SDEs in Banach spaces driven by cylindrical fractional Brownian motions

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Outline

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SDEs in Banach spaces

We consider the following evolution equation in a Banach space

\[
\begin{align*}
\text{(SDE)} & \quad \begin{cases}
    dY(t) = AY(t)dt + CdB^H(t), & t \in (0, T] \\
    Y(0) = Y_0
\end{cases}
\end{align*}
\]

where

- \( A \) is the generator of a \( C_0 \) semigroup \( (S_t)_{t \geq 0} \) on a Banach space \( V \)
- \( U \) is another Banach space
- \( C \) is a bounded and linear operator \( C : U \to V \)
- \( \{B^H(t), t \geq 0\} \) is a (cylindrical) fBm in \( U \)
- \( Y_0 \) is a given (cylindrical) random variable.

**AIM**: solve (SDE) using mild solutions, i.e. solution of the type

\[
Y(t) = S_t Y_0 + \int_0^t S_{t-s} C dB^H(s), \quad t \in [0, T].
\]
What are the problems?

$$Y(t) = S_t Y_0 + \int_0^t S_{t-s} C \, dB^H(s)$$

(a) define a fractional Brownian motion \( \{B^H(t), t \geq 0\} \) in a Banach space

(b) define a stochastic integral with respect to it

$$\int_0^T \varphi(t) \, dB^H(t)$$

for deterministic integrands \( \varphi \).
In Hilbert spaces:

(a) the noise is characterized by a series. For instance a Brownian motion \( \{ W(t), t \geq 0 \} \) in \( H \) is given by

\[
W(t) = \sum_{k=1}^{\infty} \lambda_k e_k \beta_k(t).
\]

with

- \((\beta_k)_k\) \(1\)-dim Bm
- \((e_k)_k\) basis for \( H \)
- \(\lambda_k\) real positive numbers.

(b) the stochastic integral is defined exploiting the series representation. For instance for \( \varphi(t) \in \mathcal{L}(H) \) we have

\[
\int_0^T \varphi(t) \, dW(t) = \sum_{k=1}^{\infty} \int_0^T \lambda_k \varphi(t) e_k \, d\beta_k(t)
\]

(need assumptions for it to converge in \( L^2(\Omega; H) \))
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$U$ separable Banach space.

$U^*$ topological dual: linear and continuous mapping on $U$

- A **cylindrical random variable** in $U$ is a linear map $X : U^* \to L^0_\mathbb{P}(\Omega; \mathbb{R})$.

- A **cylindrical stochastic process** in $U$ is a family of cylindrical r.v. $Y = \{Y(t), t \geq 0\}$. 
FBm: finite dimensional case

(1-dim) - continuous zero-mean Gaussian process \( \{ B^H(t), t \geq 0 \} \) on \( (\Omega, \mathcal{F}, P) \)
- Hurst parameter \( H \in (0, 1) \)
- covariance function

\[
\mathbb{E}(B^H(t)B^H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})
\]

for all \( t, s \geq 0 \).

(n-dim) - \( M \) positive symmetric \( n \times n \) matrix
- \((\cdot, \cdot)_n\) scalar product in \( \mathbb{R}^n \)
- continuous zero-mean \( \mathbb{R}^n \)-valued Gaussian process \( \{ B^H(t), t \geq 0 \} \)
- Hurst parameter \( H \in (0, 1) \)
- covariance function

\[
\mathbb{E}[(v_1, B^H(t))_n(v_2, B^H(s))_n] = (Mv_1, v_2)_n \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})
\]

for all \( t, s \geq 0 \) and \( v_1, v_2 \in \mathbb{R}^n \).
Cylindrical fBm: infinite dimensional case
A cylindrical process \((B^H(t), t \geq 0)\) in \(U\) is a cylindrical fBm with Hurst parameter \(H \in (0, 1)\) if

(i) for any \(u_1^*, \ldots, u_n^* \in U^*\) and \(n \in \mathbb{N}\), the \(\mathbb{R}^n\)-valued stochastic process
\[
\{(B^H(t)u_1^*, \ldots, B^H(t)u_n^*), t \geq 0\}
\]

is an \(n\)-dimensional fBm with Hurst parameter \(H \in (0, 1)\);

(ii) the mapping of \(B^H(1) : U^* \to L^0_P(\Omega; \mathbb{R})\) is continuous.

Special case: \(H = 1/2\) cylindrical Wiener Process.
**Theorem (series representation)**

For a cylindrical process $B^H$ the following are equivalent:

(a) $B^H$ is a cylindrical fBm with $H \in (0, 1)$;

(b) there exist a Hilbert space $H$, an orthonormal basis $(e_k)_{k \in \mathbb{N}}$, an operator $F \in L(H, U)$ and independent real valued fBm $(\beta_k^H)_{k \in \mathbb{N}}$ such that

$$B^H(t)u^* = \sum_{k=1}^{\infty} \langle Fe_k, u^* \rangle \beta_k^H(t)$$

in $L^2_{\mathbb{P}}(\Omega; \mathbb{R})$ for all $u^* \in U^*$ and all $t \geq 0$. 
\[ B^H(t)u^* = \sum_{k=1}^{\infty} \langle Fe_k, u^* \rangle \beta^H_k(t) \]

- the space \( H \) is called *Reproducing Kernel Hilbert Space* associated to \( Q \) and is unique (up to isometry).
- the operator \( Q \) is factorized through \( H \) by \( Q = FF^* \) and in some special cases \( F = Q^{1/2} \)

\[ \begin{array}{ccc}
Q : U^* & \rightarrow & U \\
Q^{1/2} = F^* & \downarrow & F = Q^{1/2} \\
H & \cup & (e_k)_k
\end{array} \]

- we have \( \langle Fe_k, u^* \rangle = [e_k, F^* u^*]_H \).
Special case: the $U$-valued fBm

An $U$-valued stochastic process $(B^H(t), t \geq 0)$ is called $U$-valued fBm if there exists a Gaussian measure on $\mathcal{B}(U)$ with covariance operator $Q : U^* \to U$ such that

- $\mathbb{E}\langle B^H(t), u^* \rangle = 0$ for all $u^* \in U^*$ and all $t \geq 0$;
- the covariance is given by

$$\mathbb{E}[\langle B^H(t), u^* \rangle \langle B^H(s), v^* \rangle] = \langle Qu^*, v^* \rangle \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H})$$

for all $t \geq 0$ and $u^*, v^* \in U^*$.

Remarks:

- see [Duncan, Jakubowski, Pasik-Duncan (2006)] (Hilbert space case)
- if $U$ is a Hilbert space then $Q : U^* \to U$ is a covariance operator if and only if $Q$ is positive, symmetric and trace class
- if $U$ is a Banach space need to be careful!
Theorem (series representation)

For an adapted $U$-valued process $B^H$ the following are equivalent:

(a) $B^H$ is a $U$-valued fBm

(b) there exist a Hilbert space $H$, an orthonormal basis $(e_k)_{k \in \mathbb{N}}$, a $\gamma$-radonifying operator $F \in L(H, U)$ and independent real valued fBms $(\beta_k^H)_{k \in \mathbb{N}}$ such that

$$W(t) = \sum_{k=1}^{\infty} F e_k \beta_k^H(t)$$

in $L^2_{\mathbb{P}}(\Omega; U)$ for all $t \geq 0$.

special cases:

- $H = 1/2$: $U$-valued Wiener process
- if $U$ is a Hilbert space: $F$ Hilbert-Schmidt $\iff F$ is $\gamma$-radonifying
cylindrical fBm $B^H(t) : U^* \to L^2(\Omega; \mathbb{R})$.
$U$-valued fBm $\tilde{B}^H(t) \in L^2(\Omega; U)$.

**Example**

One can define a cylindrical fBm $B^H(t) : U^* \to L^2(\Omega; \mathbb{R})$ by

$$B^H(t)u^* = \langle \tilde{B}^H(t), u^* \rangle$$

for all $u^* \in U^*$.

We say that a cylindrical process $Y$ is induced by an $U$-valued process $\tilde{Y}$ if $Y(t)u^* = \langle \tilde{Y}(t), u^* \rangle$ for all $u^* \in U^*$.

**Careful:** not every cylindrical process is induced by a $U$-valued process.

**QUESTION:** when is a cylindrical fBm induced by a $U$-valued process (fBm)? $\longrightarrow$ when $F$ is $\gamma$-radonifying.
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The cylindrical stochastic integral

- $B^H$ cylindrical fBm in $U$: $B^H(t)u^* = \sum_{k=1}^{\infty} \langle Fe_k, u^* \rangle \beta_k^H(t)$
- $V$ another Banach space
- $\varphi: [0, T] \to \mathcal{L}(U, V)$ deterministic

The cylindrical integral $\int_0^T \varphi(t) \, dB^H(t) = \mathcal{I}_T(\varphi)$ is a cylindrical process in $V$ defined by

$$\mathcal{I}_T(\varphi)v^* := \sum_{k=1}^{\infty} \int_0^T \langle \varphi(t)Fe_k, v^* \rangle_v \, d\beta_k^H(t)$$

for every $v^* \in V^*$.  

**Remark:** each 1-dim integral is performed as a Wiener integral. Need at least $\langle \varphi(\cdot)Fe_k, v^* \rangle_v, v^* \in \mathcal{H}_T$.  

**CAUTION:** this is the definition if the series converges in $L^2_{\mathbb{P}}(\Omega, \mathbb{R})$. 

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SDEs in Banach spaces driven by cylindrical fractional Brownian motions
Space of integrable functions

We have

\[ [e_k, F^* \varphi^*(\cdot)v^*]_{H_Q} = \langle \varphi(\cdot)F e_k, v^* \rangle_{V, V^*} \in \mathcal{H}_T \]

\[ \iff \]

\[ F^* \varphi^*(\cdot)v^* \in \mathcal{H}_T. \]

We define the space of integrable functions as

\[ \mathcal{I}_T := \{ \varphi : [0, T] \to \mathcal{L}(U, V) \text{ such that for all } v^* \in V^*, F^* \varphi^*(\cdot)v^* \in \mathcal{H}_T \} \]

**Theorem**

Let \( \varphi \in \mathcal{I}_T \). Then the integral \( \mathcal{I}_T(\varphi) \) is well defined and the following isometry property holds

\[ \| \mathcal{I}_T(\varphi)v^* \|_{L^2_{\mathbb{F}}(\Omega; \mathbb{R})}^2 = \| F^* \varphi^*(\cdot)v^* \|_{\mathcal{H}_T}^2 \]

The definition does not depend on the representation of the fBm.
When is the integral $\mathcal{I}_T(\varphi)$ induced by a classical r.v. in $V$?

Introduce the following subspace of integrable functions

$$\mathcal{B}_T := \{ \varphi \in \mathcal{I}_T \text{ such that } \exists C > 0 : \| F^* \varphi^* (\cdot) v^* \|_{\mathcal{H}_T} \leq C \| v^* \|_{V^*} \text{ for all } v^* \in V^* \}.$$ 

**Theorem**

If $\varphi \in \mathcal{B}_T$ then $\mathcal{I}_T(\varphi)$ is a zero-mean (strongly) Gaussian r.v. with covariance operator $Q_{T, \varphi}$ which is $V$-valued and can be decomposed through

$$Q_{T, \varphi} : V^* \rightarrow V$$

$$\Gamma_{T, \varphi} \downarrow \Gamma_{T, \varphi}^* \uparrow$$

$$L^2([0, T]; H_Q)$$

**Corollary**

$\mathcal{I}_T(\varphi)$ is a $V$-valued random variable $\Leftrightarrow \Gamma_{T, \varphi}^*$ is $\gamma$-radonifying.
SDEs in Banach spaces

Back to our initial problem:

\[
\begin{aligned}
\text{(SDE)} \quad \begin{cases} 
    dY(t) &= AY(t)dt + CdB^H(t), \quad t \in (0, T] \\
    Y(0) &= Y_0 \end{cases}
\end{aligned}
\]

The solution is a cylindrical process \( \{Y(t), t \in [0, T]\} \) in \( V \), i.e.

\[
Y(t) : V^* \to L^0_P(\Omega; \mathbb{R})
\]

- **weak solution:** for all \( v^* \in \text{dom}(A^*) \),

\[
Y(t)v^* = Y_0v^* + \int_0^t AY(s)v^* \, ds + CB^H(t)v^* \quad \text{where}
\]

\[
AY(t)v^* = Y(t)(A^*v^*) \quad \text{and} \quad CB^H(t)v^* = B^H(t)(C^*v^*)
\]

- **mild solution:** \( Y(t) = S_t Y_0 + \int_0^t S_{t-s} C dB^H(s) \) as cylindrical process in \( V \)
Theorem
Under some assumptions on \((S_t)_{t \geq 0}\) the Cauchy problem (SDE) has a unique cylindrical weak solution given by the mild form

\[
Y(t) = S_t Y_0 + \int_0^t S_{t-s} C \, dB^H(s)
\]

for all \(t \in [0, T]\).

Theorem
Let \(Y_0\) be a classical \(V\)-valued r.v. and let the assumptions on \(S\) hold. Then the solution \(Y(t)\) is induced by a classical process in \(V\) if and only if the (covariance) operator \(\Gamma^*_{t, S(\cdot)} C\) is \(\gamma\)-radonifying.
Thank you for your attention!