Pyramids

(Year 2016-2017)

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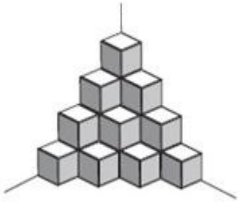


Presentation of the research topic

The tower represented in this image is built of matching cubes, of side 1, stacked one over the other and glued to the corner of a wall. Some of these cubes are not visible from this position/perspective.

a) How many cubes will a tower of height of 30 have?

b) A number $n \geq 3$ of cubes, placed side by side covers perfectly a square. For what values of n can we rearrange the cubes so that we can build a pyramid as the one in the image, without remaining any unused cubes? For every found value of n, what height has the built pyramid?

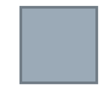
c) Build a regular triangular pyramid by overlapping some spheres of diameter 1(instead of cubes). What is the height of such a pyramid formed with 1330 balls?

d) What is the volume of the minimal tetrahedron in which the pyramid found at point c) can be inscribed?

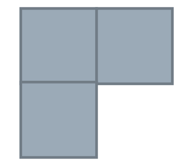
Part a)

How many cubes will a tower of height of 30 have?

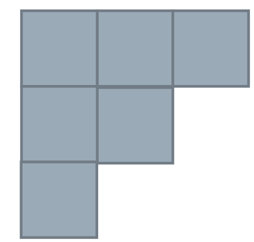
- In the first part, we found a rule of how is the figure made.
- In the first layer, seen from the top, we see a single cube.



- In the second layer we see (1+2=3) cubes.



- In the third layer we see (1+2+3=6) cubes



- And so on…
- So we find a rule of the number of the cubes that are in a layer.

- This equation is (with k the number of the layer and n the number of cubes in that layer):

$$n = 1 + 2 + 3 + \cdots + k$$

$$n = \frac{(1 + k) \cdot k}{2}$$

- So, using that equation, we found that the total number of cubes that are needed to make a tower of height 30 is:

$$n = \frac{1^2+1}{2} + \frac{2^2+2}{2} + \frac{3^2+3}{2} + \ldots + \frac{30^2+30}{2}$$

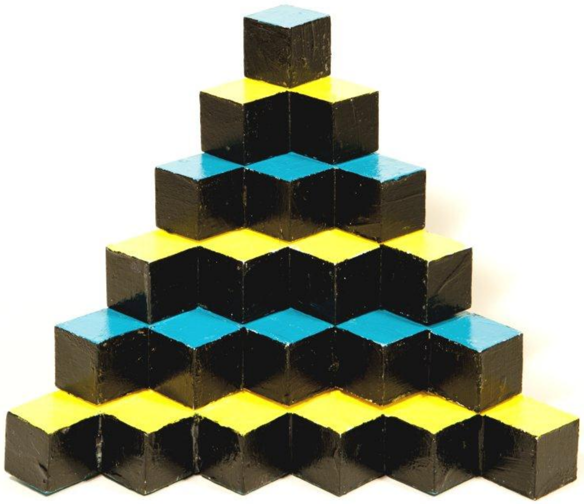
$$n = \frac{1}{2}[(1^2 + 2^2 + 3^2 + \cdots + 30^2) + (1 + 2 + 3 + \cdots + 30)]$$

$$n = \frac{1}{2}\left[(1^2 + 2^2 + 3^2 + \cdots + 30^2) + \frac{30 \cdot 31}{2}\right]$$

But, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$

So, the final number of cubes for a pyramid of height 30 is:

$$n = \frac{1}{2} \cdot \left[\frac{30 \cdot 31 \cdot 61}{6} + \frac{30 \cdot 31}{2}\right] = \frac{1}{2} \cdot (9455 + 465)$$



$$\mathrm{S} = 4960$$

Part b)

- A number $n \geq 3$ of cubes, placed side by side covers perfectly a square. For what values of n can we rearrange the cubes so that we can build a pyramid as the one in the image, without remaining any unused cubes? For every found value of n, what height has the built pyramid?

- Let's say C_n is the number of cubes in a pyramid of height n.
- If n=1, then $C_n=1$
- If n=2, then $C_n=4$
- If n=3, then $C_n=10$
- If n=4, then $C_n=20$
- If n=5, then $C_n=35$
- If n=6, then $C_n=56$

- So, we found a rule for the number of cubes used to build a pyramid of height h.

$$n = \frac{1}{2}\left[\frac{h(h+1)(2h+1)}{6} + \frac{h(h+1)}{2}\right]$$

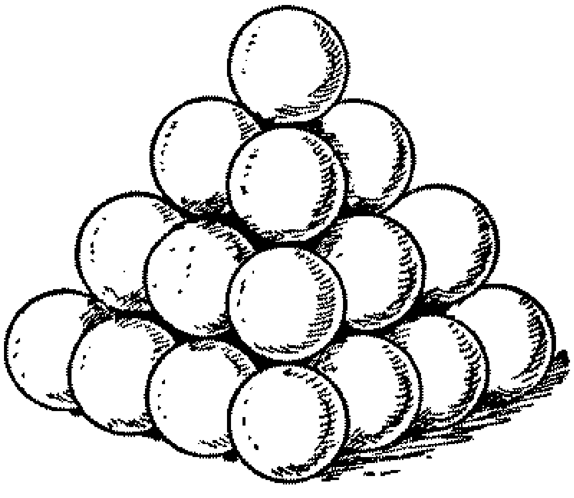
$$n = \frac{h(h+1)(h+2)}{6}$$

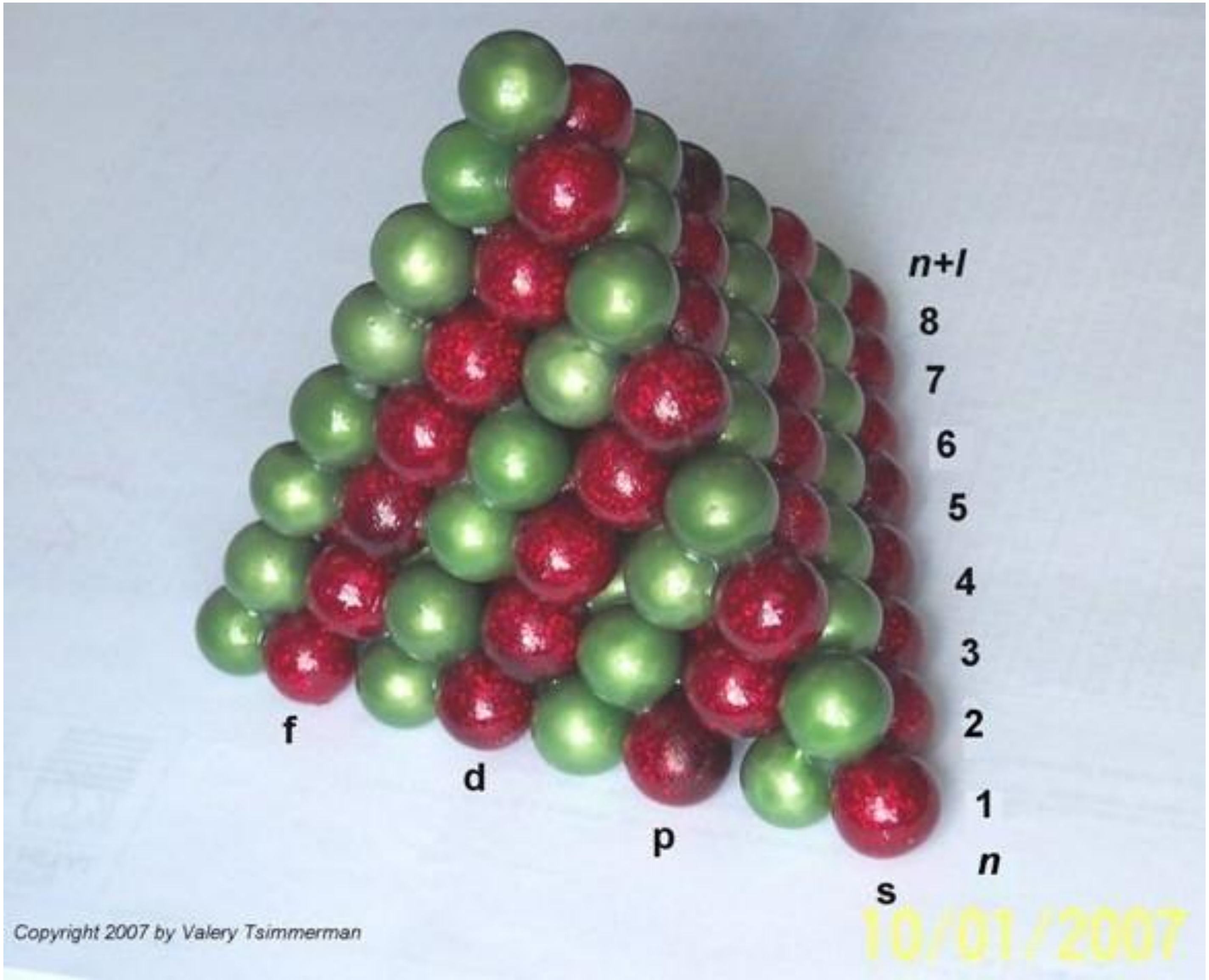
- But, n cubes should be put in a square, so n should be a perfect square.
- If the height is 1, the n is 1 and l is 1, where l is the side of the square covered by that n cubes.
- If the height is 2, the n is 4 and l is 2.
- If the height is 48, the n is 19600 and l is 140.

- We observed that this is the formula of the tetrahedral numbers. So *n* should be a tetrahedral number.
- A tetrahedral number, is a figurate number that represents a pyramid with a triangular base and three sides.
- But the only tetrahedral numbers that are making square numbers are 1,4 and 48.
- But *n* must be equal or bigger than 3, so *n* can take the values 4 or 48.

Part c)

- Build a regular triangular pyramid by overlapping some spheres of diameter 1(instead of cubes). What is the height of such a pyramid formed with 1330 balls?




n+l
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- Using the rule that we found at a):

$$1330 = \frac{1}{2}[(1^2 + 2^2 + 3^2 + \cdots + h^2) + (1 + 2 + 3 + \cdots + h)]$$

$$1330 = \frac{1}{2}\left[\frac{\mathrm{h(h+1)(2h+1)}}{6} + \frac{\mathrm{h(h+1)}}{2}\right]$$

- Where *h* is the height required.

$$2660 = \frac{\mathrm{h(h+1)(2h+1)}}{6} + \frac{\mathrm{h(h+1)}}{2}$$

$$2660 = \frac{\mathrm{h(h+1)(2h+1) + 3h(h+1)}}{6}$$

$$2660 = \frac{\mathrm{h(h+1)(2h+1+3)}}{6} = \frac{\mathrm{h(h+1)(h+2)}}{3}$$

$$\mathrm{h(h+1)(h+2)} = 7980$$

- As discussed before, the number of balls on the bottom layer is $\frac{n(n+1)}{2}$.
- Solving in $\boldsymbol{N}$ the equation

$$\frac{n(n+1)}{2} = 190$$

- We obtain $n = 19$. Therefore, the base length of a pyramid having 190 balls is 19.
- The height of the pyramid formed by stacking spheres is a function of n and d (the diameter of each sphere). We need to find the height of the pyramid in terms of n and d.
- Firstly, we consider the case of a tetrahedron made from four balls: three balls on the bottom and one ball at the top. The centres of these spheres form a tetrahedron with an edge length of d and a height of $d\sqrt{\frac{2}{3}}$.

› The total height of stack of spheres is

$$d + d\sqrt{\frac{2}{3}}.$$

› We now add a third layer to the bottom of the pyramid (add six balls) and obtain a total of 10 balls, We then consider the tetrahedron formed by the centers of the spheres at the corners of the pyramid. This tetrahedron has an edge length of $2d$ and a height of $2d\sqrt{\frac{2}{3}}$. The total height of the stack is

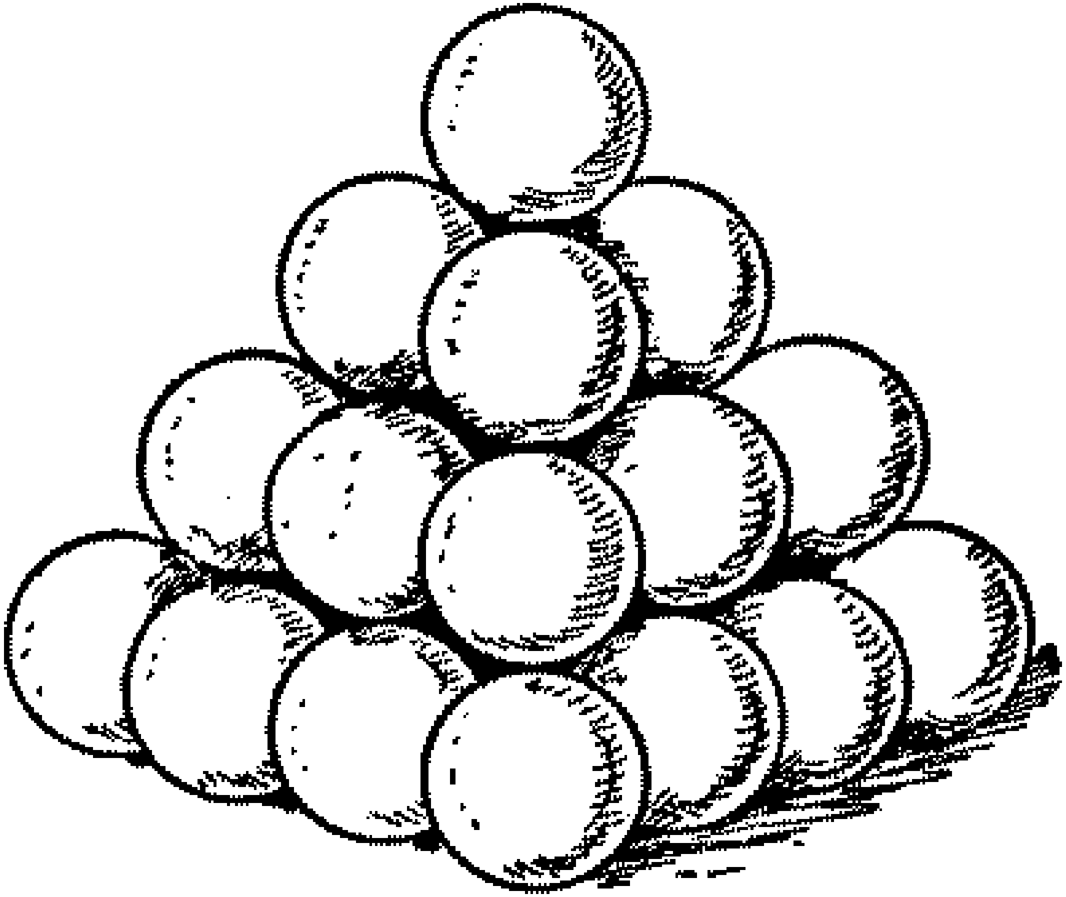
$$d + 2d\sqrt{\frac{2}{3}}$$

- In general, the total height of a pyramid with n levels is

$$d + (n-1)d\sqrt{\frac{2}{3}}$$

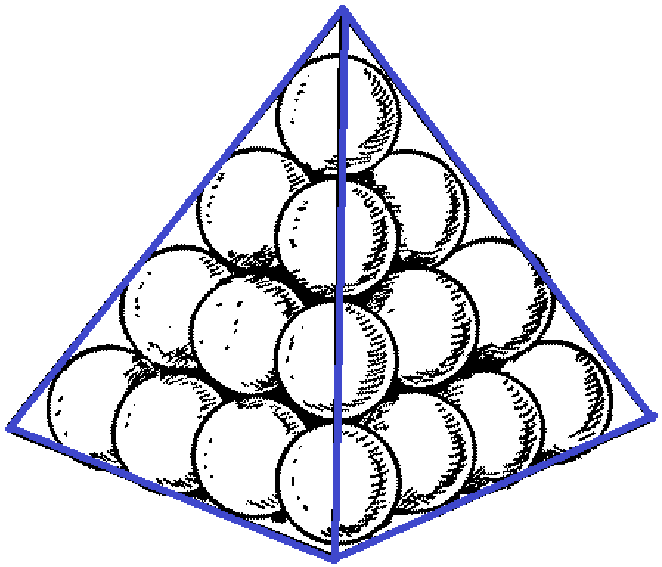
- For $n = 19$ and $d = 1cm$, we get that the height of a pyramid made of 1330 similar balls is

$$h \cong 15.7\ cm$$



Part d)

What is the volume of the minimal tetrahedron in which the pyramid found at point c) can be inscribed?



- The base of the 2 level pyramid can be inscribed in an equilateral triangle with a side length of

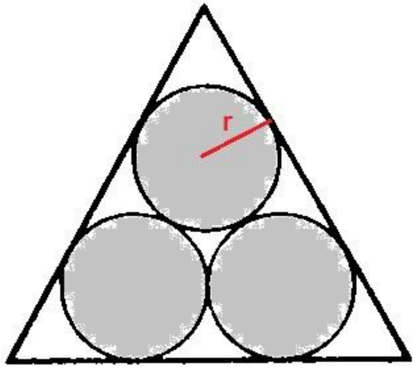
$$d + d\sqrt{3}$$

- In general, the base of the n level pyramid can be inscribed in an equilateral triangle with a side length of

$$d(n-1) + d\sqrt{3}$$

- For $n = 19$ and $d = 1cm$, we obtain that the side length of a regular pyramid is $l = 18 + \sqrt{3}$. The volume of the minimal tetrahedron is

$$V = l^3 \frac{\sqrt{2}}{12} \cong 905.42\ cm^3$$



- The volume of a single sphere is $V_{sphere} = \frac{4\pi R^3}{3} = \frac{\pi d^3}{6} = \frac{\pi}{6}$, and so, the volume of the pyramid made of 1330 spheres is

$$V_p = 665\frac{\pi}{3} \cong 696.39\ cm^3,$$

which is representing a fraction of about 77% out of the volume of the tetrahedron.