GAME OF DIFFERENCES

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THE PROBLEM

There are 4 natural numbers a, b, c, d written on a row. The differences |a-b|, |b-c|, |c-d|, |d-a| are then written on the next row.

It can be observed that the null row 0,0,0,0 has been obtained.

Is this a **coincidence** or is the null row obtained from any natural numbers a, b, c, d?

Can it be obtained from any 4 real numbers?

What happens if the rows don't have 4 numbers, but 3,5,6..?

7	**3**	**9**	**2**
4	6	7	5
2	1	2	1
1	1	1	1
0	0	0	0

POINTS:

What happens if we have:

1. ***4 natural numbers***
2. ***5,6,7... natural numbers***
3. ***4 rational numbers***
4. ***4 real numbers***

POINT 1.<FIRST SOLUTION>

We replace the numbers with their remainders modulo 2. There are $2^4=16$ ways to distribute the remainders 0 and 1 on the first row. After at most 4k steps, all the numbers on the last row will be divisible by 2^k . In conclusion, after a certain number of steps, we will reach the zero row: 0, 0, 0, 0.

All these posibilities are consisted in the following table:

1	**0**	**1**	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**1**	**0**	**0**	**1**	**1**	**1**
Or				**Or**				**Or**				**Or**			
0	1	0	0	0	0	1	0	0	0	0	1	1	0	0	0
1	1	0	0	0	1	1	0	0	0	1	1	1	0	0	1
0	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

POINT 1.<SECOND SOLUTION>

Considering $a_n, b_n, c_n, d_n \in \mathbb{N}$, let $x_n = \max\{a_n, b_n, c_n, d_n\}$. It can be demonstrated through mathematical induction that x_n is decreasing sequence. Further we prove that x_n cannot stagnate for more than 4 steps, not all zeroes on the last row. Let $a_{n+4} = x_n$.

If $a_{n+5} < a_{n+1}$, the condition is true. For $a_{n+5} = a_{n+1}$, there are two subcases: $a_{n+4} = a_{n+1}$ and $b_{n+4} = 0$, or $a_{n+4} = 0$,

If $a_{n+4} = a_{n+1}$ and $b_{n+4} = 0$, but as $b_{n+4} = |b_{n+3} - c_{n+3}|$, and $a_{n+4} = |a_{n+3} - b_{n+3}|$, it results that $b_{n+3} = c_{n+3}$ and

$|a_{n+3} - b_{n+3}| = a_{n+1}$, and we can deduce two subcases:

$a_{n+3} = a_{n+1}$, $b_{n+3} = 0$ and $a_{n+3} = 0$, $b_{n+3} = a_{n+1}$.

POINT 2:***WHAT HAPPENS IF ON THE FIRST LINE THERE ARE NOT FOUR NUMBERS, BUT THREE, FIVE, SIX OR SEVEN?***

We will show you that it is possible that, for certain configurations of the first line, to not achieve the zero configuration at the end.

But, as said before, what is going to happen if we choose to have a different amount of numbers on the first line?

IF THERE ARE 3 NUMBERS ON A ROW:

0	**1**	1
1	0	1
1	1	0
0	1	1

- cycle of length 3 -

IF THERE ARE 5 NUMBERS ON A ROW:

0	**1**	**1**	**0**	**0**
1	0	1	0	0
1	1	1	0	1
0	0	1	0	0
0	1	1	0	0

- cycle of length 5 -

IF THERE ARE 6 NUMBERS ON A ROW:

0	**0**	**0**	**1**	**0**	**1**
0	0	1	1	1	1
0	1	0	0	0	1
1	1	0	0	1	1
0	1	0	1	0	0
1	1	1	1	0	0
0	0	0	1	0	1

- cycle of length 6 -

IF THERE ARE 7 NUMBERS ON A ROW

0	**0**	**0**	**0**	**0**	**1**	**1**
0	0	0	0	1	0	1
0	0	0	1	1	1	1
0	0	1	0	0	0	1
0	1	1	0	0	1	1
1	0	1	0	1	0	1
0	0	0	0	0	1	1

- cycle of length 7 -

IF ON THE FIRST LINE THERE ARE EIGHT NUMBERS

To prove that they will become 0 after a certain number of steps as well, we have created **a C++ program**.

3.WHAT HAPPENS IF ON THE FIRST LINE THERE ARE FOUR RATIONAL NUMBERS?

$\frac{a_1}{b_1}$	$\frac{a_2}{b_2}$	$\frac{a_3}{b_3}$	$\frac{a_4}{b_4}$
d_1	d_2	d_3	d_4
…	…	….	….
x_1	x_2	x_3	x_4

We multiply the numbers from the first column and we follow the reasoning on the next slide:

Considering that for any rational numbers q_1, q_2, q_3, q_4 there is a natural number k, such that $q_1 k, q_2 k, q_3 k, q_4 k \in \mathbb{Z}$, all combinations of rational numbers can be reduced to integers. Based on that, in the case of rational numbers the properties of the table are the same as for integers, which are treated previously.

q_1	q_2	q_3	q_4
d_1	d_2	d_3	d_4

→

$q_1 k$	$q_2 k$	$q_3 k$	$q_4 k$
d_1k	d_2k	$d_3 k$	$d_4 k$

4.*Is our result true if we choose four real random numbers on the first line?*

The answer is **NO**! We can choose combinations of irrational numbers so that we will enter in a cycle and consequently, it's never going to end.

Let us consider $M = (1, t, t^2, t^3)$ for the first line and $t^3 = t^2 + t + 1$
$\Rightarrow$ t_0 =1,82928... ,the only irrational solution >1, as our theoretical survey says (Δ<0). It will result a cycle because the n+1 line will be equal with $(t_0 - 1)^{n-1}M$.

Line	**Numbers**				
1	M	1	t_0	${t_0}^2$	${t_0}^3$
2	$(t_0 - 1)M$	$t_0 - 1$	$t_0(t_0 - 1)$	${t_0}^2(t_0 - 1)$	$({t_0}^2 + t_0 + 1)(t_0 - 1)$
-	-	-	-	-	-
$n + 1$	$(t_0 - 1)^{n-1}$M	$1(t_0 - 1)^{n-1}$	$t_0(t_0 - 1)^{n-1}$	${t_0}^2(t_0 - 1)^{n-1}$	${t_0}^3(t_0 - 1)^{n-1}$

Thank you for your attention !