4) the order is not defined by a closed (convex) cone, taking into account that a minimization problem can be transformed easily into a maximization problem. Let us set

$$\alpha := \sup\{\langle \bar{x}, y\rangle \mid y \in \bar{y} + L^\perp,\ y \geq 0\},$$
$$\beta := \inf\{\langle x, \bar{y}\rangle \mid x \in L,\ x \geq \bar{x}\},$$

where, as usual, $\sup\varnothing := -\infty$ and $\inf\varnothing := \infty := +\infty$. It is easy to see that $\alpha \leq \beta$. Indeed, if $y \in \bar{y} + L^\perp$ with $y \geq 0$ and $x \in L$ with $x \geq \bar{x}$, then

$$\langle \bar{x}, y\rangle \leq \langle x, y\rangle = \langle x, y - \bar{y}\rangle + \langle x, \bar{y}\rangle = \langle x, \bar{y}\rangle;$$

if (3) or (4) is not feasible, the inequality $\alpha \leq \beta$ is obvious.

Observe that for $L = \{0\}$ one has

$$\alpha = \begin{cases} 0 & \text{if } -\bar{x} \in \operatorname{cl} K, \\ \infty & \text{if } -\bar{x} \notin \operatorname{cl} K, \end{cases} \qquad \beta = \begin{cases} 0 & \text{if } -\bar{x} \in K, \\ \infty & \text{if } -\bar{x} \notin K, \end{cases}$$

which confirms that $\alpha \leq \beta$ and shows that $\alpha < \beta$ if and only if $-\bar{x} \in (\operatorname{cl} K)\backslash K$. Moreover, if $L = X$, then $\alpha = \beta = \langle \bar{x}, \bar{y}\rangle$ for $\bar{y} \in K^\circ$ and $\alpha = \beta = -\infty$ otherwise. So we can assume that $\{0\} \neq L \neq X$ in the sequel.

We note that the pair of dual problems (1) and (2) in this general framework, even in a somewhat more general formulation, appeared in the literature long ago (see, e.g., Arrow et al. [2]); conditions for zero duality gap between these problems are given, for example, in Zălinescu [20] and Shapiro [18]. In finite-dimensional spaces they are studied mainly in connection with semidefinite programming and are called linear conic (or cone) programming problems; see Nesterov and Nemirovski's book ([14]). The problems (3) and (4) are considered in Nesterov and Nemirovskii [14], as well. Zero duality gap between these pairs of problems in finite-dimensional spaces is obtained under some interiority conditions as we shall discuss in the next section.

In the context of mathematical finance one frequently considers the pair of problems (1) and (2) in spaces of measurable functions; in such spaces the cones under consideration often have empty (algebraic) interior and so the interiority conditions cannot be envisaged. In Clark [6] the equality $\alpha = \beta$ is obtained under certain conditions which correspond to some axioms in mathematical finance: no arbitrage (NA), no approximate arbitrage (NAA), and no free lunches (NFL). In the next section we show that such (slightly modified) conditions are sufficient to have $\alpha = \beta$ in finite-dimensional spaces but are not sufficient in infinite-dimensional spaces even if K is closed; see Example 2.3. In fact, as we shall see below, in finite-dimensional spaces those conditions are equivalent to previously used interiority conditions.

2. Duality results. In the sequel we assume that L is a proper closed linear subspace of X. Let us first consider some conditions which will be used in the sequel, the framework and notation being that in the preceding section:

Condition A.1′. There exists $y_0 \geq 0$ such that $y_0 \in \bar{y} + L^{\perp}$.

Condition A.1. There exists $y_0 \geq 0$ such that $y_0 \in \bar{y} + L^{\perp}$ and $y_0 \neq 0$.

Condition A.2′. There exists $x_0 \in L \cap K$ such that $x_0 \neq 0$.

Condition A.2. There exists $x_0 \in L$ such that $\varphi(x_0) > 0$ for every $\varphi \in (L \cap K)^{\circ} \backslash \{0\}$.

In Condition A.2 the set $(L \cap K)^{\circ}$ is considered as a subset of the topological dual L^* for the dual pair (L, L^*). If $L \cap K = \{0\}$, then $(L \cap K)^{\circ} = L^*$, and so Condition A.2 does not hold; hence Condition A.2 $\Longrightarrow$ Condition A.2′ in this case. In fact, as confirmed by the author in an email communication, in the proof of Theorem 3 in Clark [6], only Condition A.2′ is used, even if in its statement (the stronger) Condition A.2 is assumed to hold. The other conditions are:

Condition A.3′. $\{x \in L \cap K \mid \langle x, \bar{y} \rangle = 0\} \subset -K$.

Condition A.3. $\{x \in L \cap K \mid \langle x, \bar{y} \rangle = 0\} = \{0\}$.

Condition B.3′. $\{y \in L^{\perp} \cap K^{\circ} \mid \langle \bar{x}, y \rangle = 0\} \subset -K^{\circ}$.

Condition A.1′ simply says that the primal problem (3) is feasible, or equivalently, that $\bar{y} \in K^{\circ} + L^{\perp}$. Of course, Condition A.1′ is weaker than Condition A.1. The example after Corollary 2.1 shows that Condition A.1′ is essential for the validity of the conclusion of the duality results. In Clark [6, p. 242] it is mentioned that Condition A.3 is the mathematical formulation of the axiom NAA, which, at its turn, is the natural topological generalization of the axiom NA.

Throughout this paragraph we assume that $\dim X < \infty$ and that K is closed. As seen in the appendix, if Condition A.3′ holds, then $\bar{y} \notin (\operatorname{cl}(L^{\perp} + K^{\circ})) \backslash (\operatorname{icr}(L^{\perp} + K^{\circ}))$. Therefore, combining Conditions A.1′ and A.3′ are satisfied iff

$$\bar{y} \in L^{\perp} + \operatorname{icr} K^{\circ} \ [= \operatorname{icr}(L^{\perp} + K^{\circ})], \tag{5}$$

which is essentially a generalized Slater condition. In a similar way, Conditions A.1′ and A.3 are satisfied iff

$$\bar{y} \in \operatorname{int}(L^{\perp} + K^{\circ}). \tag{6}$$

As in Luo et al. [12], we say that problem (3) is strongly infeasible if $\bar{y} \notin \operatorname{cl}(L^{\perp} + K^{\circ})$ (or equivalently $\operatorname{dist}(\bar{y} + L^{\perp}, K^{\circ}) > 0$) and (3) is weakly infeasible if $\bar{y} \in (\operatorname{cl}(L^{\perp} + K^{\circ})) \backslash (L^{\perp} + K^{\circ})$. Hence, if Condition A.3′ holds, then $\bar{y}$ is strongly infeasible or strongly feasible. Of course, Condition B.3′ is dual to Condition A.3′; if Condition B.3′ holds, then $\bar{x} \notin (\operatorname{cl}(L + K)) \backslash (\operatorname{icr}(L + K))$.

Lemma 2.1. *Assume that Condition* A.3 *holds and that* $y \geq 0$ *is such that* $y \in \bar{y} + L$. *Then* $\langle x, y \rangle > 0$ *for every* $x \in L \cap K \backslash \{0\}$.

Proof. Indeed, if $x \in L \cap K$, then $0 \leq \langle x, y \rangle = \langle x, y - \bar{y} \rangle + \langle x, \bar{y} \rangle = \langle x, \bar{y} \rangle$; assuming that $\langle x, y \rangle = 0$, we get $\langle x, \bar{y} \rangle = 0$, and so $x = 0$ by Condition A.3. ☐

Hence, Conditions A.1′, A.2′, and A.3 imply Condition A.1.

In the sequel we shall use the following implications:

$$P, Q \subset X \quad \Rightarrow \quad \operatorname{cl}(P + Q) = \operatorname{cl}(P + \operatorname{cl} Q) = \operatorname{cl}(\operatorname{cl} P + \operatorname{cl} Q), \tag{7}$$

$$P, Q \subset X, \quad P, Q \text{ convex cones} \quad \Rightarrow \quad (P + Q)^{\circ} = P^{\circ} \cap Q^{\circ}, \tag{8}$$

$$P, Q \subset X, \quad P, Q \text{ convex cones} \quad \Rightarrow \quad (\operatorname{cl} P \cap \operatorname{cl} Q)^{\circ} = \operatorname{cl}(P^{\circ} + Q^{\circ}), \tag{9}$$

$$S \subset W, \quad S \text{ convex cone} \quad \Rightarrow \quad (A^{-1}(\operatorname{cl} S))^{\circ} = \operatorname{cl}(A^*(S^{\circ})). \tag{10}$$

Of course, if P, Q, S are linear subspaces instead of being convex cones, in the preceding implication $^{\circ}$ can be equivalently replaced by $^{\perp}$. The implication (7) is valid in any topological vector space (and easy to prove); the implications (8) and (9) are well known (for (8) one uses just the definition, while for (9) one uses a separation theorem), and the implication (10) is stated in Zălinescu [20] for a more general situation. In fact, in the case in which S is a closed convex cone, (10) asserts

$$[Ay \in S \Rightarrow \langle c, y \rangle \geq 0] \quad \Leftrightarrow \quad c \in \operatorname{cl}(A^*(S^{\circ})), \tag{11}$$

which is the Farkas lemma when $A^*(S^{\circ})$ is closed; this is the case when $\dim W < \infty$ and S is polyhedral, that is, the intersection of a finite number of closed half-spaces.

Proposition 2.1. *Assume that* $\alpha' \in \mathbb{R}$ *and there exists* $y_0 \in \overline{y} + L^\perp$ *such that* $y_0 \geq 0$ (*that is, Condition* A.1$'$ *holds*). *Then* $\alpha' \geq \alpha$ *if and only if* $(\alpha', -\overline{x}) \in \operatorname{cl}((\mathbb{R}_+ \times K) + \{(\langle x, \overline{y}\rangle, -x) \mid x \in L\})$, *or equivalently, there exist nets* $(x_i') \subset K$, $(x_i) \subset L$ *such that* $x_i - x_i' \to \overline{x}$ *and* $\limsup \langle x_i, \overline{y}\rangle \leq \alpha'$.

Proof. For the first equivalence one can use Zălinescu [21, Theorem 4]. However, for the reader's convenience we give a direct proof. We have

$$\begin{aligned}\alpha \leq \alpha' \;&\Leftrightarrow\; [y \in \overline{y} + L^\perp,\ y \geq 0 \Rightarrow \langle \overline{x}, y\rangle \leq \alpha']\\ &\Leftrightarrow\; [t > 0,\ y \in t\overline{y} + L^\perp,\ y \geq 0 \Rightarrow \langle \overline{x}, y\rangle \leq t\alpha']\\ &\Leftrightarrow\; [t \geq 0,\ y \geq 0,\ y - t\overline{y} \in L^\perp \Rightarrow t\alpha' - \langle \overline{x}, y\rangle \geq 0].\end{aligned}$$

To obtain the implication "$\Rightarrow$" in the last equivalence we proceed as follows: take $y \geq 0$, $y \in L^\perp$; then, for $s > 0$, $y + sy_0 \geq 0$ and $y + sy_0 \in s\overline{y} + L^\perp$, so $\langle \overline{x}, y + sy_0\rangle \leq s\alpha'$; therefore, for $s \to 0$, we get $-\langle \overline{x}, y\rangle \geq 0$. Consider $B\colon \mathbb{R} \times Y \to Y$, $B(t, y) := y - t\overline{y}$ and $\varphi\colon \mathbb{R} \times Y \to \mathbb{R}$, $\varphi(t, y) := t\alpha' - \langle \overline{x}, y\rangle$; hence $B^*(x) := (\langle x, \overline{y}\rangle, -x)$. From the preceding equivalences we get

$$\begin{aligned}\alpha \leq \alpha' \;&\Leftrightarrow\; [(t, y) \in \mathbb{R}_+ \times K^\circ,\ B(t, y) \in L^\perp \Rightarrow \varphi(t, y) \geq 0]\\ &\Leftrightarrow\; \varphi \in ((\mathbb{R}_+ \times K^\circ) \cap B^{-1}(L^\perp))^\circ\\ &\Leftrightarrow\; \varphi \in \operatorname{cl}((\mathbb{R}_+ \times \operatorname{cl} K) + \operatorname{cl} B^*(L)) \;\Leftrightarrow\; \varphi \in \operatorname{cl}((\mathbb{R}_+ \times K) + B^*(L))\\ &\Leftrightarrow\; (\alpha', -\overline{x}) \in \operatorname{cl}\{(s + \langle x, \overline{y}\rangle, x' - x) \mid s \geq 0,\ x \in L,\ x' \in K\};\end{aligned}$$

the conclusion follows immediately. ☐

As seen in (A4) of the appendix, one can prove the preceding proposition using convex analysis.

From the preceding result we obtain that $\alpha = \beta$ under a closedness condition.

Corollary 2.1. *Assume that Condition* A.1$'$ *holds and the set* $(\mathbb{R}_+ \times K) + \{(\langle x, \overline{y}\rangle, -x) \mid x \in L\}$ *is closed. Then* $\alpha = \beta$ *and* β *is attained when finite.*

Proof. Because Condition A.1$'$ holds, we have that $\beta \geq \alpha > -\infty$; if $\alpha = \infty$, there is nothing to prove. Let $\alpha < \infty$. Then, by Proposition 2.1, $(\alpha, -\overline{x}) \in (\mathbb{R}_+ \times K) + \{(\langle x, \overline{y}\rangle, -x) \mid x \in L\}$; that is, $(\alpha, -\overline{x}) = (t, x') + (\langle x, \overline{y}\rangle, -x)$ with $t \geq 0$, $x' \in K$, and $x \in L$. Hence $x = \overline{x} + x' \geq \overline{x}$ and $\beta \leq \langle x, \overline{y}\rangle = \alpha - t \leq \alpha$. Therefore, $\alpha = \beta$ and $\langle x, \overline{y}\rangle = \beta$. ☐

Note that Condition A.1$'$ is essential for arriving at the conclusion in the preceding two results. Indeed, take L a proper closed linear subspace of X, $K := L$ and $\overline{y} \in Y \backslash L^\perp$. Then Condition A.1$'$ is not satisfied, whence $\alpha = -\infty$; moreover $(\mathbb{R}_+ \times K) + \{(\langle x, \overline{y}\rangle, -x) \mid x \in L\} = \mathbb{R} \times L$. Hence for $\overline{x} \in X \backslash L$ the conclusions of Proposition 2.1 and Corollary 2.1 do not hold.

The next result shows that in finite-dimensional spaces, when the cone K is also closed, we have zero duality gap under quite mild conditions.

Proposition 2.2. *Assume that Conditions* A.1$'$ *and* A.3$'$ *hold. If* $\dim L < \infty$ *and* K *is closed, then* $\alpha = \beta$ *and* β *is attained when finite.*

Proof. By Corollary 2.1, it is sufficient to show that $(\mathbb{R}_+ \times K) + S$ is closed, where $S := \{(\langle x, \overline{y}\rangle, -x) \mid x \in L\}$. Indeed, $\mathbb{R}_+ \times K$ is a closed convex cone and S is a linear subspace of $\mathbb{R} \times X$ with $\dim S = \dim L < \infty$, so S is a locally compact closed convex cone. Take $(\gamma, u) \in P := (\mathbb{R}_+ \times K) \cap S$. With $y_0 \in Y$ provided by Condition A.1$'$, we have $u = -x \in K \cap L$ and $0 \leq \gamma = \langle x, \overline{y}\rangle = \langle x, y_0\rangle = -\langle u, y_0\rangle \leq 0$. It follows that $\gamma = \langle u, \overline{y}\rangle = 0$. Since $u \in K \cap L$, from Condition A.3$'$ we get $u \in -K$, so $(\gamma, u) \in (-P)$. Hence P is a linear subspace, P being a convex cone with $P \subset (-P)$. Because $\mathbb{R}_+ \times K$ and S are closed convex cones, one of them being locally compact, by Dieudonné's theorem (see, e.g., Holmes [9, p. 118]) we obtain that $(\mathbb{R}_+ \times K) + S$ is closed. ☐

Note that in the case where $\dim X < \infty$, another simple proof can be obtained using convex analysis (see (A5) and (A7) in the appendix).

Note also that in order to have zero duality gap for problems (1) and (2) when K is closed (even for a somewhat more general formulation), in Zălinescu [20] (see Zălinescu [20, Theorem 5]) one uses a closedness condition introduced in Arrow et al. [2]. Such a condition is also used by Shapiro (see Shapiro [18, Proposition 2.6]). In Zălinescu [20, Corollary 11] it is proved that the closedness condition is satisfied if the interiority condition $(y_0 + \operatorname{int} Q) \cap A(P) \neq \varnothing$ holds (see Zălinescu [20, (6.10)]); note that this interiority condition is equivalent to the condition (2.20) in Shapiro [18]) where $\operatorname{int} Q$ is nonempty. In the context of problems (1) and (2) the interiority condition (2.20) in Shapiro [18] reads as

$$-c \in \operatorname{int}(K + \operatorname{Im} A^*).$$

However, such a condition can be found earlier in the context of linear conic and semidefinite programming problems in finite-dimensional spaces (see Nesterov and Nemirovskii [14] and in a recent paper by Nemirovski [13, Theorem 2.1]).

In this paragraph we assume that $X = Y = \mathbb{R}^n$ and K is a closed convex cone. As mentioned in Pólik and Terlaky [15], de Klerk et al. [8] established the strong duality of problems (1) and (2) under the more general condition $(\bar{y} + \operatorname{Im} A^*) \cap \operatorname{icr} K^\circ \neq \varnothing$; this condition is equivalent to

$$\bar{y} \in \operatorname{icr}(K^\circ + \operatorname{Im} A^*).$$

As mentioned in the Introduction, problems (3) and (4) were considered by Nesterov and Nemirovski [14] and the conclusion of Proposition 2.2 was obtained under the condition

$$(\bar{y} - \operatorname{int} K^\circ) \cap L^\perp \neq \varnothing \tag{12}$$

and $(-\bar{x} + L) \cap K \neq \varnothing$. Luo et al. [12, Theorem 3] obtained the conclusion of Proposition 2.2 under condition (12); it is clear that condition (12) is stronger than condition (6), which, at its turn, is stronger than condition (5). As observed above, Conditions A.1′ and A.3′ are satisfied iff Condition (5) holds; hence, Proposition 2.2 and Theorem 2.2 in Pólik and Terlaky [15] are equivalent.

If K is not closed in Proposition 2.2, its conclusion could be false (in the sense that $\alpha < \beta$) even if Conditions A.1, A.2, and A.3 hold, as the next example shows. The main result of Clark [6] is as follows.

Theorem 3 in Clark [6] (with our notations) asserts: *Suppose Conditions* A.1, A.2, *and* A.3 *hold. Then, the optimal value of the primal problem* (3) *is equal to the optimal value of its topological dual* (4).

Example 2.1. Let $X = Y = \mathbb{R}^3$, $L := \{0\} \times \{0\} \times \mathbb{R}$ (hence $L^\perp = \mathbb{R} \times \mathbb{R} \times \{0\}$) and let $K := \mathbb{R}_+(0,0,1) \cup \operatorname{int} P$ with $P := (\mathbb{R}_+)^3$. Take $\bar{x} := (-1, 0, 0)$ and $\bar{y} := (0, 0, 1)$. Then $L \cap K = \mathbb{R}_+(0,0,1)$, so Conditions A.2 and A.3 hold. Moreover $K^\circ = P$, so $\bar{y} \in K^\circ \backslash \{0\}$ (hence Condition A.1 holds), $\{y \in \bar{y} + L^\perp \mid y \geq 0\} = \mathbb{R}_+ \times \mathbb{R}_+ \times \{1\}$, and $\{x \in L \mid x \geq \bar{x}\} = \varnothing$. It follows that $\alpha = 0$ and $\beta = \infty$.

Another example with K not closed in which Conditions A.1, A.2′, and A.3 hold and $\alpha < \beta$ follows:

Example 2.2. Let $X = Y = \mathbb{R}^4$, $L := \mathbb{R} \times \{0\} \times \{0\} \times \mathbb{R}$ (hence $L^\perp = \{0\} \times \mathbb{R} \times \mathbb{R} \times \{0\}$), $K := \mathbb{R}_+ \times (\{(0,0,0)\} \cup \operatorname{int} P)$ with $P := \{(u, v, w) \in \mathbb{R}^3 \mid v, w \geq 0,\ u^2 \leq 2vw\}$ (hence $\operatorname{int} P = \{(u, v, w) \mid v, w > 0,\ u^2 < 2vw\}$). Take $\bar{x} := (0, 1, 0, 0)$ and $\bar{y} := (1, 0, 1, 0)$. Then $L \cap K = \{(s, 0, 0, 0) \mid s \geq 0\}$, so $\{x \in K \cap L \mid \langle x, \bar{y} \rangle = 0\} = \{0\}$; hence Conditions A.2′ and A.3 hold. Moreover $K^\circ = \mathbb{R}_+ \times P$, so $\bar{y} \in K^\circ \backslash \{0\}$ (showing that Condition A.1 holds), $\{y \in \bar{y} + L^\perp \mid y \geq 0\} = \{1\} \times \{0\} \times \mathbb{R}_+ \times \{0\}$, and $\{x \in L \mid x \geq \bar{x}\} = \varnothing$. It follows that $\alpha = 0$ and $\beta = \infty$.

As we have seen in Proposition 2.2, when L is finite-dimensional and K is closed, we can ensure that $\alpha = \beta$ under much weaker conditions than Conditions A.1, A.2, and A.3 (it is sufficient that Conditions A.1′ and A.3′ be satisfied). The next example shows that when L is infinite-dimensional and K is closed, Conditions A.1, A.2, and A.3 are not sufficient to have $\alpha = \beta$. In this example all the sequences are indexed by $n \in \mathbb{N}^* := \{1, 2, \ldots\}$. Of course, $\ell_2 := \{(\lambda_n) \subset \mathbb{R} \mid \sum_{n \geq 1} \lambda_n^2 < \infty\}$ and $\ell_2^+ := \{(\lambda_n) \in \ell_2 \mid \lambda_n \geq 0\ \forall\, n \geq 1\}$; consider also $\ell^\# := (\ell_2^+)^\# = \{(\lambda_n) \in \ell_2 \mid \lambda_n > 0\ \forall\, n \geq 1\}$.

Example 2.3. Consider X an infinite-dimensional real Hilbert space (e.g., $X = \ell_2$) with the orthonormal basis $(e_n)_{n \geq 1}$ and $\eta_n, \mu_n \in (0, 1)$ with $\eta_n^2 + \mu_n^2 = 1$ for every $n \geq 1$ and $(\eta_n) \in \ell_2$. Consider, similar to Cheney [4, Exercise. 39, p. 80], $z_n := \eta_n e_{2n} + \mu_n e_{2n-1}$, $z'_n := \eta_n e_{2n-1} - \mu_n e_{2n}$ for $n \geq 1$. Note that $\langle z_n, z_m \rangle = \langle z'_n, z'_m \rangle = \delta_{nm}$ (δ_{nm} being the Kronecker symbols) for $n, m \geq 1$. Consider $L := \overline{\operatorname{span}}\{e_{2n-1} \mid n \geq 1\}$, $L_1 := \overline{\operatorname{span}}\{z_n \mid n \geq 1\}$; so

$$L = \left\{ \sum_{n \geq 1} \lambda_n e_{2n-1} \,\middle|\, (\lambda_n) \in \ell_2 \right\}, \qquad L_1 = \left\{ \sum_{n \geq 1} \lambda_n z_n \,\middle|\, (\lambda_n) \in \ell_2 \right\},$$

and $L \cap L_1 = \{0\}$. It is clear that $\{e_1, e_2, \ldots, e_{2n}\} \subset \operatorname{span}\{e_1, e_3, \ldots, e_{2n-1}, z_1, \ldots, z_n\} \subset L + L_1$ therefore $X = \operatorname{cl}(L + L_1)$. Moreover, $\bar{x} := -\sum_{n \geq 1} \eta_n e_{2n} \notin L + L_1$. We have that

$$L^\perp = \left\{ \sum_{n \geq 1} \lambda_n e_{2n} \,\middle|\, (\lambda_n) \in \ell_2 \right\}, \qquad L_1^\perp = \left\{ \sum_{n \geq 1} \lambda_n z'_n \,\middle|\, (\lambda_n) \in \ell_2 \right\}.$$

Consider also $P := \{\sum_{n\geq 1} \lambda_n e_{2n-1} \mid (\lambda_n) \in \ell_2^+\} \subset L$ and $K := \operatorname{cl}(P + L_1)$; so K is a closed convex cone. We take $Y := X$, the pairing between X and Y being given by the scalar product; then

$$\begin{aligned} K^\circ &= (P+L_1)^\circ = P^\circ \cap L_1^\perp = (P + \overline{\operatorname{span}}\{e_{2n} \mid n \geq 1\}) \cap \overline{\operatorname{span}}\{z_n' \mid n \geq 1\} \\ &= \left\{ \sum_{n\geq 1} \lambda_n z_n' \;\middle|\; (\lambda_n) \in \ell_2^+ \right\}. \end{aligned}$$

It follows that

$$K = (K^\circ)^\circ = \left\{ \sum_{n\geq 1} \gamma_n e_n \;\middle|\; (\gamma_n) \in \ell_2,\ \gamma_{2n-1}\eta_n \geq \gamma_{2n}\mu_n \ \forall\, n \geq 1 \right\}. \tag{13}$$

We have that $L \cap K = P$. The inclusion $\supset$ being obvious, let $x := \sum_{n\geq 1} \lambda_n e_{2n-1} \in K$. From (13) we obtain that $\lambda_n \eta_n \geq 0$, so $\lambda_n \geq 0$; hence $x \in P$. So, for $L \cap K = P$ seen as a convex cone in L, we have that $(L\cap K)^\circ = P$ and $((L\cap K)^\circ)^\# = \{\sum_{n\geq 1} \lambda_n e_{2n-1} \mid (\lambda_n) \in \ell^\#\} \neq \varnothing$. Hence Condition A.2 is satisfied. Consider $\bar{y} := \sum_{n\geq 1} \bar{\lambda}_n z_n'$ with $(\bar{\lambda}_n) \in \ell^\#$. Then $\bar{y} \in K^\circ$ and $\{x \in L \cap K \mid \langle x, \bar{y}\rangle = 0\} = \{0\}$. These show that Conditions A.1 and A.3 are satisfied, as well. Moreover, $\{y \in K^\circ \mid y \in \bar{y} + L^\perp\} = \{\bar{y}\}$. Indeed, let $y := \sum_{n\geq 1} \lambda_n z_n'$ with $(\lambda_n) \in \ell_2^+$ such that $y - \bar{y} \in L^\perp$. Then $\sum_{n\geq 1} (\lambda_n - \bar{\lambda}_n)\eta_n e_{2n-1} - \sum_{n\geq 1} (\lambda_n - \bar{\lambda}_n)\mu_n e_{2n} = \sum_{n\geq 1} \lambda_n' e_{2n}$ with $(\lambda_n') \in \ell_2$. It follows that $(\lambda_n - \bar{\lambda}_n)\eta_n = 0$ for $n \geq 1$; hence $y = \bar{y}$. Therefore, $\alpha := \sup\{\langle \bar{x}, y\rangle \mid y \in \bar{y} + L^\perp,\ y \geq 0\} = \langle \bar{x}, \bar{y}\rangle$. On the other hand, assume that $x := \sum_{n\geq 1} \lambda_n e_{2n-1} \in L$ (with $(\lambda_n) \in \ell_2$) is such that $x \geq \bar{x}$; that is, $\sum_{n\geq 1} \lambda_n e_{2n-1} + \sum_{n\geq 1} \eta_n e_{2n} \in K$. Then $\lambda_n \eta_n \geq \eta_n \mu_n$; that is, $\lambda_n \geq \mu_n$ for every $n \geq 1$. This is a contradiction because $\mu_n \to 1$. Therefore, $\beta = \inf\{\langle x, \bar{y}\rangle \mid x \in L,\ x \geq \bar{x}\} = \infty$.

Note that Examples 2.1 and 2.3 are effectively counterexamples for Theorem 3 of Clark [6] because in a Hilbert space X every closed linear subspace L is the image of a continuous linear operator; more precisely, L is the image of the orthogonal projection onto L. In fact, in the proof of Theorem 3 of Clark [6], Condition A.2′ was used instead of Condition A.2. So a natural question is, what is needed to add besides Conditions A.1′ and A.3′ in Proposition 2.2 in order to have $\alpha = \beta$ without the closedness of K even if $\dim X < \infty$. As seen in Example 2.2 adding Condition A.2′ (or even Condition A.2 as seen in Example 2.1) is not sufficient. The next result provides such conditions; in its proof we follow the lines of the proof of Theorem 3 in Clark [6].

Proposition 2.3. *Assume that Conditions* A.1′, A.2′, *and* A.3 *hold. If* $\dim X < \infty$ *and Condition* B.3′ *holds, then* $\alpha = \beta$.

Proof. Fix $y_0 \in \bar{y} + L^\perp$ with $y_0 \geq 0$ and $x_0 \in L \cap K \backslash \{0\}$. Set $F := \{x \in L \mid \langle x, \bar{y}\rangle = 0\}$; by Condition A.3 we have that $K \cap F = \{0\}$. It is clear that $\langle \bar{x}, y_0\rangle \leq \alpha \leq \beta$. If $\beta = \langle \bar{x}, y_0\rangle$, then $\alpha = \beta$. Assume that $\beta > \langle \bar{x}, y_0\rangle$. Then $\bar{x} \notin L$; otherwise $\langle \bar{x}, \bar{y}\rangle = \langle \bar{x}, y_0\rangle \leq \langle x, y_0\rangle = \langle x, \bar{y}\rangle$ for all $x \in L$ with $x \geq \bar{x}$; therefore $\beta = \langle \bar{x}, y_0\rangle$. Take $\langle \bar{x}, y_0\rangle < \lambda < \beta$. Consider $M := L + \mathbb{R}\bar{x}$ and define $\psi\colon M \to \mathbb{R}$ by $\psi(x + t\bar{x}) := \langle x, \bar{y}\rangle + t\lambda$ for $x \in L$ and $t \in \mathbb{R}$. It is clear that ψ is linear. Moreover, $u \in M \cap K \backslash \{0\}$ implies $\psi(u) > 0$. Indeed, take $u := x + t\bar{x}$ with $u \in K\backslash\{0\}$, $x \in L$, and $t \in \mathbb{R}$. If $t = 0$, then $u = x \in L \cap K \backslash \{0\}$, so, by Lemma 2.1, $\psi(u) = \langle x, \bar{y}\rangle > 0$. If $t < 0$, then $x \geq (-t)\bar{x}$; therefore, $\langle (-t)^{-1} x, \bar{y}\rangle \geq \beta > \lambda$, so $\psi(u) = \langle x, \bar{y}\rangle + t\lambda > 0$. If $t > 0$, then

$$0 \leq \langle u, y_0\rangle = \langle x + t\bar{x}, y_0\rangle = \langle x, \bar{y}\rangle + t\langle \bar{x}, y_0\rangle < \langle x, \bar{y}\rangle + t\lambda = \psi(u).$$

Set $G := \ker \psi = \{u \in M \mid \psi(u) = 0\}$. It follows that $K \cap G = \{0\}$. Since $x_0 \in L \cap K \backslash \{0\}$, by Lemma 2.1 we get $\langle x_0, y_0\rangle > 0$, so $0 \notin \operatorname{icr} K$ (otherwise K is a linear subspace, so, because $y_0 \in K^\circ$, we must have $\langle x, y_0\rangle = 0$ for every $x \in K$). Hence $\operatorname{icr} G \cap \operatorname{icr} K = G \cap \operatorname{icr} K = \varnothing$. This shows that we can separate K and G (in the space $\operatorname{span}(G - K) = G + K - K$); that is, there exists $y \in Y$ which is not null on $\operatorname{span}(G - K)$ such that $\langle x', y\rangle \geq \langle x, y\rangle$ for all $x' \in K$ and $x \in G$. It follows that $y \in K^\circ$ and $\langle x, y\rangle = 0$ for every $x \in G = \ker \psi$. Hence there exists $\mu \in \mathbb{R}$ such that $\langle x, y\rangle = \mu\psi(x)$ for every $x \in M$. Assume that $\mu = 0$. Then $\langle x, y\rangle = 0$ for every $x \in L$; that is, $y \in L^\perp$, and $\langle \bar{x}, y\rangle = 0$. By Condition B.3′ we obtain that $y \in -K^\circ$. It follows that $\langle x, y\rangle = 0$ for every $x \in K$, which implies that $\langle x, y\rangle = 0$ for every $x \in \operatorname{span}(G - K)$, a contradiction. Therefore, $\mu \neq 0$. Since $0 \leq \langle x_0, y\rangle = \mu\langle x_0, \bar{y}\rangle$ we get $\mu > 0$, so we can take $\mu = 1$ (replacing y by $\mu^{-1} y$ if necessary). Hence $\langle x + t\bar{x}, y\rangle = \langle x, \bar{y}\rangle + t\lambda$ for $x \in L$ and $t \in \mathbb{R}$; therefore $y - \bar{y} \in L^\perp$ and $\langle \bar{x}, y\rangle = \lambda$. This shows that $\alpha \geq \lambda$. Since $\lambda \in (\langle \bar{x}, y_0\rangle, \beta)$ is arbitrary, we obtain that $\alpha \geq \beta$. This completes the proof. ☐

In (A6) of the appendix we give a proof of Proposition 2.3 (provided by one of the referees) which does not use Conditions A.2′ and A.3. Note that $L^\perp \cap K^\circ = \{0\}$ in Example 2.3, so Condition B.3′ holds. Therefore, even if K is closed, the infinite-dimensional version of Proposition 2.3 might not be true.

An inspection of the proof of Proposition 2.3 shows that we used the fact that $\dim X < \infty$ for separating the sets G and K by a closed hyperplane which does not contain $\operatorname{span}(G - K)$. In fact, if L has codimension 1 (and so $M = X$), there is no need to use a separation theorem because ψ does the job. A situation when the separation is possible is when the interior of $G - K$ is nonempty for a compatible topology, in particular when the interior of K is nonempty for such a topology. In fact, under Conditions A.1′, A.2′, and A.3, we find the desired y if and only if $x_0 \notin \operatorname{cl}(G - K)$. Indeed, if $x_0 \notin \operatorname{cl}(G - K)$, then there exists $y \in Y \backslash \{0\}$ such that $\langle x - x', y\rangle \le 0 < \langle x_0, y\rangle$ for all $x \in G$ and $x' \in K$. It follows that $y \in K^\circ \backslash \{0\}$ and $\langle x, y\rangle = 0$ for every $x \in G = \ker \psi$. Hence there exists $\mu \in \mathbb{R}$ with $\langle x, y\rangle = \mu\psi(x)$ for every $x \in M = L + \mathbb{R}\bar{x}$. In particular, $0 < \langle x_0, y\rangle = \mu\langle x_0, \bar{y}\rangle$, so $\mu > 0$; hence we can (and we do) assume $\mu = 1$. From $\langle x, y\rangle = \psi(x)$ for $x \in M$ we get $y - \bar{y} \in L^\perp$ and $\langle \bar{x}, y\rangle = \lambda$. Hence $\alpha \ge \langle \bar{x}, y\rangle = \lambda$. Conversely, assuming the existence of $y \in K^\circ$ with $y - \bar{y} \in L^\perp$ and $\langle \bar{x}, y\rangle = \lambda$ we obtain that $\langle x, y\rangle = \psi(x)$ for $x \in M$ (and $\langle x_0, y\rangle = \langle x_0, \bar{y}\rangle > 0$). Hence, for $x' \in K$ and $u \in G$ we get $\langle u - x', y\rangle = \psi(u) - \langle x', y\rangle \le 0$, so $x_0 \notin \operatorname{cl}(G - K)$.

Another situation in which the conclusion of Proposition 2.3 holds with $\dim X = \infty$ is when every positive linear functional on X is continuous and $\operatorname{icr} K$ (or more generally $\operatorname{icr}(K + L)$) is nonempty. For example, every positive linear functional on X is continuous when Y is the algebraic dual of X or when X is a Banach lattice.

Let us follow the proof of Proposition 2.3 in the case of Example 2.1 to show where the drawback was in the proof of Theorem 3 in Clark [6]. We take $y_0 := \bar{y} \in (\bar{y} + L^\perp) \cap K^\circ$. Then $F = \{x \in L \mid \langle x, \bar{y}\rangle = 0\} = \{(0, 0, 0)\}$; clearly $K \cap F = \{0\}$. Moreover, consider $0 = \langle \bar{x}, y_0\rangle < \lambda < \beta = \infty$. Then $M := L + \mathbb{R}\bar{x} = \mathbb{R} \times \{0\} \times \mathbb{R}$ and $\psi(t, 0, w) = w - t\lambda$ for $t, w \in \mathbb{R}$. Hence $G := \ker \psi = \{(t, 0, t\lambda) \mid t \in \mathbb{R}\}$, so we have confirmation that $K \cap G = \{0\}$. Moreover,

$$G - K \subset G - \operatorname{cl} K = \{(t - u, -v, t\lambda - w) \mid t \in \mathbb{R},\ u, v, w \ge 0\} = \mathbb{R} \times \mathbb{R}_- \times \mathbb{R},$$

so $\{0\} \ne K \cap L = \{0\} \times \{0\} \times \mathbb{R}_+ \subset \operatorname{cl}(G - K) = \mathbb{R} \times \mathbb{R}_- \times \mathbb{R} \ne X$.

It is possible to give examples in which Conditions A.1, A.2′, and A.3 are satisfied and to have that $\operatorname{cl}(G - K) = X$.

3. On some results about hedging prices. In Clark [5, Lemma 5] one considers a closed linear subspace L of a separated locally convex space X, a continuous linear functional π: $L \to \mathbb{R}$, and a convex cone $C \subset X$ such that $C \cap L \ne \varnothing$ and $C \cap F = \varnothing$, where $F := \{x \in L \mid \pi(x) \le 0\}$; the condition $C \cap F = \varnothing$ is denoted by NAA in Clark [5] and below. In fact, in the context of Clark [5], L is $\overline{M}$ and π is $\overline{\pi}$. One associates the so-called upper and lower hedging prices $\pi_u(x)$ and $\pi_l(x)$ to any $x \in X$ by

$$\pi_u(x) := \inf\{\pi(x') \mid x' \in L,\ x' \ge x\}, \qquad \pi_l(x) := \sup\{\pi(x'') \mid x'' \in L,\ x'' \le x\},$$

where $x' \ge x$ and $x \le x'$ mean $x' - x \in C_0 := C \cup \{0\}$. It is clear that $\pi_l(x) \le \pi_u(x)$ for every $x \in X$ (because $\pi(x'') \le \pi(x')$ for $x', x'' \in L$ with $x'' \le x'$) and $\pi_l(x) = \pi_u(x) = \pi(x)$ for every $x \in L$. In fact π_u is a sublinear functional with values in $\overline{\mathbb{R}}$ (that is, $\pi_u(0) = 0$, $\pi_u(tx) = t\pi_u(x)$ and $\pi_u(x + x') \le \pi_u(x) + \pi_u(x')$ for all $x, x' \in X$ and $t > 0$ with the convention $(+\infty) + (-\infty) := +\infty$), and $\pi_l(-x) = -\pi_u(x)$ for every $x \in X$. Moreover, setting

$$\mathcal{P} := \{\varphi \in X^* \mid \varphi|_L = \pi,\ \varphi(x) \ge 0\ \forall\, x \in C\},$$

X^* being the topological dual of X, for $\varphi \in \mathcal{P}$ one has $\pi_l(x) \le \varphi(x) \le \pi_u(x)$ for every $x \in X$; in particular

$$\mathcal{P} \ne \varnothing \implies [\pi_l(x) < +\infty,\ \pi_u(x) > -\infty\ \forall\, x \in X]. \tag{14}$$

Lemma 5 in Clark [5] asserts: *Suppose every positive linear functional on X is continuous. If NAA holds, then the following conditions are pairwise mutually equivalent:* (i) $\pi_u(x) > -\infty$ *for every* $x \in X$*;* (ii) $\pi_l(x) < +\infty$ *for every* $x \in X$*;* (iii) $\mathcal{P} \ne \varnothing$.

The implication (14) proves that (iii) implies (i) and (ii) in Clark [5, Lemma 5]. For obtaining the implication (i) $\Rightarrow$ (iii) in Clark [5] the author says "Although the algebraic Hahn–Banach theorem usually presumes f is a real-valued sublinear functional, its standard proof remains valid when f is also allowed to take the value $+\infty$." As seen in Simons [19] and Anger and Lembcke [1], the Hahn–Banach extension theorem for extended-valued sublinear functionals is not true. This means that the proof of Lemma 5 of Clark [5] is not correct. Below we provide a counterexample for the implication (i) $\Rightarrow$ (iii) in Lemma 5 of Clark [5].

Example 3.1. Consider E an infinite-dimensional linear space and p: $E \to \mathbb{R} \cup \{\infty\}$ a sublinear functional which is not minorized by any linear functional (see Simons [19], Anger and Lembcke [1] for such examples). Let $X := E \times \mathbb{R}$, $L := \{0\} \times \mathbb{R}$, $C := \operatorname{epi}_s p := \{(x, t) \in E \times \mathbb{R} \mid p(x) < t\}$, and π: $L \to \mathbb{R}$ be defined by $\pi(0, t) := t$. Taking on X the locally convex topology determined by all the seminorms on X, we have that L is a closed linear subspace, π is continuous, and any linear functional on X is continuous. Moreover, $(0,1) \in L \cap C$ and $C \cap F = \varnothing$. For $(x,s) \in X \backslash L$ (that is, $x \neq 0$), we have

$$\pi_u(x,s) = \inf\{t \in \mathbb{R} \mid (0,t) - (x,s) \in C_0\} = \inf\{t \in \mathbb{R} \mid p(-x) < t - s\} = s + p(-x),$$

so $\pi_u(x,s) > -\infty$ for all $(x,s) \in X$. Assume that there exists some $\varphi \in \mathscr{P}$. Then $\varphi(x,t) = \theta(x) + t\alpha$ for some linear functional $\theta\colon E \to \mathbb{R}$ and $\alpha \in \mathbb{R}$. Because $\varphi|_L = \pi$ and $\varphi(x,t) \geq 0$ for all $(x,t) \in \operatorname{epi}_s p$, we obtain that $\alpha = 1$ and $p(x) < t \Rightarrow \theta(x) + t \geq 0$; that is, $p(x) \geq -\theta(x)$ for every $x \in E$. The last assertion contradicts the choice of p. Hence $\mathscr{P} = \varnothing$.

In this situation it is natural to ask about sufficient conditions for having the conclusion of Clark [5, Lemma 5]. In fact $\mathscr{P} = \partial\pi_u(0)$, the subdifferential being taken in the sense of convex analysis (see, e.g., Zălinescu [22]). But a necessary and sufficient condition for the nonemptiness of the subdifferential at 0 of a proper sublinear functional is its lower semicontinuity at 0 (see Anger and Lembcke [1, Theorem (1.8)], Zălinescu [22, Theorem 2.4.14]). Of course, because $\partial\pi_u(x) \subset \partial\pi_u(0)$ for every $x \in \operatorname{int}\pi_u = L - C_0$, one has that $\partial\pi_u(0) \neq \varnothing$ if $\partial\pi_u(x) \neq \varnothing$ for some $x \in \operatorname{int}\pi_u$. In the algebraic case (in which case one takes X the locally convex topology determined by all the seminorms on X; then $X^* = X'$), one has $\partial\pi_u(x) \neq \varnothing$ at any $x \in \operatorname{icr}(\operatorname{int}\pi_u) = \operatorname{icr}(L - C_0)$ provided π_u does not take the value $-\infty$. In particular, Lemma 5 in Clark [5] holds if $\operatorname{icr} C \neq \varnothing$; hence it holds in finite-dimensional spaces without supplementary conditions. In the infinite-dimensional setting a sufficient condition, as mentioned above, is $\operatorname{icr}(L - C_0) \neq \varnothing$.

The other two results on hedging prices in Clark [5] are Lemma 6 and Theorem 7.

Lemma 6 in Clark [5] asserts: *Suppose $\mathscr{P} \neq \varnothing$ and NAA holds. Then for every $x \notin L$ and $\lambda \in \mathbb{R}$ such that $\pi_u(x) > \lambda > \pi_l(x)$, there exists some $p \in \mathscr{P}$ for which $p(x) = \lambda$.*

Theorem 7 in Clark [5] asserts: *Suppose $\mathscr{P} \neq \varnothing$ and NAA holds. Then the following conditions on a contingent claim $x \in X$ are pairwise mutually equivalent*: (i) $x \in \widehat{M}$; (ii) *x is priced by arbitrage;* (iii) $\pi_u(x) = \pi_l(x)$.

Because we refer only to (ii) and (iii) in Clark [5, Theorem 7] we recall that $x \in X$ is priced by arbitrage (see Clark [5, p. 170]) if $\{p(x) \mid p \in \mathscr{P}\}$ is a singleton. The framework of Example 2.2 can be used to give counterexamples to the results cited above.

Example 3.2. With the notation from Example 2.2, take $C := K \backslash \{0\}$ and $\pi\colon L \to \mathbb{R}$ defined by $\pi(x) := \langle x, \bar{y}\rangle$. Then $C \cap L \neq \varnothing$, $\bar{y}$ (seen as a linear function defined on X) is in $\mathscr{P}$, $F = \mathbb{R}_- \times \{0\} \times \{0\} \times \mathbb{R}$ is closed, and $C \cap F = \varnothing$; that is, condition NAA holds. Moreover, $\bar{x} \notin L$ and $\{x' \in L \mid x' \geq \bar{x}\} = \{x'' \in L \mid x'' \leq \bar{x}\} = \varnothing$; therefore $\pi_u(\bar{x}) = \infty$ and $\pi_l(\bar{x}) = -\infty$. A simple calculation shows that $\mathscr{P} = \{y \in \bar{y} + L^\perp \mid y \geq 0\} = \{1\} \times \{0\} \times \mathbb{R}_+ \times \{0\}$, so $p(\bar{x}) = 0$ for every $p \in \mathscr{P}$, contradicting Lemma 6 in Clark [5]. Because $\pi_l(\bar{x}) < \pi_u(\bar{x})$, we have that the implication (ii) ⇒ (iii) of Theorem 7 in Clark [5] is not true.

4. On some nonasymptotic Farkas lemma type results.

We use in this section the same framework and notation as in the Introduction.

The problem of the solvability (feasibility) of the equation $Ay = b$ with $y \geq 0$ is an important one, and is related to the classic Farkas lemma. Of course, the equation $Ay = b$ with $y \geq 0$ is feasible if and only if $b \in A(K^\circ)$. On the other hand, we have that $A^*z \geq 0 \Rightarrow \langle z, b\rangle \geq 0$ if and only if $b \in ((A^*)^{-1}(K))^\circ$. But, by (10),

$$((A^*)^{-1}(\operatorname{cl} K))^\circ = \operatorname{cl}(A(K^\circ)). \tag{15}$$

So, there is no chance to obtain that

$$[A^*z \geq 0 \Rightarrow \langle z, b\rangle \geq 0] \quad \Leftrightarrow \quad \exists y \in K^\circ\colon\ Ay = b \tag{16}$$

if $(A^*)^{-1}(\operatorname{cl} K) \neq \operatorname{cl}((A^*)^{-1}(K))$ or $A(K^\circ)$ is not closed (the implication ⇐ in (16) is always valid, while for the implication ⇒ one can give counterexamples).

Note that the Farkas lemma can be interpreted as an alternative theorem. In view of this remark, the literature has many results called a generalized Farkas lemma (see Jeyakumar [10] for a survey); however, in our opinion, this is an exaggeration because any time we write $X = E \cup (X \backslash E)$ for some set E, we should have a generalized Farkas lemma. We consider that a proper generalization of the Farkas lemma is when in the description of E or $X \backslash E$ instead of linear functions and operators one has sublinear functions or operators; this is because a very important use of the Farkas lemma is in deriving necessary optimality conditions in smooth optimization, and now in nonsmooth analysis (and optimization) the derivative is replaced generally by a certain directional derivative (which generally is sublinear).

The second interpretation of the Farkas lemma is as a solvability result; more precisely, the solvability of the equation $Ay = b$ with $y \geq 0$, or something similar. Viewed this way, the goal of several papers was to give characterizations for the solvability of the previous equation. The discussion above shows that when the order cone is not polyhedral, the classic characterization generally does not hold; however see Ramana [16, Theorem 19] for a Farkas lemma type result related to semidefinite programming.

In the sequel we shall deduce the solvability of the equation $Ay = b$ with $y \geq 0$ using (15). Because for $b = 0$ it is clear that the equation $Ay = b$ with $y \geq 0$ is feasible (and (16) holds), in the sequel one assumes $b \neq 0$. The next result was obtained in Lasserre [11, Theorem 5.1] in the framework of Banach spaces and, in this context, the element x_0 is taken in $\operatorname{int} K^\circ$ (which, of course, equals $K^\#$ in this case).

Proposition 4.1. *Assume that $b \neq 0$. Then $Ay = b$, $y \geq 0$ has a solution if and only if there exist a nonempty compact convex set C with $0 \notin C \subset K^\circ$, $\delta > 0$, and $x_0 \in P^\#$ such that $A^*z + x_0 \in P^\circ \Rightarrow \langle z, b\rangle \geq -\delta$, where $P := \mathbb{R}_+ C$.*

Proof. Assume that $y_0 \geq 0$ is such that $Ay_0 = b$. Take $C = \{y_0\}$. Because $b \neq 0$, C satisfies the desired conditions. Take also $x_0 \in X$ such that $\langle x_0, y_0\rangle = 1$ and $\delta := 1$. Then $x_0 \in P^\#$ and $P^\circ = \{x \in X \mid \langle x, y_0\rangle \geq 0\}$. If $A^*z + x_0 \in P^\circ$, then $0 \leq \langle A^*z + x_0, y_0\rangle = 1 + \langle z, b\rangle$; therefore $\langle z, b\rangle \geq -\delta$.

Conversely, assume that C, P, x_0, and δ are as in the statement. We can apply Zălinescu [21, Theorem 3], or, as in the proof of Proposition 2.1, because 0 is a solution of $A^*z + x_0 \in P^\circ$, we have

$$\begin{aligned}[A^*z + x_0 \in P^\circ \Rightarrow \langle z, b\rangle \geq -\delta] \;&\Leftrightarrow\; [(t, z) \in \mathbb{R}_+ \times Z,\ A^*z + tx_0 \in P^\circ \Rightarrow \langle z, b\rangle + \delta t \geq 0]\\ &\Leftrightarrow\; \varphi \in ((\mathbb{R}_+ \times Z) \cap T^{-1}(P^\circ))^\circ = \operatorname{cl}((\mathbb{R}_+ \times \{0\}) + T^*(P^{\circ\circ})),\end{aligned}$$

where $T\colon \mathbb{R} \times Z \to X$, $T(t, z) := A^*z + tx_0$, and $\varphi\colon \mathbb{R} \times Z \to \mathbb{R}$, $\varphi(t, z) := \langle z, b\rangle + \delta t$. Then $T^*(y) = (\langle x_0, y\rangle, Ay)$ for $y \in Y$ and $P^{\circ\circ} = P$ because P is a closed convex cone. Hence there exist the nets $(s_i)_{i\in I}, (t_i)_{i\in I} \subset \mathbb{R}_+$ and $(c_i) \subset C$ such that $s_i + t_i\langle x_0, c_i\rangle \to \delta$, $t_i A c_i \to b$. Since $0 \leq \langle x_0, c_i\rangle$ for $i \in I$, we have that $s_i, t_i\langle x_0, c_i\rangle \in [0, \delta]$. Because C is compact, we may (and do) assume that $c_i \to c \in C$ (hence $c \neq 0$), and $s_i \to s$, $t_i \to t$ with $s, t \in [0, \infty]$. Because $x_0 \in P^\#$ we have that $\langle x_0, c\rangle > 0$, so $s, t < \infty$. If $t = 0$ we obtain the contradiction $b = 0$. Hence $A(tc) = b$ and $tc \in P \subset K^\circ$, which shows that $y = tc$ is the desired solution. □

In the sequel, as in Clark [7], consider $F := \{A^*z \mid \langle z, b\rangle = 0\} \subset X$ and $J = \operatorname{cl}(K - F)$. Of course, J is a closed convex cone, so $((A^*)^{-1}(J))^\circ = \operatorname{cl}(A(J^\circ))$.

Lemma 4.1. *Assume that $b \neq 0$. Then $J^\circ = K^\circ \cap A^{-1}(\mathbb{R})$ and*

$$b \in A(K^\circ) \;\Leftrightarrow\; b \in A(K^\circ \cap A^{-1}(\mathbb{R}b)) \;\Leftrightarrow\; b \in \operatorname{cl}(A(K^\circ \cap A^{-1}(\mathbb{R}b))). \tag{17}$$

Proof. Because F is a linear space we have

$$y \in J^\circ \;\Leftrightarrow\; [x \in K, u \in F \Rightarrow \langle x - u, y\rangle \geq 0] \;\Leftrightarrow\; [y \in K^\circ,\ \langle u, y\rangle = 0\ \forall\, u \in F].$$

But

$$\begin{aligned}[\langle u, y\rangle = 0\ \forall\, u \in F] \;&\Leftrightarrow\; [\langle z, b\rangle = 0 \Rightarrow \langle A^*z, y\rangle = 0] \;\Leftrightarrow\; [\langle z, b\rangle = 0 \Rightarrow \langle z, Ay\rangle = 0]\\ &\Leftrightarrow\; [\exists \lambda \in \mathbb{R}\colon Ay = \lambda b] \;\Leftrightarrow\; y \in A^{-1}(\mathbb{R}b).\end{aligned}$$

Therefore, $J^\circ = K^\circ \cap A^{-1}(\mathbb{R}b)$. The first equivalence in (17) is obvious, as well as the implication $\Rightarrow$ in the second equivalence. Let $b \in \operatorname{cl}(A(K^\circ \cap A^{-1}(\mathbb{R}b)))$. Then there exists a net $(y_i)_{i\in I} \subset K^\circ \cap A^{-1}(\mathbb{R}b)$ such that $Ay_i \to b$. Since $y_i \in A^{-1}(\mathbb{R}b)$ we have that $Ay_i = \beta_i b$ for some $\beta_i \in \mathbb{R}$. Hence $\beta_i b \to b$. Since $b \neq 0$ we have that $\beta_i \to 1$. Hence $\beta_i > 0$ for some i, so $b = A(\beta_i^{-1} y_i)$ with $\beta_i^{-1} y_i \in K^\circ$; therefore $b \in A(K^\circ)$. □

Using Lemma 4.1 we obtain the following novel characterization of the solvability of the equation $Ay = b$, $y \geq 0$.

Corollary 4.1. *Let $z_0 \in Z$ be such that $\langle z_0, b\rangle > 0$. Then $b \in A(K^\circ)$ if and only if $A^*z_0 \notin (-J)$.*

Proof. Taking into account the expression of J° in Lemma 4.1 we get

$$\begin{aligned}A^*z_0 \notin (-J) \;&\Leftrightarrow\; -A^*z_0 \notin (J^\circ)^\circ \;\Leftrightarrow\; [\exists y_0 \in J^\circ\colon \langle -A^*z_0, y_0\rangle < 0]\\ &\Leftrightarrow\; \exists y_0 \in K^\circ, \beta \in \mathbb{R}\colon Ay_0 = \beta b,\ \langle A^*z_0, y_0\rangle = \beta\langle z_0, b\rangle > 0\\ &\Leftrightarrow\; [\exists y_0 \in K^\circ, \beta > 0\colon Ay_0 = \beta b] \;\Leftrightarrow\; b \in A(K^\circ).\end{aligned}$$

This completes the proof. □

Using Lemma 4.1 again we get the following result from Clark [7] which is considered to be comparable to the results in Lasserre [11] (that is, comparable to Proposition 4.1).

PROPOSITION 4.2 (CLARK [7, THEOREM 2]). *One has*

$$[\exists y \in K^\circ\colon\ Ay = b] \ \Leftrightarrow\ [A^*z \geq_J 0 \Rightarrow \langle z, b\rangle \geq 0].$$

PROOF. Indeed, using (15), with K replaced by J, and Lemma 4.1 we obtain

$$[A^*z \geq_J 0 \Rightarrow \langle z, b\rangle \geq 0] \ \Leftrightarrow\ b \in ((A^*)^{-1}(J))^\circ \ \Leftrightarrow\ b \in \mathrm{cl}(A(J^\circ)) \ \Leftrightarrow\ b \in A(K^\circ),$$

which completes the proof. ☐

In Clark [7] the condition $A^*z \geq 0 \Rightarrow \langle z, b\rangle \geq 0$ is denoted by FC (Farkas condition) and the condition $A^*z \geq_J 0 \Rightarrow \langle z, b\rangle \geq 0$ is denoted by GFC (generalized Farkas condition). It is clear that GFC ⇒ FC. Hence, from Corollary 4.1 one immediately obtains Lemma 4 in Clark [7] which asserts that GFC holds if and only if FC holds and $A^*z_0 \notin (-J)$.

We conclude with the next result obtained in Clark [7, Lemma 3] for $\dim X < \infty$ and K polyhedral.

PROPOSITION 4.3. *If $K - F$ is closed, then FC and GFC are equivalent.*

PROOF. As observed above, GFC ⇒ FC always. Assume that $K - F$ is closed and FC holds. Take z such that $A^*z \in J = \mathrm{cl}(K - F) = K - F$. Then $A^*z = x - A^*z'$ with $x \in K$ and $z' \in Z$ such that $\langle z', b\rangle = 0$. So, $A^*(z + z') = x \in K$; therefore, by FC, $0 \leq \langle z + z', b\rangle = \langle z, b\rangle$. ☐

Appendix. One of the referees suggested new proofs, using convex analysis, for Propositions 2.1–2.3, as well as alternative formulations for Conditions A.3′ and B.3′; the new proof of Proposition 2.3 shows that Conditions A.2′ and A.3 are superfluous for obtaining its conclusion. We present these new proofs below. As in §§1 and 2, X and Y are real linear spaces in separated duality.

(A1) Condition A.3′ is equivalent to $K_0 := \{x \in L \cap K \mid \langle x, \bar{y}\rangle = 0\} \subset -K_0$. Since K_0 is a convex cone, this is equivalent to the fact that K_0 is a linear subspace. Similarly, Condition B.3′ is equivalent to the fact that $\{y \in L^\perp \cap K^\circ \mid \langle \bar{x}, y\rangle = 0\}$ is a linear subspace of Y.

(A2) Let $A \subset X$ be a nonempty convex set. The quasi-relative interior of A is the set $\mathrm{qri}\,A := \{a \in A \mid \mathrm{cl}(\mathbb{R}_+(A - a))$ is a linear space$\}$ (see Borwein and Lewis [3], Zălinescu [22]). It is clear that $\mathrm{cl}(\mathbb{R}_+(A - a))$ is a linear space iff $(A - a)^\circ$ is a linear subspace of Y. In fact, if $a \in X$ and $\mathrm{cl}(\mathbb{R}_+(A - a))$ is a linear space, then necessarily $a \in \mathrm{cl}A$. Moreover, if $\dim X < \infty$, then $\mathrm{icr}A = \mathrm{qri}\,A$; see Borwein and Lewis [3] or Zălinescu [22, §1.2] for results on the quasi-relative interior. Therefore, if $\dim X < \infty$, A is a convex cone, and $a \in \mathrm{cl}A \setminus \mathrm{icr}A$, then $(A - a)^\circ = \{x^* \in A^\circ \mid \langle a, x^*\rangle = 0\}$ is not a linear subspace of X^*.

(A3) Condition A.1′ is equivalent to $\bar{y} \in K^\circ + L^\perp$. Moreover, $(K^\circ + L^\perp - \bar{y})^\circ = \{x \in L \cap \mathrm{cl}K \mid \langle x, \bar{y}\rangle \leq 0\}$. Hence, if $\bar{y} \in K^\circ + L^\perp$, then $(K^\circ + L^\perp - \bar{y})^\circ = \{x \in L \cap \mathrm{cl}K \mid \langle x, \bar{y}\rangle = 0\}$. Therefore, if Condition A.1′ is satisfied and K is closed, then Condition A.3′ holds iff $(K^\circ + L^\perp - \bar{y})^\circ$ is a linear space iff $\bar{y} \in \mathrm{qri}(K^\circ + L^\perp)$; moreover, if $\dim X < \infty$, then Condition A.3′ holds iff $\bar{y} \in \mathrm{icr}(K^\circ + L^\perp)$. Said differently, if $\dim X < \infty$ and K is closed, then combining Conditions A.1′ and A.3′ holds iff $\bar{y} \in \mathrm{icr}(K^\circ + L^\perp)$ $(= L^\perp + \mathrm{icr}K^\circ)$. Similarly, if $\dim X < \infty$ and K is closed, then Conditions A.1′ and A.3 are satisfied iff $\bar{y} \in \mathrm{int}(K^\circ + L^\perp)$.

(A4) We denote by ι_A and σ_A the indicator and support functions associated to $A \subset X$, respectively; for the other notations and results used below, see Rockafellar [17] or Zălinescu [22]. Fix $\bar{y} \in Y$ and consider

$$\phi_1, \phi_2\colon X \to \overline{\mathbb{R}}, \quad \phi_1(x) := \langle x, \bar{y}\rangle + \iota_L(x), \quad \phi_2 = \iota_{(-K)}.$$

Then

$$\phi_1^* = \iota_{(\bar{y}+L^\perp)}, \quad \phi_2^* = \iota_{K^\circ}, \quad \phi_1^* + \phi_2^* = \iota_{(\bar{y}+L^\perp)\cap K^\circ}.$$

In the sequel we assume that Condition A.1′ holds; that is, $\bar{y} \in K^\circ + L^\perp$. Then $\alpha := \sigma_{(\bar{y}+L^\perp)\cap K^\circ}$ is a proper lower semicontinuous (lsc) sublinear function on X and $\alpha = (\phi_1^* + \phi_2^*)^*$, or equivalently $\phi_1^* + \phi_2^* = \alpha^*$. Moreover, setting $\beta := \phi_1 \,\square\, \phi_2$, we have that β is a convex function and $\beta^* = \phi_1^* + \phi_2^* = \alpha^*$. (In fact $\alpha(\bar{x})$ and $\beta(\bar{x})$ are exactly the values of the problems (3) and (4), respectively.) Since α is a proper lsc convex function we have that $\alpha = \beta^{**} = \bar{\beta}$, where $\bar{\beta}$ is the lsc envelope of β. Because $\mathrm{epi}_s\,\beta = \mathrm{epi}_s\,\phi_1 + \mathrm{epi}_s\,\phi_2$, we deduce that

$$\mathrm{epi}\,\alpha = \mathrm{cl}(\mathrm{epi}\,\phi_1 + \mathrm{epi}\,\phi_2) = \mathrm{cl}(\mathbb{R}_+ \times (-K) + \{(\langle x, \bar{y}\rangle, x) \mid x \in L\});$$

the conclusion of Proposition 2.1 follows.

(A5) Assume that $\dim X < \infty$, K is closed, and Conditions A.1′ and A.3′ hold. As seen in (A3) we have that $\bar{y} \in \operatorname{icr}(K^\circ + L^\perp)$, or equivalently, $0 \in \operatorname{icr}(\operatorname{int}\phi_1^* - \operatorname{int}\phi_2^*)$. By Rockafellar [17, Theorem 16.4, Corollary 6.6.2] or Zălinescu [22, Theorem 2.8.4(viii)] we obtain that $(\phi_1^* + \phi_2^*)^* = \phi_1^{**} \square \phi_2^{**}$ and the convolution is exact; that is, $\alpha = \phi_1 \square \phi_2 = \beta$ and the infimum in the definition of β is attained (when finite). This is the conclusion of Proposition 2.2.

(A6) If $\dim X < \infty$, we have that $\beta(x) = \bar{\beta}(x)$ for all $x \in X \setminus [\overline{\operatorname{dom}\beta} \setminus \operatorname{icr}(\operatorname{dom}\beta)]$. But $\operatorname{dom}\beta = \operatorname{dom}\phi_1 + \operatorname{dom}\phi_2 = L - K$ is a convex cone and $(\operatorname{dom}\beta)^\circ = L^\perp \cap (-K^\circ)$. If $\bar{x}$ verifies Condition B.3′, then by (A2) we have that $\bar{x} \in X \setminus [\overline{\operatorname{dom}\beta} \setminus \operatorname{icr}(\operatorname{dom}\beta)]$. Hence $\alpha(\bar{x}) = \beta(\bar{x})$. This proves that the conclusion of Proposition 2.3 holds without Conditions A.2′ and A.3.

(A7) Observe that the primal problem (3) is equivalent to problem (P): minimize $F(y, 0)$ s.t. $y \in Y$, where $F(y, z) := \iota_{(\bar{y}+L^\perp)}(y + z) + \iota_{K^\circ}(y) - \langle \bar{x}, y \rangle$; it is clear that F is a proper (under Condition A.1′) and lsc convex function. When $\dim X < \infty$, the problem (P) is strongly consistent (as defined in Rockafellar [17, §29]) iff $\bar{y} \in \operatorname{icr}(K^\circ + L^\perp)$, and (P) is strictly consistent iff $\bar{y} \in \operatorname{int}(K^\circ + L^\perp)$. This shows that when $\dim X < \infty$ and K is closed, Proposition 2.2 follows using Rockafellar [17, Theorem 30.4].

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