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Abstract of PhD Thesis

MODELS OF ALGEBRAIC CURVES AND COVERS OF ALGEBRAIC CURVES

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IAŞI 2018

One of the central objects of study in arithmetic geometry are curves X defined by polynomial equations with rational coefficients. By eliminating the denominators, we may choose the polynomials definining X to have coefficients in $\mathbb{Z}$. In this way we obtain a scheme $\mathcal{X}$ of finite type over $\operatorname{Spec}\mathbb{Z}$, whose generic fiber is X. But $\operatorname{Spec}\mathbb{Z}$ consists, besides the generic point (corresponding to the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$), of a closed point for each prime p, (corresponding to the reduction map $\mathbb{Z} \to \mathbb{F}_p$), so $\mathcal{X}$ consists of the generic fiber X and, for each prime p, a special fiber $\mathcal{X}_p$, which is a curve over $\mathbb{F}_p$, defined by the reductions $\bmod p$ of the polynomials defining $\mathcal{X}$. We say that $\mathcal{X}$ is a model of X and that $\mathcal{X}_p$ is the reduction modulo p of X.

The question that we ask is essentially the following: what can we say about the curve $\mathcal{X}_p$?. For example, can we find a model $\mathcal{X}$ over $\mathbb{Z}$ such that the reductions $\mathcal{X}_p$ are smooth over $\mathbb{F}_p$? And if we cannot find such a model, what kind of singularities can we hope to find on the special fibers. There are two things that can be noticed right from the start. First, our problem is local on the base, so that we may replace $\mathbb{Z}$ with its localization at a fixed prime p. Secondly, after taking a base extension of $\mathbb{Z}$, the special fiber $\mathcal{X}_p$ might have milder singularities, as can be seen when studying the reduction of elliptic curves.

Therefore, the setup that we use consistently is the following. $\mathcal{O}_K$ will be a discrete valuation ring with fraction field K and X will be a smooth projective curve over K. By a model of X over $\mathcal{O}_K$ we mean a normal scheme $\mathcal{X}$, flat and projective over $\mathcal{O}_K$ with generic fiber X. A standard result in this direction is the semi-stable reduction theorem of Deligne and Mumford, which states that, eventually after a finite base change of K, there exists a model $\mathcal{X}$ of X that is semi-stable, i.e. its special fiber is reduced projective curve with only ordinary double points as singularities.

We say that X is marked if we fix a finite set of K-rational points on X. A model of the marked curve (X, M) is $\mathcal{X}$ is marked if the specializations of the points in M on the special fiber are all distinct.

A Belyi morphism is a finite cover $f_K : Y \to X = \mathbb{P}^1_K$ with only three branch points, which can be assumed to be $0, 1, \infty$. We mark Y and X with the ramification locus and branch locus respectively, view f_K as morphism of marked covers.

If $f_K : (Y, N) \to (X, M)$ is a finite morphism of marked curves, a model of f_K is a finite morphism of schemes $f : \mathcal{Y} \to \mathcal{X}$ such $\mathcal{Y}$ and $\mathcal{X}$ are marked models of Y respectively X and f extends f_K.

The main results of this thesis are the following:

1. **A new proof of the "easy part" of Belyi's theorem.**

Using a theorem of Fulton regarding the good reduction of a tame cover, and Abhyankhar's lemma, we give a new proof of one implication of Belyi's theorem.

2. **A characterization of the marked stable model of a marked curve.**

Given a marked smooth projective curve (X, M) over K, and $\mathcal{X}$ a fixed semi-stable model of X, we describe the minimal marked semi-stable model $\mathcal{X}'$ that dominates $\mathcal{X}$, following the ideas of Liu from the non-marked case.

3. **A characterization of all regular and semi-stable models of $\mathbb{P}^1_K$, when K is the fraction field of a complete discrete valuation ring $\mathcal{O}_K$.**

The smooth models of $\mathbb{P}^1_K$ over $\mathcal{O}_K$, while they are all abstractly isomorphic to $\mathbb{P}^1_{\mathcal{O}_K}$, are not necessarily isomorphic as models of X. The isomorphism classes of smooth models can be viewed as the vertices of a metrized tree Δ, called the tree of $PGL(2, K)$. We show that all the regular models of $\mathbb{P}^1_K$ can be identified with the finite subtrees of Δ, while the semi-stable models correspond to the finite linked subsets of Δ.

4. **A complete proof of a good reduction result for Belyi morphisms f when the order of the monodromy group of f is prime to the residue characteristic of $\mathcal{O}_K$.**

We adapt some results of Fulton to give a self-contained proof of the following fact: when the monodromy group of a Belyi map has order prime to the residue characteristic, the cover has good reduction, i.e. the stable model of f is a tamely ramified cover of smooth curves, with the same ramification data as f.

5. **A detailed proof of the structure of the reduction of a Galois Belyi morphism with Galois group of order strictly divisible by the residue characteristic of $\mathcal{O}_K$.**

The structure of the reduction of a Galois Belyi morphism $f_K : Y \to \mathbb{P}^1_K$ with Galois group G of order strictly divisible by the residue characteristic of $\mathcal{O}_K$ hsa been studied by both Raynaud and Wewers and is spread in several of their papers. We collect and synthetize all their results to give an (almost) self-contained proof of the fact that, if $f : \mathcal{Y} \to \mathcal{X}$ denotes the stable model of f_K, the special fiber $\mathcal{X}_k$ of $\bar{X}$ is at most a comb of projective lines.