Universitatea "Alexandru Ioan Cuza" din Iași
Facultatea de Matematică

Contribuții la studiul unor obiecte din geometria algebrică și teoria grupurilor prin metode combinatoriale

Rezumat teză doctorat

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Rezumat

Această lucrare este rezultatul a două programe de doctorat: unul axat pe geometrie algebrică și unul pe teoria grupurilor. Deși temele nu au o legătură directă, ele sunt unite de ideea comună a studiului unor subiecte complexe prin intermediul unor concepte ușor de explicat, precum dessins d'enfants și funcțiile aritmetice. Prima parte a tezei folosește instrumente din geometria algebrică și transformă căutarea operatorilor diferențiali algebrici într-o căutare de grafuri. A doua parte a tezei preia concepte din teoria numerelor și le transpune în studiul grupurilor finite. Cele două părți sunt tratate independent și reprezintă activitatea de cercetare în cadrul celor două programe. Rezultatele prezentate sunt cuprinse în 6 articole ([11], [12], [13], [10], [6], [14]), scrise fie în colaborare, fie individual, primele 5 dintre acestea fiind deja publicate.

Prima parte a tezei este structurată astfel: capitolul introductiv oferă inițial un istoric al subiectului abordat. Ulterior, este organizat în două secțiuni. În prima, trecem în revistă conceptele teoretice legate de operatorii diferențiali: definim diferitele tipuri de singularități, apoi explorăm rezultatele care caracterizează algebricitatea operatorilor diferențiali. În a doua, trecem în revistă câteva concepte utile legate de teorema lui Belyi și dessins d'enfants. Ne concentrăm în mod special pe "dicționarul" care prezintă corespondența dintre datele de ramificație ale funcției Belyi și datele combinatorice ale dessin-ului corespunzător.

În capitolul al doilea, analizăm operatorii Heun și prezentăm propria analiză a operatorilor algebrici cu grup de monodromie tetraedral și octaedral. Pornind de la teorema lui Klein, căutăm operatori Heun ca pull-back-uri ale operatorilor hipergeometrici din lista lui Schwarz, printr-o funcție Belyi. Urmăm algoritmul de mai jos:

- Determinăm condiții aritmetice pentru diferențele exponenților locali ai operatorului Heun.
- Stabilim gradul minim posibil pentru funcția f.
- Folosind "dicționarul" din capitolul precedent, verificăm dacă există un dessin d'enfants cu datele date.
- Dacă nu există un astfel de dessin, rezultă că operatorul Heun corespunzător nu este algebric. Mărim unul dintre exponenții locali și reluăm căutarea unui dessin d'enfants corespunzător.
- Alternativ, dacă putem construi graful, rezultă că există o funcție de pull-back și, prin urmare, un operator Heun algebric cu datele corespunzătoare. Pentru valori mici ale gradului funcției de pull-back, aceasta poate fi determinată ușor. Vom face acest lucru pentru $\deg f \leq 4$.
- Plecând de la pasul anterior, lăsăm să varieze cel puțin una dintre diferențele exponenților locali. Construim graful asociat prin adăugarea de "celule" — sub-grafuri "lipite" de graful inițial.

În acest fel, obținem familii infinite de dessins d'enfants parametrizate de numere naturale. Corespunzător, aceasta înseamnă că există familii infinite de operatori Heun algebrici.

Reușim să completăm studiul lui Vidūnas și să recuperăm unele rezultate din [5] și [8]. Rezultatele pentru operatorii cu monodromie tetraedrală sunt publicate în [10]. O analiză similară cu cea din secțiunile doi și trei poate fi făcută și pentru operatorii Heun cu monodromie icosaedrală. Studiul complet arată că există un număr numărabil de operatori Heun algebrici.

În dorința de a continua studiul operatorilor diferențiali algebrici, ne-am propus să deformăm operatori Heun "adăugând" o singularitate aparentă și să studiem ce se întâmplă cu algebricitatea soluțiilor. Astfel, ultima secțiune conține câteva calcule nepublicate privind deformări ale operatorilor Lamé, pornind de la rezultate similare din [15]. În prima subsecțiune, prezentăm calculele pas cu pas pentru non-logaritmicitatea operatorilor Lamé și recuperăm un rezultat referitor la numărul finit de operatori Lamé algebrici. În a doua subsecțiune, facem primii pași pentru a obține o procedură de reducere similară cu [15, Propoziția 2.2]. Ne propunem să continuăm acest studiu ulterior.

Această parte a tezei este structurată în trei capitole: această Introducere, Rezultatele principale și Anexa. Acestea sunt urmate de o Bibliografie dedicată acestei părți.

Capitolul de Introducere este împărțit în trei secțiuni. Prima secțiune, *Concepte din teoria numerelor*, parcurge conceptele din teoria numerelor care sunt de interes în studiul nostru.

Mai întâi, trecem în revistă *funcțiile divizorilor*. Ele permit definirea concisă a conceptelor folosite ulterior. Dintre acestea, cea mai importantă este funcția σ_z. Dat un număr complex z, funcția σ_z este suma puterii z a divizorilor unui întreg. Oferim o demonstrație a *multiplicității* ei. Cazurile particulare cele mai relevante sunt pentru $z \in \{-1, 0, 1\}$. Pentru $z = 1$, de obicei σ_1 se notează simplu σ. În afara acestora, analizăm și alte funcții definite pentru un număr natural nenul n: $\rho(n) = \sigma(n)/n$, $\phi(n)$ (funcția lui Euler), $w(n)$ (numărul divizorilor primi distincți).

În al doilea rând, folosind aceste funcții, dăm definițiile pentru numere aproape/cvasi-/perfecte. Prezentăm forma cunoscută a numerelor perfecte pare și, folosind-o, demonstrăm că ecuația

$$P_n + 1 = p^k, \quad P_n \text{ număr perfect par, } p \text{ număr prim} \tag{*}$$

are soluții numai pentru $k = 1$. Enumerăm, de asemenea, cele 4 soluții pe care le-am putut găsi computațional folosind [17] și [16].

De asemenea, trecem în revistă rezultatele de ultimă oră referitoare la probabilitatea tot mai mică a existenței numerelor perfecte impare. În mod similar, nu sunt cunoscute exemple de numere cvasi-perfecte și există un singur tip de numere aproape perfecte: puterile lui doi.

În al treilea rând, analizăm *numerele armonice*, o generalizare a numerelor perfecte. La fel ca în cazul celor din urmă, nu se cunosc numere armonice impare.

Încheiem secțiunea de teorie a numerelor, analizând conceptul de *factorizare nilpotentă* [9] și prezentăm o caracterizare a acestuia. Această factorizare își are rădăcinile în teoria grupurilor, așadar nu urmează tiparul "traducerii" într-un analog de teorie a grupurilor în secțiunea următoare.

A doua secțiune a introducerii se concentrează pe "traducerea" promisă a funcțiilor din secțiunea anterioară. Având ca punct de plecare lucrările lui Leinster [4] și de Medts [7], observăm proprietățile funcțiilor σ_z și ρ; analizăm lucrările lui Garonzi [3], respectiv Azad, Khosravi și Rashidi [1], pentru proprietăți ale lui m (suma inverselor ordinelor elementelor unui grup); și ale lui Baishya și Kumar [2], pentru analiza proprietăților lui H.

În ultima secțiune a Introducerii, reamintim patru clase de grupuri care vor apărea în studiul nostru al grupurilor aproape/cvasi-/Leinster: grupuri ZM, grupuri afine, grupuri diedrale și grupuri diciclice. Reamintim definiția lor și analizăm laticea subgrupurilor normale.

Capitolul al doilea prezintă rezultatele principale ale acestei părți a tezei. Este împărțit în trei secțiuni, acoperind cele trei subiecte abordate în cele patru articole scrise: Grupuri aproape/cvasi-Leinster ([11]), Grupuri finite cu media armonică a ordinelor elementelor întreagă ([13], [14]), Numere aproape $\mathcal{P}$ ([12]).

Începem prima secțiune prin a trece în revistă un rezultat existent al lui Leinster care ne permite să demonstrăm că singurele grupuri nilpotente aproape-/Leinster sunt cele ciclice de ordin aproape-/perfect. Demonstrăm apoi un rezultat analog pentru grupurile cvasi-Leinster. Aceasta încheie clasificarea grupurilor nilpotente aproape/cvasi-/Leinster. Ulterior, analizăm cele patru clase de grupuri non-nilpotente studiate în introducere: grupuri ZM, afine, diedrale și diciclice. Găsim exemple de grupuri aproape și cvasi-Leinster pentru primele două folosind soluțiile ecuației (*). Demonstrăm că existența grupurilor diedrale D_{2n} aproape-/cvasi-/Leinster este echivalentă cu faptul că n este impar, iar $2n$ este aproape-/cvasi-/perfect. În mod similar, pentru grupurile diciclice, demonstrăm că existența unui grup Dic_{4n} aproape-/cvasi-/Leinster este echivalentă cu n impar și $2n$ aproape-/cvasi-/perfect.

În a doua secțiune, analizăm

$$\text{grupurile finite } G \text{ care au media armonică a ordinelor elementelor } h_m(G) \in \mathbb{N}^*, \tag{**}$$

utilizând rezultatele din [13]. Demonstrăm câteva proprietăți de bază ale funcției nou introduse, precum inegalități și comportamentul față de subgrupuri. Ulterior, examinăm contraimaginea h_m^{-1} pentru diferite mulțimi și tipuri de grupuri. Începem prin a demonstra că singurele p-grupuri care satisfac (**) sunt cele ciclice de ordin $p^{\sum_{i=1}^{s} p^i}$, cu $s \in \mathbb{N}^*$, sau D_8, grupul diedral de ordin 8. În plus, arătăm că D_8 este singurul grup diedral din $h_m^{-1}(\mathbb{N}^*)$. Continuăm și demonstrăm că $h_m^{-1}(\{2\}) = \{C_4, D_8\}$ și, mai departe, că $h_m^{-1}([0,2]) = \{C_2^n \mid n \in \mathbb{N}^*\} \cup \{C_3, S_3, C_4, D_8\}$. Apoi arătăm că nu există grupuri de ordin impar și nici grupuri nilpotente în $h_m^{-1}(\{3\})$. Ultimul rezultat pe care îl demonstrăm este un criteriu de *superrezolubilitate* pentru grupuri folosind h_m. Întrebarea inițială de a determina $h_m^{-1}(\mathbb{N}^*)$ rămâne deschisă.

Ultima secțiune privește *numerele aproape* $\mathcal{P}$, adică întregi care pot caracteriza o proprietate de grup ($\mathcal{P}$) cu excepția unuia [13]. Începem reamintind un rezultat existent care caracterizează *numerele aproape ciclice*. Apoi caracterizăm *numerele aproape abeliene*. Încheiem secțiunea demonstrând că aceste numere sunt totodată și *aproape nilpotente*.

Anexa conține trei secțiuni cu algoritmi GAP și rezultatele lor. Algoritmii corespund celor trei secțiuni ale capitolului anterior. Primul cod găsește soluții ale ecuației (*). Al doilea calculează valoarea lui h_m pentru grupuri de ordin mic, permițându-ne să confirmăm unele rezultate și să găsim exemple de grupuri din $h_m^{-1}(\mathbb{N}^*)$. Al treilea și al patrulea algoritm generează primele n numere aproape abeliene și grupul non-abelian corespunzător fiecărui ordin.

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Contributions to the Study of Objects in Algebraic Geometry and Group Theory Using Combinatorial Methods

PhD Thesis Summary

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To the memory of my grandmother: "Finally our dreams came true."

Foreword

This work is the result of two PhD programs: one focused on algebraic geometry and one in group theory. The first part of the thesis uses tools from algebraic geometry and transforms the search for algebraic differential operators into a search for graphs. The second part of the thesis takes concepts from number theory and translates them to the study of finite groups. The two parts are treated independently and represent the work of the author in the two programs. The research encased is presented in 6 articles ([22], [23], [24], [46], [42], [25]) written in collaboration or individually, the first 5 of them being already published.

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Contents

Part I

Combinatorial objects in the study of differential operators with finite monodromy

Chapter 1

Algebraic geometry introduction

The problem of algebraicity of solutions of differential equations has been studied since the end of the nineteenth century. The first part of the thesis offers some contributions to this topic.

In this introduction, we mention some historical aspects relevant to the topics involved in our thesis. This is also present in [42] and [46].

Riemann studied algebraic functions [49] and the hypergeometric equation [48]. Following the work of Cauchy regarding the existence of solutions of differential equations, Bouquet and Briot studied singular points of differential equations [9] and [10].

Fuchs focused on looking at what are now called Fuchsian equations and obtained results similar to those due to Cauchy for differential equations with only regular points.

For the hypergeometric operators, Schwarz determined completely when the corresponding equation has only algebraic solutions by studying the algebraicity of the ratio of two independent solutions and gave a list of the operators with algebraic solutions [51]. Klein introduced in [34] equivalences between operators and reduced the original list of Schwarz to what is called the "basic Schwarz list". He also gave a theorem that simplified the problem of determining the algebraicity of a second order differential operator: the algebraic second operator is a pull-back of a hypergeometric algebraic operator.

Starting in 1979, Baldassari and Dwork restarted studying the problem [4, 2, 1, 3, 16, 17]. Based on some work of Brioschi, Baldassari gave necessary conditions for a Lamé operator to be algebraic [2, Page 46]. In the 2000's, Liţcanu recovered these results in a new way, using Grothendieck's theory of Belyi functions and dessins d'enfants [38], [39].

Belyi proved in 1978 the direct implication of what is known today as Belyi's Theorem [5, Theorem 4]. Starting from this result, Grothendieck introduced dessins d'enfants and the correspondence between these and transitive groups of permutation with three generators. He also presented the correspondence between dessins and what is known today as Belyi functions [23].

Using Belyi's Theorem, Liţcanu proved that the pull-back function for a Lamé is a Belyi function [39, Proposition 1.4]. The algebraicity of the Lamé operator determines restrictions on the ramification data of the pullback function. Since the pull-back function is a Belyi function, it has a corresponding dessin d'enfant. The search for the pull-back function is thus translated into the search for a dessin d'enfant with prescribed combinatorial data.

This method is described in greater detail in [42] and can be used for general second order operators with four singular points: Heun operators.

In this thesis, we use Liţcanu's method to complete the study of Heun operators by looking at those that are pull-backs of the hypergeometric operators with tetrahedral and octahedral monodromy. Afterwards, we explore deformations of second order operators by the addition of apparent singularities, and study under what conditions the deformations are also algebraic.

This first part of this thesis is structured as follows: The rest of this introductory chapter is structured into two sections. In the first, we go through the theoretical concepts related to differential operators. We go through the definitions of the different types of singularities, then we explore the results that characterize the algebraicity of differential operators. In the second, we go through some useful concepts related to Belyi's theorem and dessins d'enfantsand focus on the "dictionary" that presents the correspondence between the ramification data of the Belyi function and the combinatorial data of the corresponding dessin.

In the second chapter, we look at Heun operators and present our search for algebraic operators with tetrahedral and octahedral monodromy group. Starting from Klein's theorem, we search for the Heun operators as pull-back of the hypergeometric operators from the Schwarz list via a Belyi function. We follow the algorithm below:

- We determine arithmetic conditions for the differences of the local exponents of the Heun operator.
- We find out the minimal possible degree for the function f.
- Using the "dictionary" from the previous chapter, we check if there exists a dessin d'enfant with the given data.
- If there is no such dessin, it follows that the corresponding Heun operator is not algebraic. We increase one of the local exponents and repeat the search of a corresponding dessin d'enfants.
- Alternatively, if we can construct the graph, it follows that there exists a pull-back function and therefore an algebraic Heun operator with the corresponding data. For values of the degree of the pull-back function, the function can be easily determined. We will do this for $\deg f \leq 4$.
- Starting from the previous step, we allow at least one of the differences of the local exponents to vary. We construct the associated graph by adding "cells" - sub-graphs that are "glued" to the initial graph.

  In this way, we obtain infinite families of dessins d'enfants parametrized by natural numbers. Correspondingly, this means that there are infinite families of algebraic Heun operators.

We manage to recover some results of [39] and [45]. The results for operators with tetrahedral monodromy are published in [46].

The following chapter contains some unpublished computations for deformations of Lamé operators, following similar work of [54]. We present the non-logarithmicity conditions for Lamé operators and recover a result regarding the finite number of algebraic Lamé operators. In the second section, we start trying to find a similar reduction procedure to [54, Proposition 2.2].

1.1 Algebraic differential operators

Definition 1.1.1. *[55, Page 5] By an* ***ordinary linear differential operator*** *of* ***order*** $n \in \mathbb{N}^*$ *we understand a differential operator of the form:*

$$L = \frac{\mathrm{d}^n}{\mathrm{d}\, z^n} + a_1(z)\frac{\mathrm{d}^{n-1}}{\mathrm{d}\, z^{n-1}} + \cdots + a_n(z), \tag{1}$$

with the coefficients $a_1, \ldots, a_n : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ *being rational functions.*

The corresponding ***ordinary linear differential equation*** *for the operator* L *given in* (1) *is given by:*

$$L(y(z)) = 0 \tag{2}$$

Example 1.1.2. *[56, Page 1] The* ***Gauss hypergeometric equation*** *is given by:*

$$\frac{\mathrm{d}^2\, y}{\mathrm{d}\, z^2}(z) + \left(\frac{\gamma}{z} + \frac{\delta}{z-1}\right)\frac{\mathrm{d}\, y}{\mathrm{d}\, z}(z) + \frac{\alpha\beta}{z(z-1)}y(z) = 0, \tag{3}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ *such that* $\alpha + \beta + 1 = \gamma + \delta$.

Definition 1.1.3. *[55, Definition 1.5.2] If the solution space of equation* (2) *has a basis of algebraic solutions, we call the* ***operator*** *and the corresponding* ***differential equation algebraic****.*

Definition 1.1.4. *[55, Definitions 1.1.1, 1.1.4] Consider equation* (2). *A point* $\alpha \in \mathbb{C}$ *is called:*

- ***regular*** *if there exists* $\lim\limits_{z\to\alpha} a_i(z) \in \mathbb{C}$, *for all* $i \in \{1, 2, \ldots, n\}$*;*
- ***regular singular*** *if there exists* $\lim\limits_{z\to\alpha} (z-\alpha)^i a_i(z) \in \mathbb{C}$, *for all* $i \in \{1, 2, \ldots, n\}$.

The analogous conditions for the point $z = \infty$ *are obtained by making the substitution* $z = \frac{1}{x}$ *and studying if* $x = 0$ *is a regular or a regular singular point of the new equation.*

Definition 1.1.5. *([55, Definition 1.2.1], [47, Page 76], [6, Definition 3.4]) An ordinary linear differential equation is called **Fuchsian** if any point in* $\mathbb{P}^1(\mathbb{C})$ *is a regular or a regular singular point for the equation.*

Definition 1.1.6. *[55, Definition 1.1.7] Let* $\alpha \in \mathbb{P}^1(\mathbb{C})$ *be a regular or a regular singular point of (2).*
Let

$$\alpha_i := \begin{cases} \lim_{z\to\alpha} (z-\alpha)^i a_i(z), & \text{if } \alpha \in \mathbb{C} \\ \lim_{z\to\infty} z^i a_i(z), & \text{if } \alpha = \infty, \end{cases} \quad \text{for all } i = \{1, 2, \dots, n\}.$$

*The **indicial equation** at*

- $\alpha \in \mathbb{C}$ *is:*

$$X(X-1)\dots(X-n+1)+\alpha_1 X(X-1)\dots(X-n+2)+\dots+\alpha_{n-1}X+\alpha_n = 0. \quad (4)$$

- $\alpha = \infty$ *is:*

$$X(X+1)\dots(X-n+1) - \dots + (-1)^{n-1}\alpha_{n-1}X + (-1)^n \alpha_n = 0 \quad (5)$$

*The solutions of this equation are called **local exponents**.*

Example 1.1.7. *[55, Page 10] In the context of the above result, the local exponents at a regular point are* $0, 1, \dots, n-1$.

Remark. [47, Section 21] The regular singularities and the corresponding local exponents of a Fuchsian equation can be organized in its **Riemann P-symbol/Riemann scheme**.

Example 1.1.8. *[56, Equation 2.1] The Riemann scheme for the hypergeometric equation given by* (29) *is:*

$$\left(\begin{array}{ccc|c} 0 & 1 & \infty & z \\ \hline 0 & 0 & \alpha & \\ 1-\gamma & 1-\delta & \beta & \end{array}\right) \quad (6)$$

Proposition 1.1.9. *[55, Proposition 1.5.6] If an ordinary linear differential equation has only algebraic solutions, then the equation is Fuchsian and the local exponents are rational.*

Theorem 1.1.10 (Fuchs' relation). *Consider a Fuchsian equation of order* $n \in \mathbb{N}^*$. *Let* $\rho_1(P), \dots, \rho_n(P)$ *the set of local exponents at any* $P \in \mathbb{P}^1(\mathbb{C})$. *Then the following relation holds:*

$$\sum_{P\in\mathbb{P}^1(\mathbb{C})} \left(\rho_1(P) + \dots + \rho_n(P) - \binom{n}{2}\right) = -2\binom{n}{2} \quad (7)$$

Theorem 1.1.11 (Fuchs). *[6, Theorem 2.6] Let $\alpha \in \mathbb{P}^1(\mathbb{C})$ be a regular singularity of* (2) *and t be a local parameter at α. We distinguish two cases regarding the local exponents:*

1. *Assume that ρ is a local exponent at α such that none of the numbers $\rho+1, \rho+2, \dots$ is a local exponent. In this case, there exists $g(t) = \sum_{i\geq 0} g_i t^i$ power series in t with $g_0 \neq 0$ such that $t^\rho \cdot g(t)$ is a solution.*

2. *Let $\rho_1, \dots, \rho_n$ be the set of local exponents ordered such that the local exponents that differ by an integer occur in a decreasing order. Then there exists a nilpotent matrix $N = (N_{ij})_{1\leq i,j\leq n} \in nil(\mathcal{M}_{n\times n}(\mathbb{C}))$, $g_1, g_2, \dots, g_n$ analytic around $t = 0$ with $g_k(0) \neq 0$ $(k \in \{0, 1, \dots, n\})$ such that*

$$(t^{\rho_1} g_1, t^{\rho_2} g_2, \dots, t^{\rho_n} g_n) t^N$$

is a basis of solutions. Moreover, if $N_{ij} \neq 0$ it follows that $i \neq j \wedge \rho_i - \rho_j \in \mathbb{N}$.

Remark. In the second case, a logarithmic term appears in the solution. In this situation, we say the equation has a **logarithmic solution**.

Remark. It follows from the previous theorem that since we are searching only for algebraic solutions, we will be interested only in operators with the differences of local exponents non-integers.

Definition 1.1.12. *[55, Pages 21 - 23] Let $S \subset \mathbb{P}^1(\mathbb{C})$ be the set of singularities of* (1)*. Let z_0 be a regular point and $y_1, y_2, \dots, y_n$ be a basis of solutions around z_0. If these are continued analytically along $u \in \pi(\mathbb{P}^1(\mathbb{C}) \setminus S, z_0)$, the functions $y_1, y_2, \dots, y_n$ are transformed into new functions: $\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_n$, which are independent solutions of the equation. It follows that there exists $M_{\mathbf{y}}(u) \in \mathrm{GL}(n, \mathbb{C})$ s.t.*

$$\begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \dots \\ \tilde{y}_n \end{pmatrix} = M_{\mathbf{y}}(u) \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{pmatrix}.$$

The ***monodromy representation*** *is obtained:*

$$\begin{aligned} M_{\mathbf{y}} : \pi_1(\mathbb{P}^1 \setminus S, z_0) &\to \mathrm{GL}(n, \mathbb{C}) \\ [u] &\to M_{\mathbf{y}}(u). \end{aligned}$$

We call $M_{\mathbf{y}}(\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus S, z_0))$ the ***monodromy group*** *of the equation.*

Definition 1.1.13. *[55, Pages 26 - 27] The n dimensional* ***projective linear group*** $\mathrm{PGL}(n,\mathbb{C})$ *is the quotient group of* $\mathrm{GL}(n,\mathbb{C})$ *by the group of scalar elements:*

$$\Lambda I := \{\lambda I_n \mid \lambda \in \mathbb{C}^*\}.$$

The projection map is:

$$P : \mathrm{GL}(n,\mathbb{C}) \to \mathrm{PGL}(n,\mathbb{C})$$
$$\gamma \mapsto [\gamma] := \gamma \cdot \Lambda I.$$

Let M be the monodromy group of an n-th order Fuchsian equation. Its image through the projection map $P : \mathrm{GL}(n,\mathbb{C}) \to \mathrm{PGL}(n,\mathbb{C})$ is called the ***projective monodromy group*** *of the equation.*

Theorem 1.1.14. *[55, Theorem 2.1.8] Consider a Fuchsian equation. The following statements are equivalent:*

- *The equation is algebraic;*
- *The monodromy group of the equation is finite.*

Definition 1.1.15. *([11, Page 2756], [55, Definition 2.3.2]) Two differential operators L_1 and L_2 differential operators are said to be* ***projectively equivalent*** *if there exists θ a radical function (a product of powers of rational functions) such that $L_2 = \theta \circ L_1 \circ \theta^{-1}$.*

Proposition 1.1.16. *[55, Proposition 2.3.3, Corollary 2.3.6] Let L_1 and L_2 be* ***projectively equivalent*** *differential operators. The following are true:*

1. *If L_1 is Fuchsian, then L_2 is Fuchsian.*
2. *Let $\rho_1, \rho_2, \ldots, \rho_n$ be the local exponents of L_1 at $\alpha \in \mathbb{P}^1(\mathbb{C})$. Then the set $\{\rho_2 - \rho_1, \rho_3 - \rho_1, \ldots, \rho_n - \rho_1\}$ is the set of differences of local exponents of both L_1 and L_2 at $\alpha \in \mathbb{P}^1(\mathbb{C})$.*

Proposition 1.1.17. *[47, Page 82] Consider a Fuchsian differential equation of order n.*

1. *If it has one regular singular point, it is projectively equivalent to:*

$$\frac{\mathrm{d}^n}{\mathrm{d}\, z^n}(y(z)) = 0; \tag{8}$$

2. *If it has two regular singular points, it is projectively equivalent to* ***Euler's homogeneous equation****:*

$$\left(z^n \frac{\mathrm{d}^n}{\mathrm{d}\, z^n} + c_1 z^{n-1} \frac{\mathrm{d}^{n-1}}{\mathrm{d}\, z^{n-1}} + \cdots + c_{n-1} z \frac{\mathrm{d}}{\mathrm{d}\, z} + c_n\right) y(z) = 0, \tag{9}$$

where $c_1, c_2, \ldots, c_n \in \mathbb{C}$, $n \in \mathbb{N}^*$.

Definition 1.1.18. *([55, Definitions 2.6.1, 2.6.2], [56, Page 868]) Let* $L := \frac{\mathrm{d}^n}{\mathrm{d}\,z^n} + a_1(z)\frac{\mathrm{d}^{n-1}}{\mathrm{d}\,z^{n-1}} + \cdots + a_{n-1}(z)\frac{\mathrm{d}}{\mathrm{d}\,z} + a_n(z)$ *be a Fuchsian operator with finite monodromy. If* z *is replaced by a non-constant rational function* $f(x) \in \mathbb{C}(x)$*, then* L *becomes:*

$$L_f := \left(\frac{\mathrm{d}}{f'(x)\,\mathrm{d}\,x}\right)^n + \cdots + a_{n-1}(f)\frac{\mathrm{d}}{f'(x)\,\mathrm{d}\,z} + a_n(f) \tag{10}$$

an n*-th order operator with derivation with respect to* x*, called the* ***proper rational pull-back*** *of* L *by* $z = f(x)$*.*

If L' *is a differential operator with respect to* x *that is projectively equivalent with* L_f*,* L' *is called a* ***rational pull-back*** *of* L *by* $z = f(x)$*.*

Proposition 1.1.19. *[55, Proposition 2.6.3] Let* L_f *be a proper rational pull-back of* L *by* f*. If* L *is Fuchsian, then* L_f *is Fuchsian.*

Proposition 1.1.20. *([56, Lemma 2.1], [55, Proposition 2.8]) Let* L' *be a rational pull-back of the Fuchsian operator* L *by* $z = f(x) \in \mathbb{C}(x)$*. Let* $\rho_1(\alpha)$ *and* $\rho_2(\alpha)$ *be two local exponents of* L *in* $\alpha = f(\tilde{\alpha}) \in \mathbb{P}^1(\mathbb{C})$*. Let* $e_{\tilde{\alpha}}$ *be the ramification index of* f *in* $\tilde{\alpha}$*. Then there exist the local exponents* $\rho_1(\tilde{\alpha})$ *and* $\rho_2(\tilde{\alpha})$ *of* L' *in* $\tilde{\alpha}$ *such that:*

$$\rho_1(\tilde{\alpha}) - \rho_2(\tilde{\alpha}) = e_{\tilde{\alpha}}(\rho_1(\alpha) - \rho_2(\alpha)). \tag{11}$$

Proposition 1.1.21. *([11, Page 2756]) Let* $L = \frac{\mathrm{d}^2}{\mathrm{d}\,z^2} + a_1(z)\frac{\mathrm{d}}{\mathrm{d}\,z} + a_2(z)$ *be a second order differential operator. Then it is projectively equivalent to one in normalized form:*

$$\frac{\mathrm{d}^2}{dz^2} + B(z), \quad \textit{where } B : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}).$$

Example 1.1.22. *[39, Equation (1.2)] The normal form of the hypergeometric operator is given by:*

$$H_{\lambda,\mu,\nu} = \frac{\mathrm{d}^2}{\mathrm{d}\,z^2} + \frac{1-\lambda^2}{4z^2} + \frac{1-\mu^2}{4(z-1)^2} + \frac{\lambda^2+\mu^2-\nu^2-1}{4z(z-1)}, \ \lambda,\mu,\nu \in \mathbb{C}. \tag{12}$$

Let us note that all Fuchsian equation of order 2 are equivalent to the hypergeometric equation is the only Fuchsian equation of order 2.

Theorem 1.1.23. *([55, Theorem 1.6.3], [39, Page 97]) Consider a second order differential operator with a basis of solutions* f_1*,* f_2*. The following are equivalent:*

1. f_1*,* f_2 *are algebraic (i.e. the operator is algebraic)*

2. *The Wronskian $W(f_1, f_2)(z)$ and the ratio of solutions $(f_1/f_2)(z)$ are algebraic.*

3. *The Wronskian $W(f_1, f_2)(z)$ is algebraic and the projective monodromy group is finite.*

Schwarz and Klein created a list of the hypergeometric operators with algebraic solutions, what is called the "basic Schwarz list" [34], [39, Page 97]:

$$\begin{array}{cc} (\lambda, \mu, \nu) & \text{Projective monodromy group} \\ \hline (1/n, 1, 1/n) & C_N,\ N \in \mathbb{N}^* \\ (1/2, 1/n, 1/2) & D_N,\ N \in \mathbb{N}^* \\ (1/2, 1/3, 1/3) & A_4 \\ (1/2, 1/3, 1/4) & S_4 \\ (1/2, 1/3, 1/5) & A_5 \end{array}. \tag{13}$$

Theorem 1.1.24 (Klein). *[39, Theorem 1.2] Let L be a second order differential operator in normal form on $\mathbb{P}^1(\mathbb{C})$ with finite projective monodromy group G. Then there exists a unique hypergeometric operator H belonging to (13), having G as projective monodromy group, such that L is a pull-back of H via a rational function $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. Moreover, the function f is also unique, modulo Möbius transformations leaving the operator H invariant and permuting its singular points.*

Proposition 1.1.25. *[42, Proposition 2.7] Let L be a Fuchsian operator as in Theorem 1.1.24, and S its set of singularities. The following hold true:*

- *f is only ramified above 0, 1 and ∞.*
- *$f(S) \subset \{0, 1, \infty\}$.*
- *Let $\alpha \in \{0, 1, \infty\}$ with $\rho_1(\alpha), \rho_2(\alpha)$ its local exponents, and $z_0 \notin S$ with $f(z_0) = \alpha$. Then the ramification index of f at z_0 is $e_{z_0} = \frac{1}{|\rho_1(\alpha) - \rho_2(\alpha)|}$.*

Proposition 1.1.26. *[42, Proposition 2.5] Consider L, L' two differential operators such that L' is the pull-back of L by the function f. Let*

- *$\Delta_L = \sum_{\alpha \in \mathbb{P}^1(\mathbb{C})} (|\rho_1(\alpha) - \rho_2(\alpha)| - 1)$, where $\rho_1(\alpha), \rho_2(\alpha)$ are the local exponents of L in α*
- *$\Delta_{L'} = \sum_{\alpha' \in \mathbb{P}^1(\mathbb{C})} (|\rho_1(\alpha') - \rho_2(\alpha')| - 1)$, where $\rho_1(\alpha'), \rho_2(\alpha')$ are the local exponents of L' in α'.*

It follows that

$$\Delta_{L'} + 2 = \deg f \cdot (\Delta_L + 2) \tag{14}$$

These results constitute the starting point of our analysis in the next chapter. Before proceeding to the next section, we give a few more facts.

Definition 1.1.27. *[47, Page 69] Consider a differential equation, P a singular point for it. The point P is called an **apparent singularity** if there is a basis of solutions around P of the differential equation that are holomorphic.*

1.2 Dessins d'enfants, Belyi morphisms

Theorem 1.2.1 (Belyi's Theorem for Riemann surfaces). *[22, Theorem 3.1] Let X be a compact Riemann surface. The following are equivalent:*

1. *X is defined over $\overline{\mathbb{Q}}$*
2. *There is a morphism of Riemann surfaces $f : X \to \mathbb{P}^1(\mathbb{C})$ with at most three critical values.*

Definition 1.2.2. *[22, Page 170] Given X a compact Riemann surface, a **Belyi function/morphism** is a morphism of Riemann surfaces $f : X \to \mathbb{P}^1(\mathbb{C})$ with at most three critical values. The pair (X, f) is called a **Belyi pair**.*

Up to Möbius transformations, the critical values can be taken to be $\{0, 1, \infty\}$.

Definition 1.2.3. *[37, Definitions 1.3.6, 1.5.2] A **map** is a pair $(X, \mathcal{D})$ where $\mathcal{D}$ is a multi-graph embedded into a surface X such that*

- *the vertices are distinct points of the surface;*
- *the edges are curves on the surface that intersect only at the vertices;*
- *the set $X \setminus \mathcal{D}$ is a disjoint union of connected components, called **faces**, each homeomorphic to an open disk.*

*The map $(X, \mathcal{D})$ is called a **hypermap/dessin d'enfant** if $\mathcal{D}$ is a bicoloured graph. We shall consider the two colours to be white and black. The dessin is called **clean** if all white vertices have degree 2.*

Theorem 1.2.4 (Grothendieck correspondence). *[23, 22] There is a bijective correspondence between the set of abstract dessins d'enfants and the isomorphism classes of Belyi pairs.*

Remark. [42, Page 48] In short, if one starts with a Belyi couple (X, f), then $f^{-1}([0, 1])$ is a graph embedded on X, the vertices being the elements of the set $f^{-1}(\{0, 1\})$ (the two colours). The other way around, if one has a dessin embedded in a surface X, it is easy to construct (topologically) a map $f : X \to \mathbb{P}^1(\mathbb{C})$ (seen as the topological sphere) with three branching points. Then the Riemann Existence Theorem guarantees the existence of a Riemann surface structure on X such that f becomes a rational function with three critical values. Moreover, Belyi's Theorem says that this Riemann surface can be defined over a number field.

This correspondence can be summarized in the following ”dictionary”:

Dessin d'enfant	**Belyi function with critical values** $\{0, 1, \infty\}$
White vertices	$f^{-1}(\{0\})$
Black vertices	$f^{-1}(\{1\})$
Faces	$f^{-1}(\{\infty\})$
Degree of white vertex v_w	Branching order of point $P_{v_w} \in f^{-1}(\{0\})$
Degree of black vertex v_b	Branching order of point $P_{v_b} \in f^{-1}(\{1\})$
Order of the face φ	Branching order of point $P_\varphi \in f^{-1}(\{\infty\})$
Number of edges	Degree of f
Edges of the graph	Sheets of $f^{-1}([0, 1])$

Table 1.1: Grothendieck correspondence between dessins d'enfants and Belyi functions

We will switch without mentioning between the first three lines of the table.

We have the following finiteness results:

Theorem 1.2.5. *[40, Proposition 3.2] Let m be a fixed positive real number. The set of Belyi pairs (X, f) (up to automorphisms of X and of $\mathbb{P}^1(\mathbb{C})$) of degree at most m is finite.*

In particular, the set of Belyi functions $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that the critical values are 0,1 and ∞ and $f(\{0, 1, \infty\}) \subset \{0, 1, \infty\}$, of bounded degree, is finite.

Proposition 1.2.6. *[42, Proposition 3.2] If M is a fixed constant and G is one of the finite groups appearing in the Schwarz list, there are finitely many isomorphism classes of couples (X, L) where L has the projective monodromy group G, no apparent singularity and $\Delta_L \leq M$.*

Theorem 1.2.7. *[41, Theorem 1.4] The set of isomorphism classes of couples (X, L), where L has finite monodromy and no apparent singularity, is countable.*

Chapter 2

Algebraic Heun operators

2.1 Generalities about Heun operators

Definition 2.1.1. *[50, Page 7] The **Heun equation** is the canonical second-order Fuchsian differential equation on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ with 4 regular singularities:*

$$\frac{\mathrm{d}^2 u(z)}{\mathrm{d}\, z^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-t}\right)\frac{\mathrm{d}\, u(z)}{\mathrm{d}\, z} + \frac{\alpha\beta z - q}{z(z-1)(z-t)} u(z) = 0, \tag{1}$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, t, q \in \mathbb{C}, t \neq 0, 1$ such that $\alpha + \beta + 1 = \gamma + \delta + \varepsilon$.

Proposition 2.1.2. *[50, Page 7] The Riemann scheme of* (1) *is given by:*

$$\left(\begin{array}{cccc|c} 0 & 1 & t & \infty & x \\ \hline 0 & 0 & 0 & \alpha & \\ 1-\gamma & 1-\delta & 1-\varepsilon & \beta & \end{array}\right) \tag{2}$$

Proposition 2.1.3. *[14] The normal form of the Heun equation is :*

$$\frac{\mathrm{d}^2 y(z)}{dz^2} - \left(\frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-t} + \frac{D}{z^2} + \frac{E}{(z-1)^2} + \frac{F}{(z-t)^2}\right) y(z) = 0. \tag{3}$$

Remark. [50, Page 7] Any equation with 4 regular singular points can be reduced to the Heun equation.

Definition 2.1.4. *Let us denote with $\mathcal{H}(a, b, c, d)$ the Heun operator given by* (3) *with local exponent differences $a, b, c, d \geq 0$. We will name these the **parameters** of the Heun operator for the regular singularities* 0, 1, *t*, ∞*.*

Remark. We denote the sum of the parameters with S:

$$S = a + b + c + d \tag{4}$$

2.1.1 Description of the algorithm

Remark. We want to find conditions for Heun operators to be algebraic. We use Klein's theorem and we search for Heun operators as pull-back of a hypergeometric operator with finite monodromy from the Schwarz list:

$$\mathcal{H}_{a,b,c,d} = f^*(H_{\lambda,\mu,\nu}). \tag{5}$$

From Theorem 1.1.25 it follows that we can look for the pull-back function as a Belyi function. Using the Grothendieck correspondence, we can instead focus on searching dessins d'enfants with the associated data.

From Proposition 1.1.20 and 1.1.25, the possible values for the ramification indices are given by:

$z\backslash x$	0	1	t	∞	Non-singular points over critical values
0	$\frac{a}{\lambda}$ or 0	$\frac{b}{\lambda}$ or 0	$\frac{c}{\lambda}$ or 0	$\frac{d}{\lambda}$ or 0	points of order $\frac{1}{\lambda}$
1	$\frac{a}{\mu}$ or 0	$\frac{b}{\mu}$ or 0	$\frac{c}{\mu}$ or 0	$\frac{d}{\mu}$ or 0	points of order $\frac{1}{\mu}$
∞	$\frac{a}{\nu}$ or 0	$\frac{b}{\nu}$ or 0	$\frac{c}{\nu}$ or 0	$\frac{d}{\nu}$ or 0	points of order $\frac{1}{\nu}$

Table 2.1: Possible values for the ramification indices

The columns corresponding to the values 0, 1, t, ∞ have only one non-zero value corresponding to the image of the value through the pull-back function. The lines correspond to the critical values: 0, 1, ∞ and their ramification set. The sum of the values on each line is equal to the degree of f. Using this, we can compute the number of non-singular points in the last column of the table.

Proposition 2.1.5. *Any algebraic Heun operator has the sum of its parameters greater than* 2.

Proof. The degree of f can be computed using equation 14:

$$(S-4+2)=\deg f(\lambda+\mu+\nu-3+2) \iff \tag{6}$$

$$S-2=\deg f(\lambda+\mu+\nu-1) \iff \deg f=\frac{S-2}{\lambda+\mu+\nu-1} \tag{7}$$

Since for operators in Schwarz list $\lambda+\mu+\nu>1$, it follows from (7) that $S>2$.

We describe the algorithm used in the analysis:

Algorithm 2.1.6. *[46, Page 217] We ignore permutations between the columns and also permutations between the second and third line, since the corresponding functions can be obtained from the pull-back functions corresponding to the graphs below by composing with a Möbius transformation. Therefore we will only study the essentially different possibilities. In each case:*

- *We determine arithmetic conditions for the parameters.*
- *We find out the minimal possible degree for the function f (from equation (7) this is equivalent to determining the minimal value of S).*
- *We check if there exists a dessin d'enfant with the given data.*
- *If there is no possible graph, it follows that the Heun operator with the predetermined parameters is not algebraic.*
- *If we can construct the graph, it follows that there exists a pull-back function and therefore an algebraic Heun operator with the corresponding data. For small values of the degree of the pull-back function, the function can be determined by symbolic computations. We will do this for* $\deg f \leq 4$. *It is important to note that when determining f, we also fix the value of the fourth singular point t.*
- *Starting from the minimal graph, we allow at least one of the parameters to vary. We construct the associated graph by adding cells. In this way, we obtain infinite families of dessins d'enfants parametrized by natural numbers. Correspondingly, this means that there are infinite families of algebraic Heun operators.*

2.1.2 Results for Lamé operators

Remark. [46] In a series of papers ([39, 38, 40, 41]), Liţcanu uses these techniques for studying the Lamé operators, a special case of Heun operators:

$$L_n = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}\sum_{i=1}^{3} \frac{1}{x-e_i}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{n(n+1)x+B}{4\left(x-e_1\right)\left(x-e_2\right)\left(x-e_3\right)}. \tag{8}$$

Their Riemann schemes are:

$$\begin{pmatrix} e_1 & e_2 & e_3 & \infty \\ 0 & 0 & 0 & -\frac{n}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{n+1}{2} \end{pmatrix}$$

We can suppose that three of the singular points are 0,1 and ∞. Let $\lambda \in \mathbb{C}$ be the fourth singular point.

Let E_λ be the elliptic curve described by the equation:

$$y^2 = x(x-1)(x-\lambda).$$

Then the pull-back of L_n by the projection

$$\pi_\lambda : E_\lambda \to \mathbb{P}^1 \quad , \quad \pi(x,y) = x$$

is a second order operator

$$L_n = D^2 - [n(n+1) + B]$$

having one singular point, 0_E, with the local exponents $(-n, n+1)$. Here $D = y\frac{\mathrm{d}}{\mathrm{d}x}$.

Theorem 2.1.7. *[39, Theorem 3.4]*

1. *There is no Lamé operator with cyclic projective monodromy group.*

2. *There is no Lamé operator with tetrahedral projective monodromy group.*

3. *If the projective monodromy group of L_n is octahedral, then*

$$n \in \frac{1}{2}\left(\mathbb{Z} + \frac{1}{2}\right) \cup \frac{1}{3}\left(\mathbb{Z} + \frac{1}{2}\right)$$

4. *If the projective monodromy group of L_n is icosahedral, then*

$$n \in \frac{1}{3}\left(\mathbb{Z} + \frac{1}{2}\right) \cup \frac{1}{5}\left(\mathbb{Z} + \frac{1}{2}\right)$$

5. *If the projective monodromy group of L_n is dihedral, then $n \in \mathbb{Z}$. If the projective monodromy group of L_n is finite and $n \in \mathbb{Z}$, then this group is dihedral of order at least 6.*

Theorem 2.1.8. *[41, Theorem 2.2] Let $n \notin \mathbb{Z} + \frac{1}{2}$ and G be a finite group. The set of isomorphism classes of elliptic curves on which there exists a Lamé operator L_n with projective monodromy G is finite, and on each such curve there exist finitely many such operators.*

Corollary 2.1.9. *[41, Corollary 2.3] If $n \notin \mathbb{Z} + \frac{1}{2}$, there are finitely many Lamé operators L_n with a full set of algebraic solutions.*

Theorem 2.1.10. *[38, Theorems 4.8, 4.9]*

1. *The number $\mathcal{C}(1, N)$ of non-homographic covers $f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ which transform by pull-back a hypergeometric operator H with dihedral projective monodromy group of order $2N$ into a Lamé operator L_1 is*

$$\mathcal{C}(1, N) = \frac{(N-1)(N-2)}{6} + \frac{2\varepsilon}{3}$$

where $\varepsilon = 1$ if $3 \mid N$ and $\varepsilon = 0$ if not.

2. *If $L(1, N)$ is the number of non-homographic Lamé operators L_1 with dihedral projective monodromy group of order $2N$, then*

$$\mathcal{C}(1, N) = \sum_{N' \mid N, N' \neq 2} L(1, N)$$

2.2 Graphs for tetrahedral case

In this section, we follow the work in [46] and we look for Heun operators as pull-back of the hypergeometric operator with tetrahedral monodromy:

$$\mathcal{H}_{a,b,c,d} = f^*(H_{\frac{1}{2},\frac{1}{3},\frac{1}{3}}).$$

In this case, the degree of f follows from (7):

$$\deg f = 6(S-2).$$

The possible ramification indices are given by:

$z \backslash x$	0	1	t	∞	
0	$2a$ or 0	$2b$ or 0	$2c$ or 0	$2d$ or 0	points of order 2
1	$3a$ or 0	$3b$ or 0	$3c$ or 0	$3d$ or 0	points of order 3
∞	$3a$ or 0	$3b$ or 0	$3c$ or 0	$3d$ or 0	points of order 3

Table 2.2: Possible values for the ramification indices in the tetrahedral case

Considering lines 2 and 3 are identical, there are 9 essentially different cases. In this summary, we present the analysis of the more interesting ones.

1.

	0	1	t	∞	
∞	$2a$	$2b$	$2c$	$2d$	$2(S-3)$ points of order 2
0					$2(S-2)$ points of order 3
1					$2(S-2)$ points of order 3

We want to determine the possible values for the local exponents. Without loss of generality, we can assume

$$a \geq b \geq c \geq d \tag{9}$$

Since $a, b, c, d \in \mathbb{Q} \setminus \mathbb{N}$ and the indices are positive integers, it follows that

$$a, b, c, d \in \mathbb{N} + \frac{1}{2}. \tag{10}$$

In addition, $S \geq 3$.

We can see this as the data for a 3-regular graph, with $2(S-2)$ vertices of each colour. For such a graph, the order of a face cannot exceed the number of vertices of one colour: $2a \leq 2(S-2)$. It follows that $b+c+d \geq 2$. The minimal solution is given by:

$$a = b = \frac{3}{2}, c = d = \frac{1}{2}, S = 4, \deg f = 12. \tag{11}$$

The corresponding table is given by

	0	1	t	∞	
∞	3	3	1	1	2 points of order 2
0					4 points of order 3
1					4 points of order 3

and its attached graph is

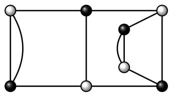

Figure 2.1: Graph corresponding to $a = b = \frac{3}{2}, c = d = \frac{1}{2}, \deg f = 12$

It has 4 black and 4 white vertices of order 3. There are 2 faces of order 1, 2 faces of order 2 and 2 faces of order 3.

As a side note, let us observe that the Lamé operators (Heun operators with parameters: $b = c = d = \frac{1}{2}, a = n + \frac{1}{2}$, where $n \in \mathbb{N}$) would fall under this case. Since we obtained that $b + c + d \geq 2$, it follows that there are no Lamé operators which are pull-backs of the tetrahedral hypergeometric one, recovering the result in [4] and [39].

We want to vary some of the parameters. Allowing $a = \frac{3}{2} + m, b = \frac{3}{2} + k$, $(m, k \in \mathbb{N})$, we get the following data:

$$a = \frac{3}{2} + m, b = \frac{3}{2} + k, c = d = \frac{1}{2}, S = 4 + k + m, \deg f = 12 + 6k + 6m$$

	0	1	t	∞	
∞	$3 + 2m$	$3 + 2k$	1	1	$2 + 2m + 2k$ points of order 2
0					$4 + 2k + 2m$ points of order 3
1					$4 + 2k + 2m$ points of order 3

We represent below a graph corresponding to the case $m = k = 1$:

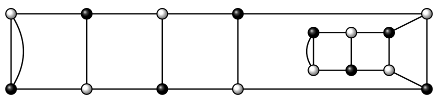

Figure 2.2: Graph corresponding to $a = b = \frac{5}{2}, c = d = \frac{1}{2}, \deg f = 24$

The graphs corresponding to the different values for m and k are obtained by inserting m respectively k cells of type:

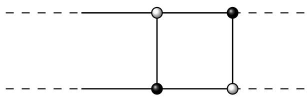

Figure 2.3: First type of added cell

2.

	0	1	t	∞	
0	$2a$	$2b$	$2c$		$2(S-3)+d$ points of order 2
1				$3d$	$2(S-2)-d$ points of order 3
∞					$2(S-2)$ points of order 3

Since the values in the table are positive integers, it follows that $2(S-2)-d, 2(S-2) \in \mathbb{N}$. Subtracting the two, it follows that $d \in \mathbb{Z}$, which contradicts Remark 3.

3.

	0	1	t	∞	
0	$2a$	$2b$			$3(S-2)-(a+b)$ points of order 2
∞			$3c$	$3d$	$2(S-2)-(c+d)$ points of order 3
1					$2(S-2)$ points of order 3

All the values in the table have to be positive integers and since the local exponent differences cannot be integers, it follows that $\Rightarrow$

$$a, b \in \mathbb{N} + \frac{1}{2} \text{ and } c, d \in \frac{\mathbb{N}}{3}, 2S \in \mathbb{N}, S > 2 \tag{12}$$

Since the local exponent differences are non-integers, it follows that $a, b \in \mathbb{N} + \frac{1}{2}$. Therefore

$$a + b \in \mathbb{N} \Rightarrow c + d \in \mathbb{N} \Rightarrow S \in \mathbb{N}. \tag{13}$$

Without loss of generality, we can assume $a \geq b$ and $c \geq d$. Using (13), it follows that the minimal value for S is 3. For this minimal values, there are four possible sub-cases:

(a) $a + b = 1 \Rightarrow a = b = \frac{1}{2}$,
$c + d = 2 \Rightarrow$

i. $c = \frac{4}{3}, d = \frac{2}{3}$
or

ii. $c = \frac{5}{3}, d = \frac{1}{3}$.

(b) $a + b = 2 \Rightarrow a = \frac{3}{2}, b = \frac{1}{2}$,
$c + d = 1 \Rightarrow c = \frac{2}{3}, d = \frac{1}{3}$.

Let us analyze them:

(a) i.

	0	1	t	∞	
0	1	1			2 points of order 2
∞			4	2	0 points of order 3
1					2 points of order 3

The corresponding graph has 4 white vertices (2 of order 2, 2 of order 1), 2 black vertices of order 3 and 2 :

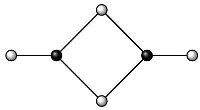

Figure 2.4: Graph corresponding to $a = b = \frac{1}{2}$, $c = \frac{4}{3}$,$d = \frac{2}{3}$, $S = 3$, $\deg f = 6$

Again, we would like to allow at least one parameter to vary, namely c:

	0	1	t	∞	
0	1	1			$3m + 2$ points of order 2
∞			$4 + 3m$	2	n points of order 3
1					$2m + 2$ points of order 3

We were not able to proceed as in the first case, instead we found ourselves distinguishing between the parity of m.

- Taking $m = 2k, m \in \mathbb{N}$, we get: $S = 2k + 3, \deg f = 12k + 6$.
  For $k = 0$, the corresponding graph is the one in Figure 2.4. For $k \in \mathbb{N}^*$, k cells given in Figure 2.5 will be added:

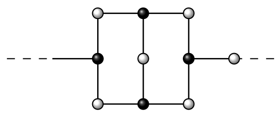

Figure 2.5: Second type of added cell

We give the graph for $k = 1$ in Figure 2.6

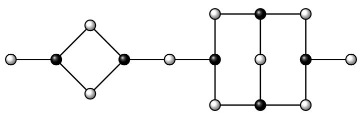

Figure 2.6: $a = b = \frac{1}{2}, c = \frac{10}{3}, d = \frac{2}{3}$

- Taking $m = 2k+1, k \in \mathbb{N}$, we get: $S = 2k+3, \deg f = 12k+6$. The graph for $k = 0$ is given in Figure 2.7:

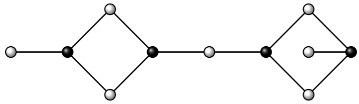

Figure 2.7: $a = b = \frac{1}{2}, c = \frac{7}{3}, d = \frac{2}{3}$

The graphs for different values of k are obtained again by adding k cells given in Figure 2.5.

For the following sub-cases in this case, we will proceed similarly to the previous analysis. More precisely, we will allow again c to vary. We will present just the minimal cases, i.e. the graphs for c and $c+1$. The infinite families can be again obtained by adding cells given in Figure 2.5.

ii. $a = b = \frac{1}{2}, c = \frac{5}{3}, d = \frac{1}{3} \Rightarrow S = 3, \deg f = 6$

	0	1	t	∞	
0	1	1			2 points of order 2
∞			5	1	0 points of order 3
1					2 points of order 3

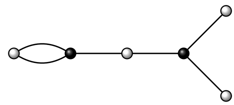

Figure 2.8: $a = b = \frac{1}{2}, c = \frac{5}{3}, d = \frac{1}{3}$

Increasing c by 1, we get:

	0	1	t	∞	
0	1	1			5 points of order 2
∞			8	1	1 point of order 3
1					4 points of order 3

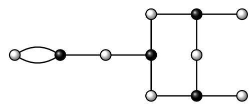

Figure 2.9: $a = b = \frac{1}{2}, c = \frac{8}{3}, d = \frac{1}{3}$

(b) $a = \frac{3}{2}, b = \frac{1}{2}, c = \frac{2}{3}, d = \frac{1}{3}, S = 3, \deg f = 6$

	0	1	t	∞	
0	3	1			1 point of order 2
∞			2	1	1 point of order 3
1					2 points of order 3

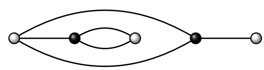

Figure 2.10: $a = \frac{3}{2}, b = \frac{1}{2}, c = \frac{2}{3}, d = \frac{1}{3}$

	0	1	t	∞	
0	3	1			4 points of order 2
∞			5	1	2 points of order 3
1					4 points of order 3

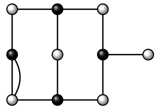

Figure 2.11: $a = \frac{3}{2}, b = \frac{1}{2}, c = \frac{5}{3}, d = \frac{1}{3}$

4.

	0	1	t	∞	
0	$2a$	$2b$			$3(S-2)-(a+b)$ points of order 2
∞			$3c$		$2(S-2)-c$ points of order 3
1				$3d$	$2(S-2)-d$ points of order 3

We can assume without loss of generality that $a \geq b$ and $c \geq d$. All the values in the table have to be positive integers. It follows that $a, b \in \mathbb{N} + \frac{1}{2} \Rightarrow a + b \in \mathbb{N} \Rightarrow 3S \in \mathbb{N}$. In addition $c, d \in \frac{\mathbb{N}}{3}$ and $2(S-2)-c, 2(S-2)-d \in \mathbb{N}$. Subtracting the two, it follows that

$$c - d \in \mathbb{N} \tag{14}$$

To determine the minimal value of S, let us remark that

$$a + b \geq 1 \Rightarrow 3(S-2) \geq 1 \Rightarrow 3S \geq 7 \Rightarrow S = \frac{7}{3}.$$

For this minimal value, it follows that $a = b = \frac{1}{2}, c + d = \frac{4}{3} \xRightarrow{(14)} c = d = \frac{2}{3}$. Let us draw the graph in this case:

	0	1	t	∞	
0	1	1			0 points of order 2
1			2		0 points of order 3
∞				2	0 points of order 3

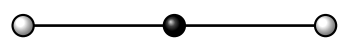

Figure 2.12: $a = b = \frac{1}{2}$, $c = d = \frac{2}{3}$

Let us construct the function f.

Since $\deg f = 2$ and $f(0) = f(1) = 0$ and the $f(\infty) = \infty$ of order 2, it follows that

$$f(z) = kz(z-1),\ k \in \mathbb{C}^*.$$

Since $f(z) - 1$ has solution the double point t, it follows that $kz(z-1) - 1$ is a perfect square. Therefore the second order polynomial $kz^2 - kz - 1$ has its discriminant equal to 0. It follows that $k^2 + 4k = 0 \Rightarrow k = -4 \Rightarrow$

$$f(z) = 4z(1-z). \tag{15}$$

The fourth singular point becomes $t = \frac{1}{2}$.

We allow the parameter d to vary, $d = \frac{2}{3} + m, m \in \mathbb{N}$. Since once more we distinguish between the parity of m, we also need to show the corresponding graph for $d = \frac{2}{3} + 1$. The graphs for other values of m will be obtained by adding $\left[\frac{m}{2}\right]$ cells given in Figure (2.5) to one of the two initial examples.

$$a = b = \frac{1}{2}, c = \frac{2}{3}, d = \frac{5}{3}, S = \frac{10}{3}, \deg f = 8$$

	0	1	t	∞	
0	1	1			3 points of order 2
1			2		2 points of order 3
∞				5	1 point of order 3

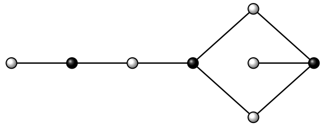

Figure 2.13: $a = b = \frac{1}{2}, c = \frac{5}{3}, d = \frac{2}{3}$

9.

	0	1	t	∞	
0					$3(S-2)$ order 2
1	$3a$	$3b$			$2(S-2)-(a+b)$ order 3
∞			$3c$	$3d$	$2(S-2)-(c+d)$ order 3

Looking again at the last column of the table:

$$\left.\begin{array}{l} 2S - 4 - (a+b) \geq 0 \\ 2S - 4 - (c+d) \geq 0 \end{array}\right\} \overset{+}{\Longrightarrow} 3S \geq 8 \Rightarrow S \geq \frac{8}{3}$$

For this minimal value $2 \cdot \frac{8}{3} - 4 \geq a + b \Rightarrow$

$$a + b \leq \frac{4}{3} \tag{16}$$

Analogously

$$c + d \leq \frac{4}{3} \tag{17}$$

On the other hand, $a, b, c, d \in \frac{\mathbb{N}}{3} \setminus \mathbb{N}$. It follows that

$$a = b = c = d = \frac{2}{3}, S = \frac{8}{3}, \deg f = 4$$

	0	1	t	∞	
0					2 order 2
1	2	2			0 order 3
∞			2	2	0 order 3

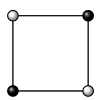

Figure 2.14: Graph for $a = b = c = d = \frac{2}{3}$

We want to determine the function f. Using the ramification data, $f(z) = \frac{((z-k)(z-l))^2}{s(z-t)^2}$ and $f(z) - 1$ has numerator $z^2(z-1)^2$. Solving the system, we get: $f(z) = \frac{(z^2-x+\frac{1}{2})^2}{(z-\frac{1}{2})^2}$.

We would like again to allow one of the parameters to vary (in this particular case it does not matter which one is allowed to vary). Assume $a = \frac{2}{3} + m, m \in \mathbb{N}$.

	0	1	t	∞	
0					$2+3n$ order 2
∞	$2+3n$	2			n order 3
1			2	2	$2n$ order 3

For this graph, let us take $m = 1$, and see the this as the data for a clean dessin with 3 faces of orders $5, 2$ and 3 and 4 vertices: 2 of order 2 and 2 of order 3. We try to construct the graph. Since there are no vertices of order 4 or higher, the face of order 2 looks like this:

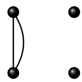

Figure 2.15: Step one for drawing the graph for $a = \frac{5}{3}, b = c = d = \frac{2}{3}$

Since the graph is connected, at least one of the vertices in the face of order 2 is connected to one of the others:

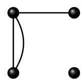

Figure 2.16: Step one for drawing the graph for $a = \frac{5}{3}, b = c = d = \frac{2}{3}$

The vertex still left unconnected must have at least one edge connecting it to the vertex that is not part of the face of order 2:

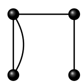

Figure 2.17: Step three for drawing the graph for $a = \frac{5}{3}, b = c = d = \frac{2}{3}$

For the drawing already obtained there is no way to obtain a face of order 3 while preserving the required order of faces, therefore there is no graph with the given ramification data.

For $m = 2$, unfortunately we cannot proceed as in the previous cases and "add" cells to the graph, therefore we have to make a similar study to case 8.e, and produce new "minimal" examples, for $a = \frac{8}{3}$ and $a = \frac{11}{3}$. For each of these, we add $\left[\frac{m}{2}\right]$ cells given in Figure 2.5.

$$a = \frac{8}{3}, b = c = d = \frac{2}{3}, S = \frac{14}{3}, \deg f = 16$$

	0	1	t	∞	
0					8 points of order 2
∞	8	2			2 points of order 3
1			2	2	4 points of order 3

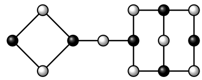

Figure 2.18: $a = \frac{8}{3}, b = c = d = \frac{2}{3}$

$$a = \frac{11}{3}, b = c = d = \frac{2}{3}, S = \frac{17}{3}, \deg f = 22$$

	0	1	t	∞	
0					11 points of order 2
∞	11	2			3 points of order 3
1			2	2	6 points of order 3

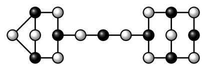

Figure 2.19: $a = \frac{11}{3}, b = c = d = \frac{2}{3}$

The full analysis proves the result:

Proposition 2.2.1. *[46] There are pull-back functions of any degree greater than 2 from the tetrahedral hypergeometric operator to an algebraic Heun operator.*

Remark. From the analysis above, it follows that there is a countable number of algebraic Heun operators.

2.3 Graphs for the octahedral case

In this section, we look for Heun operators as pull-back of the hypergeometric operator with octahedral monodromy:

$$\mathcal{H}_{a,b,c,d} = f^*(H_{\frac{1}{2},\frac{1}{3},\frac{1}{4}}).$$

Formula (7) becomes:

$$\deg f = 12(S - 2).$$

Obviously $S \in \frac{N^*}{12}$. The possible ramification indices are given by:

$z\backslash x$	0	1	t	∞	
0	$2a$ or 0	$2b$ or 0	$2c$ or 0	$2d$ or 0	points of order 2
1	$3a$ or 0	$3b$ or 0	$3c$ or 0	$3d$ or 0	points of order 3
∞	$4a$ or 0	$4b$ or 0	$4c$ or 0	$4d$ or 0	points of order 4

Table 2.3: Possible values for the ramification indices in the octahedral case

We ignore permutations between the columns since the corresponding functions can be obtained from the pull-back functions corresponding to the graphs below by composing with a Möbius transformation. Therefore we will only study the 15 essentially different possibilities. In this summary we look at the first 2, the rest of the analysis being similar:

1.

	0	1	t	∞	
∞	$2a$	$2b$	$2c$	$2d$	$5S-12$ points of order 2
0					$4(S-2)$ points of order 3
1					$3(S-2)$ points of order 4

Since all the values in the cells are positive integers and the local exponent differences are non-integers, it follows that $a, b, c, d \in \mathbb{N} + \frac{1}{2} \Rightarrow S \in \mathbb{N}$. Since $5S > 12$, it follows that the minimal value of S is 3. Assuming $a \geq b \geq c \geq d$, it follows that $a = \frac{3}{2}, b = c = d = \frac{1}{2}$. The corresponding ramification data and graph are given by:

	0	1	t	∞	
∞	3	1	1	1	3 points of order 2
0					4 points of order 3
1					3 points of order 4

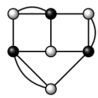

Figure 2.20: $a = \frac{3}{2}, b = c = d = \frac{1}{2}$

Similarly to the tetrahedral case, we want to allow one of the parameters to vary. We will take $a = \frac{3}{2} + m, m \in \mathbb{N}$. Again, we find ourselves distinguishing between the odd and the even case. We need to construct a graph, for $m = 1$:

	0	1	t	∞	
∞	5	1	1	1	3 points of order 2
0					8 points of order 3
1					6 points of order 4

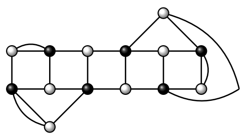

Figure 2.21: $a = \frac{3}{2}, b = c = d = \frac{1}{2}$

Starting from the two minimal cases, we need to add the following cell $[\frac{m}{2}]$ times:

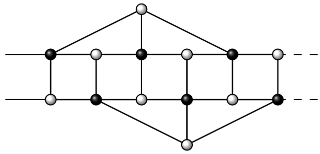

Figure 2.22: Fourth type of added cell

The graph for $m = 2$ is given by:

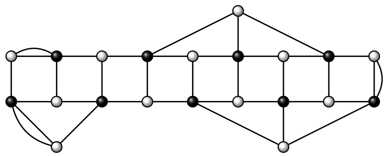

Figure 2.23: $a = \frac{7}{2}, b = c = d = \frac{1}{2}$

This case includes the Lamé operator. We proved that for any values of $a = \frac{3}{2} + m, m \in \mathbb{N}$, there is a Lamé algebraic operator with octahedral monodromy. Identifying with the parameter n of the Lamé operator, it follows that for $n \in \mathbb{N}^*$, there is a Lamé algebraic operator with octahedral monodromy, which is covered by the result in [39].

2.

	0	1	t	∞	
∞	$2a$	$2b$	$2c$		$5S + d - 12$ points of order 2
0				$3d$	$4(S-2) - d$ points of order 3
1					$3(S-2)$ points of order 4

Since all the values in the table are positive integers and the parameters are non-integers, it follows that

$$a, b, c \in \mathbb{N} + \frac{1}{2}, \tag{18}$$

and $4S - d, 3S \in \mathbb{N}$. Subtracting the last two, it follows that $S - d = a + b + c \in \mathbb{N}$ which contradicts (18). It follows that there are no algebraic Heun operators in this case.

Chapter 3

Deformations of second order differential operators

In this chapter, we follow the computations from [54], provide a small correction to the calculations and produce a small new result for modified Lamé differential operators.

3.1 Non logarithmicity conditions for second order differential operators

Proposition 3.1.1. *Consider the second order differential operator:*

$$L^* = z^2 \frac{d^2}{d\,z^2} + zp^*(z)\frac{d}{d\,z} + q^*(z) \tag{1}$$

with p^* and q^* analytic in a neighborhood of $z = 0$:

$$p^*(z) \quad = \quad \sum_{m=0}^{\infty} p_m z^m \tag{2}$$

$$q^*(z) \quad = \quad \sum_{m=0}^{\infty} q_m z^m \ . \tag{3}$$

Consider the correspondent differential equation:

$$L(u(z)) = 0 \tag{4}$$

and its solution:

$$u(z) = z^r \sum_{m=0}^{\infty} c_m z^m = \sum_{m=0}^{\infty} c_m z^{m+r} \ , \tag{5}$$

where $r \in \mathbb{R}$ and $c_m \in \mathbb{C}$ for $m \in \mathbb{N}$. The condition for non-logarithmicity of the solution at $z = 0$ is given by:

$$R_{n_0} = \sum_{k=0}^{n_0-1} c_k[(r_2 + k)p_{n_0-k} + q_{n_0-k}] = 0 \ . \tag{6}$$

Using this result, we can recover a result of Brioschi and Halphen:

Proposition 3.1.2. *Consider the Lamé differential operator:*

$$L_n = \left(\frac{d}{dz}\right)^2 + \frac{1}{2}\sum_{i=1}^{3}\frac{1}{z-e_i}\frac{d}{dz} - \frac{n(n+1)z+B}{4\prod_{i=1}^{3}(z-e_i)} \tag{7}$$

with Riemann scheme:

$$\left(\begin{array}{cccc|c} e_1 & e_2 & e_3 & \infty & \\ 0 & 0 & 0 & -\frac{n}{2} & z \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{n+1}{2} & \end{array}\right). \tag{8}$$

Suppose $n \in \mathbb{Z}+\frac{1}{2}$. The operator L_n has non-logarithmic solutions if and only if the accessory parameter B and the finite singular points satisfy a polynomial equation $f(B, e_1, e_2, e_3) = 0$, $f \in \mathbb{Q}[X, Y, Z, T]$.

3.2 Second order differential equations with apparent singularities

Proposition 3.2.1. *Consider the following differential operator:*

$$\begin{aligned} L^* = \left(\frac{d}{dz}\right)^2 + \left[\frac{1}{2}\left(\frac{1}{z}+\frac{1}{z-1}+\frac{1}{z-\lambda}\right) - \frac{1}{z-t}\right]\frac{d}{dz} + \\ \frac{1}{z(z-1)}\left(\mu\mu' + \frac{B}{z-\lambda} + \frac{A}{z-t}\right) \end{aligned} \tag{9}$$

with Riemann scheme

$$\left(\begin{array}{ccccc|c} 0 & 1 & \lambda & t & \infty & \\ 0 & 0 & 0 & 0 & -\frac{n}{2} & z \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 2 & \frac{n+1}{2} & \end{array}\right) \tag{10}$$

The condition for non-logarithmicity at t is given by:

$$A^2 - \left[\frac{1}{2}(2t-1) + \frac{t(t-1)}{t-\lambda}\right]A + t(t-1)\left(\mu\mu' + \frac{B}{t-\lambda}\right) = 0 \tag{11}$$

Algebraic Geometry - Bibliography

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Part II

Number Theoretic Functions in Group Theory

Chapter 4

Group Theory Introduction

There has been a recent trend to translate number theory concepts into group theory. Starting from this, a plethora of research has appeared and this second part of the thesis is inscribed in this effort. This part of the thesis gathers the results of four articles ([22], [23], [24], [25] written in collaboration with my advisor, prof. dr. Marius Tărnăuceanu, 3 of them already published.

The cornerstone of our research is Tom Leinster's study of what would later be called Leinster groups [16], an analogue of perfect numbers (integers that are equal to the sum of their proper divisors).

The study of Leinster groups proved to be intrinsically linked to the one of perfect groups. The cyclic Leinster groups are the ones of perfect order. In addition, while Leinster dihedral groups are in bijection with odd perfect numbers, there exist Leinster groups of odd order [15].

If we deviate slightly from the definition of perfect numbers, we get the concepts of almost and quasi-perfect numbers [13].

We continue the analogy initiated by Leinster, by introducing the concepts of almost and quasi-Leinster groups. We classify the nilpotent almost and quasi-Leinster groups.

Afterwards, we look at harmonic numbers, which can be seen as a generalisation of perfect numbers. We saw the value in looking at the harmonic mean of the element orders of the group. This continues the current effort of studying different means of element orders: arithmetic in [14, 12] and geometric in [12]. We search for groups that have the harmonic mean of element orders a positive integer by a combination of a theoretical approach and the output of codes.

The last group theoretical problem we focus on is to deduce properties of the group from the number theoretical properties of the order of the group. Specifically, we start from what are called $\mathcal{P}$ numbers, numbers that determine a specific property of the group of which they are order of. While there are classical results in this field: a group with the order a positive integer that is coprime with its own Euler totient function is cyclical [9], we deviated from the beaten path by looking at "almost" $\mathcal{P}$ numbers.

This part of the thesis is structured into three chapters: this Introduction, the Main results and the Appendix. These are followed by a Bibliography for this part.

The Introduction chapter is divided into two sections. The first section, Number Theory Concepts, predictably goes through the number theoretic concepts.

Firstly, we go through divisor functions. Secondly, using these functions, we give the definitions for almost/quasi-/perfect numbers. We present the known form of perfect even numbers and using this, we prove that the equation

$$P_n + 1 = p^k, P_n \text{ perfect even number}, p \text{ prime number} \tag{*}$$

has solutions only for $k = 1$. We also enumerate the 4 solutions that we were able to find computationally using [33] and [31].

We also go through the conjectures stating that there are no odd perfect numbers, no quasi-perfect numbers, no odd harmonic numbers and only one type of almost perfect numbers: powers of two.

Ending the section of number theory, we look at the concept of nilpotent factorization [21] and explores a characterization of this concept.

The second section of the introduction focuses on the promised "translation" of the functions in the previous sections. We follow the work of Leinster [16] and de Medts [18] to observe the properties of σ_z and ρ, the work of Garonzi [11], Azad, Khosravi, and Rashidi [2] to look at properties of m (the sum of inverses of order of elements of a group), and Baishya, and Kumar [3] to analyze the properties of H.

The second chapter presents the main results of this part of the thesis. It is divided into three sections, covering the results of the three articles mentioned in the beginning ([22], [23], [24]).

The first section is focused on almost/quasi-Leinster groups. We first review an existing result of Leinster that allows us to prove that the only almost/Leinster nilpotent groups are the cyclic ones of almost/perfect order. We then prove an analogous result for quasi-Leinster groups. This concludes the classification of almost/quasi-/Leinster nilpotent groups. Afterwards, we look at the four classes of non-nilpotent groups studied in the introduction: ZM-groups, affine groups, dihedral groups and dicyclic groups. We find examples of almost and quasi-Leinster groups for the first two using the solutions to (*). We prove that the existence of almost/quasi-/Leinster dihedral groups D_{2n} groups is equivalent to n being odd, and $2n$ being almost/quasi-/perfect. Similarly, for dicyclic groups, we prove that the existence of an almost/quasi-/Leinster group Dic_{4n} is equivalent to n being odd and $2n$ being almost/quasi-/perfect.

In the second section, we look at:

$$\text{finite groups } G \text{ that have an integer harmonic mean } h_m(G) \in \mathbb{N}^*, \tag{**}$$

following the results in [23]. We prove some basic properties of the newly introduced function, like inequalities, behaviour for subgroups. Afterwards, we look at the counterimage h_m^{-1} for different sets and types of groups. We start by proving that the only p-groups that check (**) are cyclic of order

$p^{\sum_{i=1}^{s} p^i}$ with $s \in \mathbb{N}^*$ or D_8, the dihedral group of order 8: D_8. In addition, we prove that D_8 is the only dihedral group in $h_m^{-1}(\mathbb{N}^*)$. We then proceed to prove that $h_m^{-1}(\{2\}) = \{C_4, D_8\}$ and further that $h_m^{-1}([0,2]) = \{C_2^n \mid n \in \mathbb{N}^*\} \cup \{C_3, S_3, C_4, D_8\}$. We proceed to prove that there are no groups of odd order and no nilpotent groups in $h_m^{-1}(\{3\})$. The last result we prove is a criterion for supersolvability for groups using h_m. The starting question of determining $h_m^{-1}(\mathbb{N}^*)$ is still open.

The last section looks at almost $\mathcal{P}$ numbers, integers that can characterize a group property ($\mathcal{P}$) except one [24]. We begin by recalling an existing result that characterizes almost cyclic numbers. Then, we characterize almost abelian numbers. We finish the section by proving that these numbers are also the almost nilpotent ones.

Notation 4.1. We denote **the set of positive integers** by:

$$\mathbb{N}^* = \{1, 2, 3, \dots\}.$$

Definition 4.0.1. *[18, Page 17] Given a positive integer n, by **proper divisor** we understand a positive divisor of n not equal to the number itself.*

4.1 Number Theory Concepts

Definition 4.1.1. *[13, Page 71] For a positive integer n, the following functions can be defined:*

- *the sum of its divisors:*
$$\sigma(n) = \sum_{d \mid n} d \tag{1}$$

- *the alliquot sum (the sum of the proper divisors):*
$$s(n) = \sum_{\substack{d \mid n \\ d \neq n}} d = \sigma(n) - n. \tag{2}$$

Definition 4.1.2. *[13, Page 71] Given a positive integer n, and a complex number $z \in \mathbb{C}$, the **divisor function** of n is given by:*

$$\sigma_z(n) = \sum_{d \mid n} d^z \tag{3}$$

Besides the case $z = 1$, there are two other relevant cases for our work:

- [3] the number of divisors of n :

$$\tau(n) = \sigma_0(n) = \sum_{d|n} d^0 = \sum_{d|n} 1 \tag{4}$$

- the abundancy index of n:

$$\rho(n) = \sigma_{-1}(n) = \sum_{d|n} d^{-1} = \frac{\sigma(n)}{n} \tag{5}$$

Proposition 4.1.3. *[1, Page 827], [16, Page 1] The divisor function is multiplicative: Given $m, n \in \mathbb{N}^*$ positive coprime integers, it follows that:*

$$\sigma_z(mn) = \sigma_z(m)\sigma_z(n). \tag{6}$$

Other two functions of interest are the following:

Definition 4.1.4. *[13, Page 138] For a positive integer n, **Euler's totient function** is given by:*

$$\phi(n) = \sum_{\substack{d \leq n \\ \gcd(d,n)=1}} 1 \tag{7}$$

Notation 4.2. [13, Page xiv] For a positive integer n, $w(n)$ denotes its number of distinct prime factors.

4.1.1 Almost, quasi-, perfect numbers

Using the previously introduced functions, we describe some concepts that will be used throughout the thesis.

Definition 4.1.5. *[13, Page 71] A positive integer n is called **perfect** if it is equal to its alliquot sum:*

$$n = s(n). \tag{8}$$

Alternatively, n is perfect if its abundancy index is equal to 2:

$$\rho(n) = 2. \tag{9}$$

Definition 4.1.6. *[13, Page 13] A positive prime n of the form $2^p - 1$ is called a **Mersenne prime**.*

Proposition 4.1.7 (Euclid & Euler). *[13, Page 71] There is a one-to-one correspondence between even perfect numbers and Mersenne primes.*

While this gives the exact form of even perfect numbers, this does not say anything about their number. Until now, 52 Mersenne primes have been discovered [32]. The first 50 of them are confirmed in their position, while there may be others between the 50^{th} and the 52^{th}.
This raises the question:

Question 4.1. [13, Page 14] Are there infinitely many even perfect numbers? (equivalently, infinitely many Mersenne primes)

When it comes to odd perfect numbers, it is conjectured that there are none:

Conjecture 4.1. [13, Page 71] There are no odd perfect numbers.

Proposition 4.1.8. *[22, Corollary 2.2] Given p a prime, P_n an even perfect number and $k \in \mathbb{N}^*$, the equation:*

$$P_n + 1 = p^k \tag{10}$$

has solutions only for $k = 1$.

Proof. We need to solve the following

$$p^k - 1 = 2^{r-1}(2^r - 1), \tag{11}$$

where $2^r - 1$ is a Mersenne prime $r \geq 2$.

Let us assume by contradiction that k is even, i.e. $k = 2l, l \in \mathbb{N}^*$. Equation (11) becomes:

$$(p^l - 1)(p^l + 1) = 2^{r-1}(2^r - 1).$$

Numbers $2^l - 1$ and $2^l + 1$ are even consecutive numbers. It follows that $r - 1 \geq 2 \iff r \geq 3$.

Since $2^r - 1$ is prime, it follows that $p^l - 1 = 2^{r-2}$ and $p^l + 1 = 2^{r+1} - 2$. Subtracting these, it follows that $2^{r+1} - 4 = 2^{r-2}$, which yields a contradiction, hence k is odd.

Let us now prove that k can actually be just 1.

Assume by contradiction, that $k > 1$. In this case, equation (11) becomes:

$$2^{r-1}(2^r - 1) = (p-1)(p^{k-1} + \cdots + p + 1)$$

Since q is odd, it follows that p is odd, therefore $p^{k-1} + \cdots + p + 1$ is odd, and is thus equal to $2^r - 1$. In addition, $2^{r-1} = p - 1$. We get that: $2p - 3 = p^{k-1} + \cdots + p + 1$. Since p is odd, this has no solutions.

This concludes the proof.

Example 4.1.9. *[22, Example 2.3] We found* 4 *solutions for:*

$$(P_n + 1) \text{ is prime.} \qquad \text{(perfect+1=prime)}$$

These are:

$$n \in \{6 = P_1, 28 = P_2, 33550336 = P_5, 137438691328 = P_7\}. \tag{12}$$

They were obtained using a GAP script [26].

Definition 4.1.10. *[13, Page 74] A positive integer n is called*

- ***abundant*** *if it is smaller than its alliquot sum (it is abundant in divisors):*
$$n < s(n). \tag{13}$$
  Alternatively, n is abundant if
$$\rho(n) > 2. \tag{14}$$

- ***deficient*** *if it is greater than its alliquot sum (it is deficient in divisors):*
$$n > s(n). \tag{15}$$
  Alternatively, n is deficient if
$$\rho(n) < 2. \tag{16}$$

Definition 4.1.11. *[13, Page 74] An abundant number n is* ***quasi-perfect*** *if it is equal to the sum of its non-trivial divisors. This can be written as:*

$$\sigma(n) = 2n + 1. \tag{17}$$

A deficient number n is ***almost perfect*** *if it is equal to the sum of its divisors plus one:*

$$\sigma(n) = 2n - 1. \tag{18}$$

Example 4.1.12. *[13, Page 74] The only known almost perfect numbers are powers of* 2.

There are no known examples of quasi-perfect numbers.

Question 4.2. [13, Page 74] Are there any other almost perfect numbers besides powers of two?

Question 4.3. [13, Page 74] Are there any quasi-perfect numbers?

4.1.2 Harmonic divisor numbers

Definition 4.1.13. *[13, Page 75] An* ***harmonic divisor number (Ore number)*** *is a positive integer $n \in \mathbb{N}^*$ for which the harmonic mean of their divisor is again a positive integer:*

$$H(n) = \frac{n\sigma_0(n)}{\sigma_1(n)} \in \mathbb{N}^*. \tag{19}$$

Proposition 4.1.14 (Ore). *[19, Page 617] Every perfect number is a harmonic number.*

Conjecture 4.2 (Ore). [19, Page 619] There are no odd harmonic divisor numbers.

4.1.3 Nilpotent factorization

Definition 4.1.15. *[21, Definition] Consider a positive integer n and its prime factorization:*

$$n = p_1^{n_1} \cdots p_k^{n_k}.$$

The number n is said to have ***nilpotent factorization*** *if $p_i^l \not\equiv 1 \bmod p_j$ for all positive integers i, j and l with $1 \leq l \leq n_i$.*

Proposition 4.1.16. *[21, Page 3] A positive integer is coprime with its Euler totient function if and only if it is square free and has nilpotent factorization.*

4.2 Group Theory functions

Most of the concepts in the previous section can be ”translated” to analogues from group theory.

Definition 4.2.1. *[18, Definition 2.1] We define the following functions for finite groups:*

- ***sum of orders of normal subgroups***

$$G \mapsto \sigma(G) = \sum_{N \lhd G} |N| \tag{20}$$

- ***average of orders of normal subgroups***

$$G \mapsto \rho(G) = \frac{\sigma(G)}{|G|}. \tag{21}$$

Proposition 4.2.2. *[18, Observation 3.1] The following statements hold:*

1. $\rho(G) > 1$, *for all* $G \neq \{1\}$;
2. $\rho(G/N) \leq \rho(G)$, *for all* $N \lhd G$;

Definition 4.2.3. *Given a finite group and a complex number* $z \in \mathbb{C}$, *we define the following function for finite groups:*

$$G \mapsto \sigma_z(G) = \sum_{N \lhd G} |N|^z. \tag{22}$$

Definition 4.2.4. *[16, Page 5] A pair of groups are called* ***coprime*** *if they have no composition factor in common. In particular, if two groups have coprime orders then they are coprime.*

Proposition 4.2.5. *The function* σ_z *is multiplicative, i.e.*

$$G_1, G_2 \text{ coprime} \Rightarrow \sigma_z(G_1 \times G_2) = \sigma_z(G_1)\sigma_z(G_2). \tag{23}$$

Corollary 4.2.6. *Let* G_1, G_2 *be two groups with coprime orders. It follows that*

$$\sigma_z(G_1 \times G_2) = \sigma_z(G_1)\sigma_z(G_2).$$

Proposition 4.2.7. *[16, Example 2.1] For a cyclic finite group* C_n, *the following hold:*

$$\sigma(C_n) = \sigma(n) \tag{24}$$

and

$$\delta(C_n) = \delta(n). \tag{25}$$

In addition, $\sigma(G) \geq \sigma(n)$, *for any nilpotent group* G *of order* n.

Definition 4.2.8. *[3, Page 1] A finite group is called* ***Leinster*** *if*

$$\rho(G) = 2. \tag{26}$$

Definition 4.2.9. *[18, Definition 2.1]*

- *A finite group* G *is called* ***abundant*** *if* $\rho(G) > 2$.
- *A finite group* G *is called* ***deficient*** *if* $\rho(G) < 2$.

Definition 4.2.10. *[22, Definition 2.4]*

- *An abundant group* G *is called* ***quasi-Leinster*** *if*

$$\sigma(G) = 2|G| + 1. \tag{27}$$

- *A deficient group G is called **almost Leinster** if*

$$\sigma(G) = 2|G| - 1. \tag{28}$$

Example 4.2.11. *[22, Example 2.5] The non-abelian groups $G_{p,q}$ of order pq, $p < q$ primes are not almost/quasi/Leinster.*

Definition 4.2.12. *[3, Section 2] We define the following function for finite groups:*

$$G \mapsto H(G) = |G|\frac{\tau(G)}{\sigma(G)}. \tag{29}$$

Definition 4.2.13. *[3, Definition 2] A finite group G is called **harmonic** if $H(G) \in \mathbb{N}^*$.*

Proposition 4.2.14. *[3, Proposition 2.2] Let G be a Leinster group. Then G is harmonic iff $\tau(G)$ is even.*

Definition 4.2.15. *[11, Page 678] Let $r, s \in \mathbb{R}$. We consider the following sum for finite groups:*

$$G \mapsto \sum_{g\in G} \frac{o(x)^s}{\phi(o(x))^r}. \tag{30}$$

Definition 4.2.16. *[2, Page 1] We define the following function for finite groups:*

$$G \mapsto m(G) = \sum_{x\in G} \frac{1}{o(x)}. \tag{31}$$

Lemma 4.2.17. *[2, Lemma 1.3] If G is a finite non-cyclic group of order $n \in \mathbb{N}^*$, then the following inequality holds:*

$$m(G) > m(C_n). \tag{32}$$

Lemma 4.2.18. *[2, Lemma 2.2] If G is a finite group and H a subgroup of G, $H \leq G$, then $m(H) \leq m(G)$ with equality iff $H = G$.*

Lemma 4.2.19. *[2, Lemma 2.2, Lemma 2.3] If G is a finite group and N a normal subgroup of G, $N \lhd G$, then $m(G/N) \leq m(G)$ with equality iff $N = 1$.*

Lemma 4.2.20. *[2, Lemma 2.4] If G is a finite group, p a prime, and $P \in \mathrm{Syl}_p(G)$ a normal and cyclic subgroup of G. Then $m(Px) \geq m(P)/o(Px)$, where $Px \in G/P$, and the equality holds iff x centralizes P.*

Lemma 4.2.21. *[2, Lemma 2.6] If G_1 and G_2 are finite groups, then $m(G_1 \times G_2) \geq m(G_1)m(G_2)$. Also*

$$m(G_1 \times G_2) = m(G_1)m(G_2) \text{ iff } \gcd(|G_1|, |G_2|) = 1.$$

Definition 4.2.22. *[24, Equation (1)] We define* ***the harmonic mean of element orders of a finite group****, i.e.*

$$G \text{ finite group } \mapsto h_m(G) := \frac{|G|}{m(G)}\,. \tag{33}$$

Proposition 4.2.23. *[24, Proposition 2.1] The following properties hold:*

1. *If $|G| = n \in \mathbb{N}^*$, then $h_m(G) \leq h_m(C_n)$. Equality holds iff $G \cong C_n$.*
2. *For $H \leq G$, $h_m(G) \leq [G : H]h_m(H)$. Equality holds iff $H = G$.*
3. *For $N \lhd G$, it follows that $h_m(G) \leq |N|h_m(G/N)$. Equality holds iff N is the trivial subgroup.*
4. *If P is a normal cyclic Sylow p-subgroup of G: $P \in \mathrm{Syl}_p(G)$, then $h_m(G) \leq h_m(P)h_m(G/P)$. Equality holds iff P is central in G.*
5. *h_m is multiplicative: for all finite groups G_1, G_2 of coprime orders, we have $h_m(G_1 \times G_2) = h_m(G_1)h_m(G_2)$. This shows that the study of the function h_m for finite nilpotent groups can be reduced to p-groups.*

Chapter 5

Main results

5.1 Almost and quasi Leinster Groups

Proposition 5.1.1. *[18, Proposition 3.2] If G is a nilpotent group, the following are equivalent:*

1. $\delta(G) \leq 2$
2. *The group G is cyclic and $|G|$ is perfect or deficient.*

Corollary 5.1.2. *[22, Corollary 3.2] Let G be a nilpotent group. The following are equivalent:*

1. *The group G is almost/Leinster.*
2. *The group G is cyclic and $|G|$ is almost/perfect.*

Now, we can prove a similar result for quasi-Leinster groups.

Proposition 5.1.3. *[22, Proposition 3.3] Let G be a nilpotent group. Then G is quasi-Leinster iff G is cyclic and $|G|$ is quasi-perfect.*

Proof. The converse statement follows from Proposition 4.2.7.

Let us prove the direct. Consider the prime factorization of the order of the group:

$$|G| = \prod_{i=1}^{k} p_i^{n_i}, \qquad p_i \text{ primes.} \tag{1}$$

Assume that G is quasi-Leinster, i.e.

$$\sigma(G) = 2|G| + 1. \tag{2}$$

The group G is nilpotent, therefore it can be written as the direct product of its Sylow p-subgroups:

$$G = \prod_{i=1}^{k} G_i \tag{3}$$

where $|G_i| = p_i^{n_i}$, for all $i = 1, 2, \ldots, k$. Obviously, the subgroups are coprime, so by the multiplicativity of σ (Corollary 4.2.6), we obtain:

$$\sigma(G) = \prod_{i=1}^{k} \sigma(G_i). \tag{4}$$

Putting together (1), (2), (3), (4), we get:

$$\prod_{i=1}^{k} \sigma(G_i) = 2\prod_{i=1}^{k} |G_i| + 1.$$

Let us assume by contradiction that G is not cyclic, i.e. one of the Sylow p-subgroups is not cyclic.

Without loss of generality, let G_1 be not cyclic. It follows that $n_1 \geq 2$. From Sylow's theorems, it follows that the number of subgroups of order $p_1^{n_1-1}$ of G_1 is greater or equal than $1 + p_1$. In addition, since G_1 is a p-group, it follows that these subgroups are normal, thus the following inequality holds:

$$\rho(G_1) \geq \frac{1 + p_1^{n_1} + (p_1+1)p_1^{n_1-1}}{p_1^{n_1}} = 2 + \frac{1}{p_1} + \frac{1}{p_1^{n_1}}.$$

Since G is quasi-Leinster, it follows that

$$\rho(G) = \frac{\sigma(G)}{|G|} = \frac{2|G|+1}{|G|} \overset{(1)}{=\!=} 2 + \frac{1}{p_1^{n_1} \dots p_k^{n_k}} \tag{5}$$

On the other hand,

$$\begin{aligned} \rho(G) &\overset{\rho \text{ is multiplicative}}{=\!=\!=\!=\!=\!=} \rho(G_1)\dots\rho(G_k) \\ &\geq \left(2 + \frac{1}{p_1} + \frac{1}{p_1^{n_1}}\right)\left(1 + \frac{1}{p_2^{n_2}}\right)\dots\left(1 + \frac{1}{p_k^{n_k}}\right) \\ &> 2 + \frac{1}{p_1^{n_1}\dots p_k^{n_k}}. \end{aligned} \tag{6}$$

Clearly, (5) and (6) yield a contradiction. Thus G_1 is cyclic. Analogously, all G_i, $i = 2, \dots, k$ are cyclic, and **therefore G is cyclic**. Proposition 4.2.7 gives that $|G|$ is quasi-Leinster and concludes the proof.

Proposition 5.1.4. *[22, Propostion 4.3] There exist quasi-Leinster ZM-groups.*

Proof. **We look for particular cases where the computations for σ are easier.**
Let us assume that m prime and let

$$n = d = \text{ord}_m(r). \tag{7}$$

We compute $\sigma(ZM(m,n,r))$:

$$\sigma(ZM(m,n,r)) = m\sigma(n) + \sigma\left(\frac{n}{d}\right) \overset{(7)}{\Longleftrightarrow} \sigma(ZM(m,n,r)) = m\sigma(n) + 1. \tag{8}$$

Since we search for quasi-Leinster groups, it follows that $\sigma(ZM(m,n,r)) = 2mn+1$ which implies $\sigma(n) = 2n$, i.e. n is perfect.

In addition, it follows from Fermat's little theorem [20], that any $r \in \{2,\dots,m\}$ verifies $m-1 = \mathrm{ord}_m(r)$. Therefore, $m-1 = n$ which is perfect.

Using Remark 4.1.9, we get 4 known values for n, and the corresponding values for m. For each of these, we have $n-2$ solutions. We write just one of them, for $r=3$: $(m,n,r) = (7,6,3)$ and $(m,n,r) = (29,28,3)$, $(m,n,r) = (33550337, 33550336, 3)$, $(m,n,r) = (137438691329.137438691328, 3)$.

Proposition 5.1.5. *[22, Proposition 5.2]*

1. *There are no Leinster affine groups.*
2. *The cyclic group of order* 2 *is the only almost Leinster affine group.*
3. *If q is odd, an affine group $\mathrm{Aff}(F_q)$ is quasi-Leinster iff q is prime and $q-1$ is perfect.*

Proof. We have:

$$N(\mathrm{Aff}(\mathbb{F}_q)) = \{1\} \cup \{N_H | H \leq \mathbb{F}_q^*\}.$$

Then

$$\sigma(\mathrm{Aff}(\mathbb{F}_q)) = 1 + \sum_{H \leq \mathbb{F}_q^*} |N_H| = 1 + q \sum_{H \leq \mathbb{F}_q^*} |H| \xlongequal{\mathbb{F}_q^* \text{ cyclic}} 1 + q\sigma(q-1).$$

We obtain that:

- $\mathrm{Aff}(\mathbb{F}_q)$ is Leinster iff $1 + q\sigma(q-1) = 2q(q-1) \Rightarrow q|1$, which is absurd.
- $\mathrm{Aff}(\mathbb{F}_q)$ is quasi-Leinster iff $1+q\sigma(q-1) = 2q(q-1)-1 \Rightarrow q|2 \Rightarrow q = 2$, i.e. $\mathrm{Aff}(\mathbb{F}_q) \cong C_2$.
- $\mathrm{Aff}(\mathbb{F}_q)$ is almost Leinster iff $1+q\sigma(q-1) = 2q(q-1)+1 \Rightarrow \sigma(q-1) = 2(q-1)$, i.e. $q-1$ is perfect.

  Since no odd perfect numbers are known, we analyse the case when q is odd. Using Proposition 4.1.8, we get that q is prime. From Remark 4.1.9, we obtain 4 solutions: $q \in \{7, 29, 33550337, 137438691329\}$.

We recover the result regarding Leinster dihedral groups in [16, Example 2.4].

Proposition 5.1.6. *[22, Proposition 6.2] The existence of quasi-/almost-/Leinster generalised dihedral groups is equivalent to the existence of odd quasi-/almost-/perfect numbers.*

Proof. We describe the dihedral group:

$$D_{2n} = \{1, r, \ldots, r^{n-1}, s, rs, \ldots, r^{n-1}s\}, \tag{9}$$

where r is the rotation of angle $\frac{2\pi}{n}$ and s is the reflection about the line determined by a vertex and the center of the polygon.

If we assume D_{2n} is almost/quasi-/Leinster, we get:

$$\sigma(D_{2n}) = 2 \cdot 2n \pm a, a \in \{0, 1\}. \tag{10}$$

We distinguish the following two cases:

- n **odd**: It follows that $N(D_{2n}) = \{D_{2n}\} \cup \{\langle r^d \rangle \mid d|n\}$ and therefore $\sigma(D_{2n}) = \sigma(n) + 2n$. Then equation (10) becomes

  $$\sigma(n) + 2n = 2 \cdot 2n \pm a, a \in \{0, 1\},$$

  which is equivalent to n being almost-/quasi-/perfect.

- n **even**: We have:

  $$N(D_{2n}) = \{D_{2n}\} \cup \{\langle r^d \rangle \mid d|n\} \cup \{\langle r^2, s\rangle, \langle r^2, rs\rangle\}$$

  and therefore $\sigma(D_{2n}) = \sigma(n) + 4n$. It follows that

  $$\sigma(n) + 4n = 2 \cdot 2n \pm a, a \in \{0, 1\},$$

  which has no solution.

Proposition 5.1.7. *[22, Proposition 7.2]*

- *There is one Leinster dicyclic group:* Dic_{12}.
- *The existence of quasi-Leinster dicyclic groups is equivalent to the existence of quasi-perfect numbers of type* $4k + 2, k \in \mathbb{N}$.
- *The existence of almost Leinster dicyclic groups is equivalent to the existence of odd almost perfect numbers.*

Proof. If Dic_{4n} is almost/quasi-/Leinster, we get

$$D(\mathrm{Dic}_{4n}) = 8n + a, a \in \{-1, 0, 1\}. \tag{11}$$

We compute $\sigma(\mathrm{Dic}_{4n})$, taking into account the parity of n.

n **even** In this case $D(\mathrm{Dic}_{4n}) = D(2n) + 8n$. Together with (11), we get $\sigma(2n) + 8n = 8n + a, a \in \{-1, 0, 1\}$, which has no solution.

n **odd** In this case, $\sigma(\mathrm{Dic}_{4n}) = \sigma(2n) + 4n$. It follows that $\sigma(2n) + 4n = 8n + a, a \in \{-1, 0, 1\}$, i.e.:

$$\sigma(2n) = 4n + a, a \in \{-1, 0, 1\}. \tag{12}$$

1. If $\sigma(2n) = 4n - 1$, then $2n$ is almost perfect.
2. If $\sigma(2n) = 4n$, then $2n$ is perfect.
3. If $\sigma(2n) = 4n + 1$, then $2n$ is quasi-perfect.

Remark. In the context of the previous proof, when n is odd, Equation (12) has no known solutions for $a \in \{-1, 1\}$.

For $a = 0$, or equivalently, $2n$ perfect, there is just one solution, $n = 3$, which yields the unique Leinster group Dic_{12}.

Remark. Sometimes, the dicyclic groups are called generalized quaternion groups, see [27, Page 252]. Our results above recover the result for Leinster generalized quaternion groups from [4, Proposition 2.1.].

5.2 Finite groups with integer harmonic mean of element orders

Lemma 5.2.1. *[25, Lemma 1.2] Let G be a finite group, $C(G) = \{H \leq G \mid H \text{ cyclic}\}$ and p the greatest prime divisor of $|G|$. Then the following inequality holds:*

$$h_m(G) \geq \frac{p|G|}{(p-1)|C(G)| + 1}. \tag{13}$$

Equality holds iff G is a p-group.

Proof. Let $d_1 = 1, d_2 = p, \ldots, d_r$ be the orders of the elements in G.

For all $i \in \overline{1, r}$, we introduce the following notations:

$$n_i = |\{a \in G \mid o(a) = d_i\}| \text{ and } n_i' = |\{H \in C(G) \mid |H| = d_i\}|.$$

Then the following relations hold:

$$\begin{aligned} m(G) = \sum_{i=1}^{r} \frac{n_i}{d_i} &= \sum_{i=1}^{r} n_i' \frac{\varphi(d_i)}{d_i} = 1 + \sum_{i=2}^{r} n_i' \frac{\varphi(d_i)}{d_i} \\ &= 1 + \sum_{i=2}^{r} n_i' \prod_{q \mid d_i, q \text{ prime}} \left(1 - \frac{1}{q}\right) \\ &\leq 1 + \frac{p-1}{p}(n_2' + \cdots + n_r') = \frac{(p-1)|C(G)| + 1}{p}, \end{aligned}$$

which give (13).

The equality case happens iff p is the only prime divisor of $|G|$, which holds iff G is a p-group.

Remark ([25]). An initial incorrect version of this lemma was published in [24]. The proofs for Theorem 5.2.6 and Theorem 5.2.7 were based on the initial incorrect version on the lemma. New proofs have been provided.

Remark. [24, Remark, Page 4] Since $|C(G)| \leq |G|$, (13) gives the anticipated lower boundary for h_m in terms of the greatest prime divisor p of $|G|$:

$$h_m(G) \geq \frac{p|G|}{(p-1)|G|+1}\,.$$

Theorem 5.2.2. *[24, Theorem 2.4] Let G be a finite p-group. Then $h_m(G) \in \mathbb{N}$ iff G is cyclic of order $p^{\sum_{i=1}^{s} p^i}$, with $s \in \mathbb{N}^*$, or $G \simeq D_8$.*

Proof. Let $|G| = p^n$, $\exp(G) = p^m$ and

$n_i = |\{a \in G \mid o(a) = p^i\}|$ and $n'_i = |\{H \in C(G) \mid |H| = p^i\}|$, for all $i \in \overline{0, m}$.

Lemma 5.2.1 implies that

$$h_m(G) = \frac{p^{n+1}}{(p-1)|C(G)|+1}\,,$$

which leads to the following equivalences:

$$\begin{aligned} h_m(G) \in \mathbb{N} \text{ iff } & (p-1)|C(G)|+1 = p^t \text{ with } t \leq n+1 \\ \text{iff } & |C(G)| = p^{t-1} + p^{t-2} + \cdots + p + 1, \text{ with } t \leq n. \end{aligned} \tag{14}$$

For $p = 2$, this formula becomes

$$|C(G)| = 2^t - 1. \tag{15}$$

We distinguish the following two cases:

Case 1: p odd. There are two possibilities: G can be cyclic or not.

1. Suppose G is not cyclic.

   Using Theorem 1.10 from [6], it follows that $\begin{cases} n'_1 \equiv p+1 \pmod{p^2} \\ n'_2, \ldots, n'_m \equiv 0 \pmod{p} \end{cases}$ and so, recalling that the trivial subgroup is an element of $C(G)$, we obtain $|C(G)| \equiv 2 \pmod{p}$. Thus there are no solutions.

2. Suppose G is cyclic.
   From (14), it follows that $n+1 = p^{t-1}+p^{t-2}+\cdots+p+1,$ with $t \leq n \xRightarrow{s=t-1} n = \sum_{i=1}^{s} p^i$.

Case 2: $p = 2$

1. Suppose G is not cyclic and G is not of maximal class.
   Using Theorem 1.17 from [6], it follows that $\begin{cases} n'_1 \equiv 3 \pmod 4 \\ n'_2, \ldots, n'_m \equiv 0 \pmod 2 \end{cases}$
   and so $|C(G)| \equiv 0 \pmod 2$. Thus there are no solutions.
2. Suppose G is cyclic.
   Therefore we have $|C(G)| = n+1$. Using (15), it follows that $n = 2^t - 2 \xlongequal{s=t-1} \sum_{i=1}^{s} 2^i$ with $1 \leq s \leq n+1$.
3. Suppose G is of maximal class
   It follows that $G \in \{D_{2^n}, Q_{2^n}, SD_{2^n}\}$. Using [29], the following analysis is obtained:
   (a) $G \cong D_{2^n} \Rightarrow |C(G)| = 2^{n-1}+n \Rightarrow 2^{n-1}+n+1|2^{n+1}$. It follows that the left hand side has to be a power of 2: $2^{n-1}+n+1 = 2^t$. This happens when $2^{n-1} = n+1$. Since $2^{n-1} > n+1,$ for all $n > 3$ (it can be shown inductively), it can be proven by direct computation that the only solution is $n = 3$ and therefore $G \cong D_8$.
   (b) $G \cong Q_{2^n} \Rightarrow |C(G)| = 2^{n-2}+n \Rightarrow 2^{n-2}+n+1|2^{n+1}$. Reasoning similarly as above, it follows that there are no solutions.
   (c) $G \cong SD_{2^n} \Rightarrow |C(G)| = 3 \cdot 2^{n-3}+n \Rightarrow 3 \cdot 2^{n-3}+n+1|2^{n+1}$. The left hand side must be a power of 2: $3 \cdot 2^{n-3}+n+1 = 2^s$, therefore $n+1$ is divisible by 2^{n-3}. Using $2^{n-3} > n+1,$ for all $n > 5$ (it can be again shown inductively), it can be proved that there are no solutions.

The proof of Theorem 2.3 is now complete.

Corollary 5.2.3. *[24, Corollary 2.5] D_8 is the only non-cyclic p-group with integer harmonic mean of element orders.*

Proposition 5.2.4. *[24, Proposition 2.6] D_8 is the only dihedral group with integer harmonic mean of element orders.*

Proof.

Let D_{2n} be a dihedral group of order $2n$ such that $h_m(D_{2n}) \in \mathbb{N}^*$. (16)

Let $n = p_1^{n_1} \cdot p_2^{n_2} \cdots p_k^{n_k}$ be the decomposition of n as a product of prime factors, where $p_1 < p_2 < \cdots < p_k$.

Let us use the description from (9) for the dihedral group: $D_{2n} = \{1, r, \ldots, r^{n-1}, s, sr$ where r is the rotation of order $\frac{2\pi}{n}$ and s is the reflection about the line determined by a vertex and the center of the polygon. Obviously, $\langle r \rangle \cong C_n$ and $o(sr^i) = 2$, for all $i \in \overline{0, n-1}$.

If we compute $h_m(D_{2n})$, we get

$$h_m(D_{2n}) = \frac{|G|}{\sum_{i=0}^{n-1} \frac{1}{o(r^i)} + \sum_{i=0}^{n-1} \frac{1}{o(sr^i)}} = \frac{2n}{m(C_n) + \frac{n}{2}}.$$

Let us denote

$$\alpha := \frac{2n}{m(C_n) + \frac{n}{2}} \in \mathbb{N}^*$$

because of assumption (16). Since $m(C_n) > 0$, it follows that $\alpha < 4$ and therefore three cases can occur:

1. $\alpha = 1$

   Then $m(C_n) = \frac{3n}{2}$, contradicting $m(C_n) = \sum_{a \in C_n} \frac{1}{o(a)} \leq \sum_{a \in C_n} 1 = n$.

2. $\alpha = 2$

   It follows that

$$m(C_n) = \frac{n}{2}. \tag{17}$$

   Clearly $C_n \cong \prod_{i=1}^{k} C_{p_i^{n_i}}$. According to Proposition 4.2.23 e), it follows that

$$m(C_n) = \prod_{i=1}^{k} m(C_{p_i^{n_i}}). \tag{18}$$

   Each factor in the right-hand side can be computed. Fix $i \in \overline{1, k}$. For each $j \in \overline{1, n_i}$, there are $\varphi(p_i^j)$ elements of order p_i^j. Therefore

$$\begin{aligned} m(C_{p_i^{n_i}}) &= 1 + \sum_{j=1}^{n_i} \frac{\varphi(p_i^j)}{p_i^j} = 1 + \sum_{j=1}^{n_i} \frac{p_i^j(1 - \frac{1}{p_i})}{p_i^j} = 1 + \sum_{j=1}^{n_i} \left(1 - \frac{1}{p_i}\right) \\ &= \frac{p_i - 1 + 1}{p_i} + \frac{n_i}{p_i}(p_i - 1) \\ &= \frac{(n_i + 1)(p_i - 1) + 1}{p_i}. \end{aligned}$$

   Using (17) and (18), it follows that

$$\prod_{i=1}^{k} ((p_i - 1)(n_i + 1) + 1) = \frac{1}{2} p_1^{n_1+1} \cdot \cdots \cdot p_n^{n_k+1}.$$

Since the left-hand side is an integer, it follows that $p_1 = 2$, so

$$(n_1+2)\prod_{i=2}^{k}((p_i-1)(n_i+1)+1) = 2^{n_1}p_2^{n_2+1}\dots p_k^{n_k+1}$$

Bernoulli's inequality gives:

$$p_i^{n_i+1} = (1+(p_i-1))^{n_i+1} > (p_i-1)(n_i+1)+1, \text{for all } i=\overline{2,k},$$

therefore $n_1 + 2 > 2^{n_1}$, i.e. $n_1 = 1$.

If $k > 1$ it follows that

$$3\prod_{i=2}^{k}((p_i-1)(n_i+1)+1) = 2p_2^{n_2+1}\cdot\dots\cdot p_k^{n_k+1},$$

therefore $p_2 = 3$ and

$$3(2n_2+3)\prod_{i=3}^{k}((p_i-1)(n_i+1)+1) = 2\cdot 3^{n_2+1}p_3^{n_3+1}\cdot\dots p_k^{n+1}.$$

Then $2n_2 + 3 \geq 2 \cdot 3^{n_2}$, which does not yield solutions. Thus the assumption that $k > 1$ does not hold, and consequently the only solution is $k = 1$ and $n_1 = 2$, which gives $n = 4$.

3. $\alpha = 3$

   The analysis is analogous to the previous case.

Remark. [24, Remark, Page 7] Let us note that we can build nilpotent groups G with $h_m(G) \in \mathbb{N}$ as direct products of p-groups of the type of the ones in Theorem 5.2.2.

In addition, we can construct non-nilpotent groups G with this property, for example $G = \mathrm{SL}(2,3) \times C_{7^7}$:

$$h_m(G) = h_m(\mathrm{SL}(2,3)) \cdot h_m(C_{7^7}) = \frac{24}{7}\cdot 7^6 = 24\cdot 7^5.$$

In what follows, we will study the integer values of the function h_m.

Obviously, we have $h_m(G) = 1$ iff G is the trivial group.

In addition, Proposition 5.2.4 and its proof give that $h_m(C_4) = h_m(D_8) = 2$. We can show that these are the only groups G with $h_m(G) = 2$.

Theorem 5.2.5. *([5, Theorems 2, 3] [28, Theorems 1, 2]) Let G be a finite group such that $|C(G)| = |G| - \Delta$ with $\Delta \in \{0, 1, 2, 3\}$.*

1. *If $\Delta = 0$, then G is an elementary abelian 2-group.*

2. *If $\Delta = 1$, then G is one of the following groups: C_3, C_4, S_3, D_8.*

3. *If $\Delta = 2$, then G is one of the following groups: C_6, $C_2 \times C_4$, D_{12}, $C_2 \times D_8$.*

4. *If $\Delta = 3$, then G is one of the following groups: C_5, Q_8, D_{10}.*

Theorem 5.2.6. *[24, Theorem 2.7] Let G be a finite group. Then $h_m(G) = 2$ iff $G \cong C_4$ or $G \cong D_8$.*

Theorem 5.2.7. *[24, Theorem 2.8] The finite non-trivial groups G with $h_m(G) \leq 2$ are $C_2^n, n \in \mathbb{N}, C_3, S_3, C_4$ and D_8.*

Proof of Theorems 2.7 and 2.8.. Let G be a finite group with $h_m(G) \leq 2$ and denote by $n_d(G)$ the number of elements of order d in G, $\forall\, d \in \mathbb{N}$. Then

$$\begin{aligned}\frac{|G|}{2} \leq m(G) &= 1 + \frac{n_2(G)}{2} + \frac{n_3(G)}{3} + \dots \\ &\leq 1 + \frac{n_2(G)}{2} + \frac{n_3(G) + \dots}{3} \\ &= 1 + \frac{n_2(G)}{2} + \frac{|G| - 1 - n_2(G)}{3}\end{aligned}$$

and so

$$|G| \leq n_2(G) + 4.$$

Since $|C(G)| \geq 1 + n_2(G)$, we get

$$|C(G)| \geq |G| - 3,$$

i.e. G is one of the groups in Theorem 5.2.5. It is now easy to check that $h_m(G) \leq 2$ iff G is an elementary abelian 2-group, C_3, C_4, S_3 or D_8. Also, $h_m(G) = 2$ iff G is C_4 or D_8.

Next we will focus on finite groups G with $h_m(G) = 3$. Note that the smallest example of such a group is $SmallGroup(12, 1)$. We are not able to determine all these groups, but we can prove that they have even order and are not nilpotent.

Proposition 5.2.8. *[24, Proposition 2.9] There are no finite groups G of odd order with $h_m(G) = 3$.*

Proof. Let G be a finite group of odd order such that $h_m(G) = 3$. Then $3 \mid |G|$. Let $H \leq G$ with $|H| = 3$. We identify the following cases:

Case 1: $\exp(G) = 3$

Then

$$h_m(G) = \frac{|G|}{1 + \frac{|G|-1}{3}} = \frac{3|G|}{|G|+2} \neq 3,$$

which is a contradiction.

Case 2: $\exp(G) \neq 3$

We will prove that

$$G \text{ has at least 6 elements of order } \geq 5 \tag{19}$$

We identify the cases:

1. $3^2 \,|\, \exp(G)$

   Then G has at least a cyclic subgroup of order 3^2 and so at least $6 = \varphi(3^2)$ elements of order 9.

2. $3^2 \nmid \exp(G)$.

   Let p the smallest prime $\neq 3$ such that $p \,|\, \exp(G)$. If $p \geq 7$, then G has at least a cyclic subgroup of order p and consequently at least $\varphi(p) = p - 1$ elements of order p. If $p = 5$, there are two sub-cases:

   (a) G has only one subgroup K with $|K| = 5$. It follows that $K \lhd G$, therefore $KH \leq G$ and $|KH| = 15$. Then KH is cyclic and it possesses $\varphi(15) = 8$ elements of order 15.

   (b) G has at least 3 subgroups of order 5. Then G has at least $3\varphi(5) = 12$ elements of order 5.

   This concludes the proof of (19). We get

$$h_m(G) \geq \frac{|G|}{1 + \frac{6}{5} + \frac{|G|-7}{3}} = \frac{15|G|}{5|G|-2} > 3,$$

a contradiction which completes the proof.

Proposition 5.2.9. *[24, Proposition 2.10] There are no finite nilpotent groups G with $h_m(G) = 3$.*

Proof. Assume that G is a finite nilpotent group such that $h_m(G) = 3$. If G is a p-group, then $p = 3$ and the conclusion follows from Proposition 5.2.8. If G is not a p-group, then it can be written as a direct product of at least two p-groups, say $G = G_1 \times \cdots \times G_k$ with $k \geq 2$. Since

$$h_m(G) = h_m(G_1) \cdots h_m(G_k),$$

we get $h_m(G_i) < 2$, for all $i = 1, ..., k$. Now Theorem 5.2.7 implies that $G_i = C_2^n$ for some $n \in \mathbb{N}$ or $G_i = C_3$, and therefore $h_m(G_i) = \frac{2^{n+1}}{2^n+1}$ for some $n \in \mathbb{N}$ or $h_m(G_i) = \frac{9}{5}$. We remark that any product of these numbers is not 3, contradicting our assumption.

We can also prove a criterion for supersolvability using h_m. For this, we need another existing result. Before this, we introduce some notations:

Notation 5.1. Let $d_1 = 1, \dots, d_r$ be the orders of the elements in G, increasingly ordered.

For all $i \in \overline{1, r}$, we introduce the following notations:

$$n_i = |\{a \in G \mid o(a) = d_i\}| \text{ and } n'_i = |\{H \in C(G) \mid |H| = d_i\}|.$$

Theorem 5.2.10. *([30], [17]) Let G be a non-abelian finite group. If $n_2(G) > \frac{1}{2}|G| - 1$, then one of the following holds:*

1. *$G \cong D(A)$ is a generalized dihedral group, where A is abelian;*
2. *$G \cong D_8 \times D_8 \times E$, where E is an elementary abelian 2-group;*
3. *$G \cong H(r) \times E$, where $H(r)$ is a central product of r copies of D_8 and E is an elementary abelian 2-group;*
4. *$G \cong S(r) \times E$, where $S(r) = \langle x_1, y_1, ..., x_r, y_r, z \mid x_i^2 = y_i^2 = z^2 = 1$, all pairs of generators commute except $[z, x_i] = x_i y_i\rangle$ and E is an elementary abelian 2-group.*

In particular, a finite group G with $n_2(G) > \frac{1}{2}|G| - 1$ is supersolvable.

Now we can proceed to our criterion:

Theorem 5.2.11. *[25, Theorem 2.1] Let G be a finite group. If $h_m(G) < \frac{72}{31}$, then G is supersolvable. Moreover, if $h_m(G) = \frac{72}{31}$, then either $G \cong A_4$ or G is a generalized dihedral group.*

Proof. Assume that $h_m(G) < \frac{72}{31}$. Then, similarly with the proof of Theorems 5.2.6 and 5.2.7, we have

$$\begin{aligned}\frac{31}{72}|G| < m(G) &= 1 + \frac{n_2(G)}{2} + \frac{n_3(G)}{3} + \dots \\ &\le 1 + \frac{n_2(G)}{2} + \frac{n_3(G) + \dots}{3} \\ &= 1 + \frac{n_2(G)}{2} + \frac{|G| - 1 - n_2(G)}{3}\end{aligned}$$

implying that

$$n_2(G) > \frac{7}{12}\,|G| - 4.$$

If $|G| \geq 36$, then $\frac{7}{12}\,|G| - 4 \geq \frac{1}{2}\,|G| - 1$ and so $n_2(G) > \frac{1}{2}\,|G| - 1$. Then G is supersolvable by Theorem 5.2.10.

Assume that $|G| \leq 35$ and G is not supersolvable. Then $|G| \in \{12, 24\}$. Since A_4 is the unique non-supersolvable group of order 12, if $|G| = 12$, we get $G \cong A_4$. Thus $h_m(G) = \frac{72}{31}$, a contradiction. Similarly, if $|G| = 24$ we get $G \cong S_4$, $G \cong A_4 \times C_2$ or $G \cong \mathrm{SL}(2,3)$ and so $h_m(G) = \frac{72}{29} > \frac{72}{31}$, $h_m(G) = \frac{48}{17} > \frac{72}{31}$ or $h_m(G) = \frac{24}{7} > \frac{72}{31}$, a contradiction. Thus G is supersolvable in all cases.

Assume now that $h_m(G) = \frac{72}{31}$. Then both 2 and 3 divide $|G|$. By repeating the above reasoning, we obtain

$$n_2(G) \geq \frac{7}{12}\,|G| - 4$$

and therefore $n_2(G) > \frac{1}{2}\,|G| - 1$ for $|G| \geq 37$. In this case G is a generalized dihedral group by Theorem B. If $|G| \leq 36$, then $|G| \in \{6, 12, 18, 24, 30, 36\}$. By computing the harmonic mean of element orders for all groups of these orders using [31], we conclude that the unique possibility is $G \cong A_4$. This completes the proof.

Finally, we note that the results so far leave the following open question:

Question 5.1. [24, Question 2] What is $\mathrm{Im}(h_m) \cap \mathbb{Z}$?

5.3 Almost $\mathcal{P}$-numbers

Definition 5.3.1. *[10, Page 3]A positive integer n is called a **$\mathcal{P}$-number** if all groups of order n are in class $\mathcal{P}$.*

Some of the most obvious particular cases for this definition (cyclic, abelian, nilpotent) use the concept of nilpotent factorization given in Definition 4.1.15. This allows to state the characterizations as follows:

Proposition 5.3.2. *[21, Theorem 1, Pages 3, 4] A positive integer n is*

- *a nilpotent number iff it has nilpotent factorization.*
- *an abelian number iff it is a cube–free number with nilpotent factorization.*
- *a cyclic number iff it is a square–free number with nilpotent factorization*

The statement for cyclic numbers can also be given as follows:

Proposition 5.3.3. *[21, Page 4] A positive integer n is cyclic iff the number and its Euler totient function $\varphi(n)$ are coprime, that is*

$$\gcd(n, \varphi(n)) = 1.$$

Example 5.3.4. *[9, Page 1] All prime numbers are cyclic numbers.*

If we loosen the hypothesis in Definition 5.3.1, we reach the following definition.

Definition 5.3.5. *[23, Definition 1.6] A positive integer n is called an **almost $\mathcal{P}$-number** if all but one group of order n (up to isomorphism) are in class $\mathcal{P}$.*

This definition has started from the following result:

Theorem 5.3.6. *[7, Theorem 6.7] Let G be a group of order $n = p_1^{n_1} \dots p_k^{n_k}$, where $p_1 < p_2 < \dots < p_k$. Then there are exactly two groups (up to isomorphism) of order n iff $k \geq 2$ and one of the following scenarios occurs:*

$$\begin{aligned} &\text{for all } l \in \{1, \dots, k\} : n_l = 1, \text{ and} \\ &\text{there exists a unique pair } (i, j) \in \{1, \dots, k\}^2 \text{ such that } p_i | p_j - 1 \end{aligned} \tag{20}$$

or

$$\begin{aligned} &\text{there exists a unique } j \in \{1, \dots, k\} \text{ such that } n_j = 2, \text{ and} \\ &\text{for all } l \in \{1, \dots, k\} \setminus \{j\} : n_l = 1, \text{ and} \\ &\text{there exists a unique } i \in \{1, \dots, k\} \setminus \{j\} : p_i \mid p_j - 1, \ p_i \nmid p_j + 1, \text{ and} \\ &\text{for all } (\alpha, \beta) \in \{1, \dots, k\}^2 \setminus \{(i, j)\} : p_\alpha \nmid p_\beta - 1 \end{aligned} \tag{21}$$

We can rephrase the result above using Definition 5.3.5:

Corollary 5.3.7. *[23, Corollary 1.8] A positive integer $n = p_1^{n_1} \dots p_n^{n_k}$ is almost cyclic iff $k \geq 2$ and either (20) or (21) hold.*

Lemma 5.3.8. *Given $p < q$ primes and $n \in \mathbb{N}^*$ such that $p|(q^n - 1)$. Then there exists a non-abelian group of order pq^{n-1}: $C_{q^n} \rtimes C_p$.*

Lemma 5.3.9. *Given p a prime and $n \in \mathbb{N}, n \geq 3$, there are at least 2-non-abelian groups of order p^n.*

Theorem 5.3.10. *[23, Theorem 2.1] A positive integer $n = p_1^{n_1} \dots p_n^{n_k}$ is almost abelian iff either (20) or*

$$\begin{aligned} &\text{there exists a unique } j \in \{1, \dots, k\} \text{ such that } n_j = 2, \text{ and} \\ &\text{for all } l \in \{1, \dots, k\} \setminus \{j\} : n_l = 1, \text{ and} \\ &\text{there exists a unique } i \in \{1, \dots, k\} \setminus \{j\} : p_i \mid p_j + 1, \text{ and} \\ &\text{for all } (\alpha, \beta) \in \{1, \dots, k\}^2 \setminus \{(i, j)\} : p_\alpha \nmid p_\beta - 1 \end{aligned} \tag{22}$$

Proof. "$\Rightarrow$" Let us suppose that n is almost abelian. Since there is just one non-abelian group of order n, it follows that:

$$\text{For any divisor } d \text{ of } n : d|n, \text{ there is at most one non-abelian group of order } d. \tag{23}$$

We begin with observing that Lemma 5.3.9 gives us that n **is cube-free**.

Knowing this, since n is not abelian, it follows that

$$\text{there exist } i, j \in \{1, \dots, k\} \text{ such that } p_i | p_j^{n_j} - 1 \tag{24}$$

and

$$n_r \leq 2, \text{for all } r \in \{1, .., k\} \tag{25}$$

Now, let us note that **there is just one pair checking** (24). Indeed, if there are two distinct pairs (i, j) and (α, β) such that $p_i|(p_j^{n_j} - 1)$ and $p_\alpha|(p_\beta^{n_\beta} - 1)$, then it follows that there are two distinct non-abelian groups of order $p_i p_j^{n_j} p_\alpha p_\beta^{n_\beta}$:

$$((C_{p_\beta})^{n_\beta} \rtimes C_{p_\alpha}) \times C_{p_i p_j^{n_j}} \text{ and } C_{p_\alpha p_\beta^{n_\beta}} \times ((C_{p_j})^{n_j} \rtimes C_{p_i})$$

which proves that there is at most one pair.

In this step, let us analyse **which of the exponents can be** 2. From (24), it follows from Lemma 5.3.8, that there exists a nontrivial semi-direct product $(C_{p_j})^{n_j} \rtimes C_{p_i}$.

- If $n_r = 2$, for $r \in \{1, 2, \dots, k\} \setminus \{i, j\}$, it follows that there are two non-abelian groups of order $p_i p_j^{n_j} p_r^2$:

  $$((C_{p_j})^{n_j} \rtimes C_{p_i}) \times (C_{p_r})^2 \text{ and } ((C_{p_j})^{n_j} \rtimes C_{p_i}) \times C_{p_r^2}$$

  which gives us a contradiction with (23).

- If $n_i = 2$ then we will have two non-abelian groups of order $p_i^2 p_j^{n_j}$:

  $$(C_{p_j})^{n_j} \rtimes (C_{p_i})^2 \text{ and } ((C_{p_j})^{n_j} \rtimes C_{p_i}) \times C_{p_i},$$

  which again gives a contradiction with (23).

- If $n_j = 2$, let us assume that

  $$p_i|(p_j - 1). \tag{26}$$

  It follows that there are two distinct non-abelian groups of order $p_i p_j^2$:

  $$(C_{p_j})^2 \rtimes C_{p_i} \text{ and } C_{{p_j}^2} \rtimes C_{p_i}$$

  which contradicts our hypothesis. It follows that $p_i \nmid (p_j - 1) \xrightarrow{p_i|(p_j^2-1)}$ $p_i|(p_j + 1)$ which gives (22).

Now, **if all the exponents are** 1 (in particular $n_j = 1$), (20) is satisfied, which concludes the direct proof.

"$\Leftarrow$" For the converse, we will prove by **induction over** n that:

$$\text{If } n \text{ has at least two non-prime factors and satisfies either (20) or (22),}$$
$$\text{it follows that there is a unique non-abelian group of order } n. \quad (27)$$

The **base case** is $n = 6$. There is just one non-abelian group of order 6: the symmetric group S_3.

Let us proceed to the **inductive step**. Assume that (27) holds for any positive integer with at least two factors $n' < n$. Let G be a non-abelian group of order n. We order the prime factors: $p_1 < p_2 < \dots p_k$.

It follows that $j \geq 2$. Indeed, if $j = 1$, we have two possibilities:

- $p_i | p_1 - 1$ which implies $p_i < p_1$, a contradiction.
- $p_i | p_1 + 1 \xRightarrow{p_i > p_1} p_i = p_1 + 1 \Rightarrow p_1 = 2$ and $p_i = p_2 = 3 \Rightarrow p_j | p_i - 1$ which is a contradiction.

Thus $j \geq 2$ and therefore $n_1 = 1$, hence the p_1-Sylow subgroups of G are cyclic of order p_1. Thus, by the Burnside normal p-complement theorem, G has a p_1-normal complement, i.e. there exists $H \triangleleft G$ with $|H| = n/p_1$ such that $G = H \rtimes C_{p_1}$. We identify two cases:

- n satisfies (20).
  - If $i = 1$, it follows that H is cyclic and we get: $G = C_{p_2 \dots p_k} \rtimes C_{p_1} \cong (C_{p_j} \rtimes C_{p_1}) \times C_{\frac{p_2 \cdots p_k}{p_j}}$
  - If $i \geq 2$, there are again two possibilities:
    * H abelian $\Rightarrow H$ cyclic which is analogous with above.
    * H non-abelian $\Rightarrow H \cong (C_{p_j} \rtimes C_{p_i}) \times C_{\frac{p_2 \cdots p_k}{p_i p_j}} \Rightarrow$
      $$\begin{aligned} G &\cong ((C_{p_j} \rtimes C_{p_i}) \times C_{\frac{p_2 \cdots p_k}{p_i p_j}}) \rtimes C_{p_1} \\ &\cong ((C_{p_j} \rtimes C_{p_i}) \rtimes C_{p_1}) \times C_{\frac{p_2 \cdots p_k}{p_i p_j}} \end{aligned}.$$
      Since $|\operatorname{Aut}(C_{p_j} \rtimes C_{p_i})| = p_j(p_j - 1)$ and $p_1 \nmid p_j(p_j - 1)$ the semidirect product $(C_{p_j} \rtimes C_{p_i}) \rtimes C_{p_1}$ is trivial therefore
      $$\begin{aligned} G &\cong ((C_{p_j} \rtimes C_{p_i}) \times C_{\frac{p_2 \cdots p_k}{p_i p_j}} \\ &\cong (C_{p_j} \rtimes C_{p_i}) \times C_{\frac{p_1 \cdots p_n}{p_i p_j}}. \end{aligned}$$
      It follows that in this case there is a single non-abelian group of order n which is: $(C_{p_j} \rtimes C_{p_i}) \times C_{\frac{n}{p_i p_j}}$.

- n verifies (22).
  - If $i = 1$, then H is abelian and we get
  $$H \cong C_{p_j^2} \times C_{\frac{n}{p_1 p_j^2}} \text{ or } G \cong ((C_{p_j})^2 \rtimes C_{p_1}) \times C_{\frac{n}{p_1 p_j^2}}.$$
  We get two possibilities:
    - $G \cong ((C_{p_j})^2 \rtimes C_{p_1}) \times C_{\frac{n}{p_1 p_j^2}}$.
    Since $p_1 \nmid |\operatorname{Aut}((C_{p_j})^2)| = p_j(p_j - 1)$, it follows that $(C_{p_j})^2 \rtimes C_{p_1}$, therefore $G \cong C_n$ which is a contradiction.
    - $G \cong ((C_{p_j})^2 \rtimes C_{p_1})$ which is non-abelian.
  - If $i \geq 2$, then we again have two cases:
    - $H \cong ((C_{p_j})^2 \rtimes C_{p_i}) \times C_{\frac{n}{p_1 p_i p_j^2}}$, then:
    $$G \cong (((C_{p_j})^2 \rtimes C_{p_i}) \rtimes C_{p_1}) \times C_{\frac{n}{p_1 p_i p_j^2}}.$$
    From [8], it follows that $|\operatorname{Aut}((C_{p_j})^2 \rtimes C_{p_i}| = 2(p_j^2 - 1)p_j^2 \not\vdots\, p_1$, therefore:
    $$\begin{aligned} G &\cong (((C_{p_j})^2 \rtimes C_{p_i}) \times C_{p_1}) \times C_{\frac{n}{p_1 p_i p_j^2}} \\ &\cong ((C_{p_j})^2 \rtimes C_{p_i}) \times C_{\frac{n}{p_i p_j^2}}. \end{aligned}.$$
    - H abelian. We identify two possibilities:
      - $H \cong C_{p_2} \times \cdots \times C_{p_j} \times \cdots \times C_{p_k} \cong C_{\frac{n}{p_1}} \Rightarrow G$ cyclic which is false.
      - $H \cong C_{p_2} \times \ldots C_{p_j}^2 \times \cdots \times C_{p_k} \cong (C_{p_j})^2 \times C_{\frac{n}{p_1 p_j^2}}$
      Let us observe that
      $$p_1 \nmid |\operatorname{Aut}((C_{p_j})^2 \times C_{\frac{n}{p_1 p_j^2}})| = (p_j^2 - 1)(p_j^2 - p_j) \cdot \varphi\left(\frac{n}{p_1 p_j^2}\right). \tag{28}$$
      $G \cong ((C_{p_j})^2 \times C_{\frac{n}{p_1 p_j^2}}) \rtimes C_{p_1} \cong ((C_{p_j})^2 \times C_{\frac{n}{p_1 p_j^2}}) \times C_{p_1} \overset{(28)}{\cong} (C_{p_j})^2 \times C_{\frac{n}{p_j^2}}$.
      This means G is abelian which is false.

  Therefore, also in this case there is only one non-abelian group of order n.

Corollary 5.3.11. *Let $n = p_1^{n_1} \cdot \dots \cdot p_k^{n_k}$ where $2 = p_1 < \dots < p_k$. It follows that n is almost abelian iff $k = 2$ and $n_1 = n_2 = 1$.*

Corollary 5.3.12. *Let $n = p_1 \cdot \dots \cdot p_k$, where $p_1 < \dots < p_k$. Then n is almost abelian iff there is a unique pair $(i, j) \in \{1, \dots, k\}^2$ such that $p_i | p_j - 1$.*

Remark. If n is almost cyclic, n is either abelian or almost abelian.

Theorem 5.3.13. *A number $n = p_1^{n_1} \cdot \dots \cdot p_k^{n_k}$ is almost nilpotent iff n is almost abelian, i.e. $k \geq 2$ and n checks* (20) *or* (22)*.*

Proof. "$\Leftarrow$" The converse follows from Theorem 5.3.10 since all the groups constructed in the proof are non-nilpotent.
"$\Rightarrow$"Let us assume n is almost nilpotent. It follows that for all $d|n$, there exists at most one non-nilpotent group of order d. Since n is non-nilpotent, it follows that there is $(i, j) \in \{1, \dots, k\}^2$ and $1 \leq d_j \leq n_j$ such that $p_i | p_j^{d_j} - 1$. It follows that $\alpha_j = n_j$. Otherwise there would be two non-nilpotent non-isomorphic groups of order $p_j^{n_j} p_i$:

$$((C_{p_j})^{\alpha_j} \rtimes C_{p_i}) \times C_{p_j^{n_j - d_j}} \text{ and } (C_{p_j})^{n_j} \rtimes C_{p_i}.$$

Furthermore, the pair (i, j) is unique. Otherwise, if there were two pairs $(i', j') \neq (i, j)$ such that $p_{i'} | p_{j'}^{n_j} - 1$, again there would be two non-nilpotent non-isomorphic groups of order n:

$$((C_{p_j})^{n_j} \rtimes C_{p_i}) \times C_{\frac{n}{p_i p_j^{n_j}}} \text{ and } ((C_{p_{j'}})^{n_{j'}} \rtimes C_{p_{i'}}) \times (C_{\frac{n}{p_{i'} p_{j'})^{n_{j'}}}}.$$

Let us observe that $n_r = 1$, for all $r \neq i, j$. Indeed, otherwise there would exist at least two distinct groups P_r and Q_r of order $p_r^{n_r}$, which would give two non-nilpotent, non-isomorphic groups of order $p_j^{n_j} p_i p_r^{n_r}$:

$$((C_{p_j})^{n_j} \rtimes C_{p_i}) \times P_r \text{ and } ((C_{p_j})^{n_j} \rtimes C_{p_i}) \times Q_r.$$

Analogously, we can show that $n_i = 1$.
If $n_j = 1$, (20) holds.
If $n_j \geq 2$, then $n_j = 2$, since otherwise we would two non-nilpotent, non-isomorphic groups of order $p_j^{n_j} p_i$. Thus

$$p_i | p_j^2 - 1. \tag{29}$$

In addition, if $p_1 | p_j - 1$, there are two non-nilpotent, non-isomorphic groups of order $p_j^2 p_i$:

$$(C_{p_j})^2 \rtimes C_{p_i} \text{ and } C_{p_j^2} \rtimes C_{p_i}.$$

It follows that $p_i \nmid p_j - 1 \xRightarrow{(29)} p_i | p_j + 1$, therefore we get (22), which concludes the proof.

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