PROBABILISTIC REPRESENTATION OF A GENERALIZED POROUS MEDIA TYPE EQUATION. THE DETERMINISTIC AND STOCHASTIC CASES

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It includes mainly joint work with

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References

Ph. Blanchard, M. Röckner, F. Russo:  
Probabilistic representation for solutions of an irregular porous media type equation.  

V. Barbu, M. Röckner, F. Russo:  
Probabilistic representation for solutions of an irregular porous media type equation: the degenerate case.  
N. Belaribi, F. Cuvelier, F. Russo:

N. Belaribi, F. Cuvelier, F. Russo:
**About Fokker-Planck equation with measurable coefficients:** application to the fast diffusion equation.
HAL-INRIA-00645483.
http://hal.inria.fr/hal-00645483/fr/
Other recent available preprints
http://uma.ensta-paristech.fr/~russo
Outline

1. Motivations from self-organized criticality.
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3. The deterministic PDE.
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1 Motivations from the self-organized criticality.

Self-organized criticality, sandpiles and anomalous diffusions.

Pattern idea: The evolution of a sand layer.

- Two phases: avalanche and regular sand perturbation.
- Avalanch:
1. Points of the layer above some threshold $e_c$ provocate an avalanche.
2. Points of the layer below the threshold $e_c$ remain still.

Regular sand perturbation.

The system is perturbed by a noise modeling a regular perturbation.
Basic literature

6 Ph. Blanchard ...
6 Recent work: Barbu-Blanchard-Da Prato-Röckner.
Bantay-Janosi: inspired by the discrete BTW (Bak-Tang-Wiesenfeld) model, propose the following evolution.

\[
\begin{align*}
\dot{u} &= \frac{\gamma}{2} \Delta (H(u - e_c)u) + \text{"noise"} \\
\phi(0, x) &= u_0(x)
\end{align*}
\]

(1)

\(u(t, \cdot)\): level of the layer at time \(t\).

The “noise” (often Gaussian in the literature) should model the regular perturbation effect.
The two phases do not appear with the same time scale since “avalanches” occur much faster.

Therefore, it has a sense to study separately the two phases.
2 The model and connection with SPDEs

In the literature, the model appears to be additive. However, we aim at considering *multiplicative noise*.

### 2.1 Objects of investigation

\[
\begin{aligned}
\partial_t u &= \frac{1}{2} \Delta(\beta(u)) + \xi(t, x)u(t, x) \quad \text{in} \quad L^1(\mathbb{R}^d) \\
u(0, x) &= u_0(x)
\end{aligned}
\] (2)
6 β increasing possibly discontinuous.
6 There is λ > 0 such that \( \lim_{u \to \mp\infty} \beta(u) + \lambda u = \mp\infty \).
6 \( \beta(0) = 0 \) and \( \beta \) continuous at zero.

Consequence:
1. There is \( \Phi : \mathbb{R}^* \to \mathbb{R}_+ \) such that
\[
\beta(u) = \Phi^2(u)u, \; u \neq 0.
\]
2. Most of the time we will also assume.

**Hypothesis (G).** There is \( c > 0 \) such that \( |\beta(u)| \leq c|u| \).
In this case \( \Phi \) is bounded.
Remark 1  If $\beta : \mathbb{R} \to \mathbb{R}$ is monotone (possibly discontinuous), it is possible to complete $\beta$ into a monotone graph.

$\Phi : \mathbb{R}_* \to \mathbb{R}$ can also be associated with a graph. We prolongate it to $\mathbb{R}$, setting

$$\Phi(0) := \left[\liminf_{u \to 0^+} \Phi(u), \limsup_{u \to 0^+} \Phi(u)\right].$$
For instance, if $\Phi(u) = H(u - e_c)$, then $\beta(u) = H(u - e_c)u$

the corresponding graph is

$$
\beta(u) = \begin{cases} 
0 & : u < e_c \\
[0, e_c] & : u = e_c \\
u & : u > e_c
\end{cases}
$$

Moreover $\Phi(u) = H(u - e_c)$

$\xi(t, x)$ some noise, “”white” in time, regular in space.
2.2 First step

\[ \xi = 0. \]

Only the “avalanch phase” is considered.
3 The deterministic PDE

3.1 The evolution

We suppose for a moment that $\beta$ is single-valued (i.e. continuous).

\[
\begin{align*}
\partial_t u &= \frac{1}{2} \Delta (\beta(u)) \quad \text{in} \quad L^1(\mathbb{R}^d) \\
 u(0, x) &= u_0(x)
\end{align*}
\]

We are interested in the following distributional sense.
\[ \int_{\mathbb{R}^d} \varphi(y) u(t, y) \, dy = \int_{\mathbb{R}^d} \varphi(y) u_0(y) \, dy \]

\[ + \frac{1}{2} \int_0^t ds \int_{\mathbb{R}^d} \Delta \varphi(y) \beta(u(s, y)) \, dy. \]

**Proposition 2** Let \( u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \). Then there is a unique solution in the sense of distributions \( u \in C^0([0, T]; L^1(\mathbb{R}^d)) \cap L^\infty([0, T] \times \mathbb{R}^d) \) of (3)
3.2 The evolution in the discontinuous case

Proposition 3  Let $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then there is a unique solution in the sense of distributions $u \in (L^1 \cap L^\infty)([0, T] \times \mathbb{R}^d)$ of

\[
\begin{cases}
\partial_t u \in \frac{1}{2} \Delta(\beta(u)), \\
u(0, x) = u_0(x),
\end{cases}
\]

(4)

that is, there exists a unique couple $(u, \eta_u) \in ((L^1 \cap L^\infty)([0, T] \times \mathbb{R}^d))^2$ such that
\[ \int u(t, x) \varphi(x) dx = \int u_0(x) \varphi(x) dx \]
\[ + \frac{1}{2} \int_0^t ds \int \eta_u(s, x) \Delta \varphi(x) dx, \]
\[ \forall \varphi \in \mathcal{S}(\mathbb{R}^d) \quad \text{and} \]
\[ \eta_u(t, x) \in \beta(u(t, x)) \quad \text{for} \]
\[ dt \otimes dx \quad \text{a.e. } (t, x) \in [0, T] \times \mathbb{R}^d. \]

Furthermore, \( \|u(t, \cdot)\|_\infty \leq \|u_0\|_\infty \) for every \( t \in [0, T] \) and there is a unique version of \( u \) such that
\[ u \in C([0, T]; L^1(\mathbb{R}^d)) \subset L^1([0, T] \times \mathbb{R}^d). \]
Remark 4 1) If $\beta$ is continuous then we can take
\[ \eta_u(s, x) = \beta(u(s, x)). \]

2) If $\Phi \geq c_0 > 0$, we will say that $\beta$ is non-degenerate.
If $\lim_{u \to 0^+} \Phi(u) = 0$, we will say that $\beta$ is degenerate.

This result applies in the Heaviside case where
$\Phi(u) = H(u - e_c)$ and in the non-degenerate case
$\Phi(u) = H(u - e_c) + \varepsilon$. 
3.3 Proof of Proposition 3.

There are three steps.

1. **Uniqueness.** Benilan-Crandall, 1981. They state the result for $\beta$ continuous, but a remark extends the result to the general case.

2. **Existence.** Use of the theory of non-linear semigroups (Barbu, Brezis ...).
   We set $E = L^1(\mathbb{R}^d)$ (Banach space).
   We consider the multivalued map $A : \mathcal{D}(A) \subset E \to \mathcal{P}(E)$, defined by $Ag = -\frac{1}{2} \Delta(\beta(g))$. It is $m$-accretive.
In particular \((I + \lambda A)^{-1} : E \to E, \lambda > 0\), is a contraction.

(4) becomes

\[
\begin{cases}
0 & \in u'(t) + Au(t) \\
u(0) & = u_0
\end{cases}
\] (6)
3.4 Difficulties

6 \( E = L^1(\mathbb{R}^d) \) is not a Hilbert space, not reflexive, not even uniformly convex. So difficult, to show the existence of a strong solution so of an absolutely continuous \( u : [0, T] \rightarrow E \) with

\[
u(t) \in u_0 + \int_0^t Au(s)ds.
\]

6 Existence of a \( C^0 \)-solution, i.e. a uniform limit of \( \varepsilon \)-solutions. See for instance Showalter (1997).
An $\varepsilon$-solution is a discretization

$$\mathcal{D} = \{0 = t_0 < t_1 < \ldots < t_N = T\}$$

and an $E$-valued step function

$$v(t) = \begin{cases} u_0 : & t = t_0 \\ v_j : & t \in ]t_{j-1}, t_j] \end{cases}$$

for which $t_j - t_{j-1} \leq \varepsilon$ for $1 \leq j \leq N$, and

$$0 \in \frac{v_j - v_{j-1}}{t_j - t_{j-1}} + Av_j, 1 \leq j \leq N.$$
A $C^0$-solution is a solution in the sense of distributions.
4 An associated Fokker-Planck equation

- $\mathcal{M}(\mathbb{R}^d)$: the space of all signed measures on $\mathbb{R}^d$ with finite total variation.
- $\mathcal{M}_+(\mathbb{R}^d)$: the subset of all non-negative finite measures on $\mathbb{R}^d$.
- Let $a : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ be a Borel non-negative function.
Theorem 5  (N. Belaribi-FR)  
(generalizing Ph. Blanchard-M.Röckner-FR)  
Let $z_i : [0, T] \to \mathcal{M}(\mathbb{R}^d), i = 1, 2$, be continuous with respect to the weak topology on finite measures on $\mathcal{M}(\mathbb{R}^d)$. Let $z^0$ be an element of $\mathcal{M}(\mathbb{R}^d)$. Suppose that both $z_1$ and $z_2$ solve the problem $\partial_t z = \Delta (az)$ in the sense of distributions with initial condition $z(0, \cdot) = z^0$, i.e.
\[
\begin{aligned}
\left\{ \begin{array}{c}
\partial_t z &= \partial^2_{xx}(a z) \\
z(0, \cdot) &= z^0,
\end{array} \right.
\end{aligned}
\]

(7)

Then \( z := (z_1 - z_2)(t, \cdot) \) is identically zero for every \( t \), under the following requirement.
Hypothesis 6 (B)

There is \( \tilde{z} \in L^1_{loc}([0, T] \times \mathbb{R}^d) \), such that \( z(t, \cdot) \) admits \( \tilde{z}(t, \cdot) \) as density for almost all \( t \in [0, T] \). \( \tilde{z} \) will still be denoted again by \( z \).

Moreover, either (B1) or (B2) below is fulfilled.

(B1)

1. \( \int_{[0,T] \times \mathbb{R}^d} |z(t, x)|^2 \, dt \, dx < +\infty \),
2. \( \int_{[0,T] \times \mathbb{R}^d} |\partial_t z|^2(t, x) \, dt \, dx < +\infty \).
(B2) We suppose $d = 1$. For every $t_0 > 0$, we have

1. $\int_{[t_0,T] \times \mathbb{R}} |z(t, x)|^2 \, dt \, dx < +\infty,$

2. $\int_{[0,T] \times \mathbb{R}} |az|(t, x) \, dt \, dx < +\infty,$

3. $\int_{[t_0,T] \times \mathbb{R}} |az|^2(t, x) \, dt \, dx < +\infty.$
Remark 7  Let $d = 1$.

1) If $a$ is non-degenerate (i.e. $a \geq \text{const} > 0$) the third assumption of Hypothesis(B2) implies the first one.

2) If $a$ is bounded, then the second item of Hypothesis(B2) is always verified because

$$
\sup_{t\in[0,T]} \| z(t, \cdot) \|_{\text{var}} < +\infty.
$$

3) If $z(t, x) \in L^\infty([t_0, T] \times \mathbb{R})$, then the first item of Hypothesis(B2) is always verified.

4) So, Theorem 5, is a generalization of a result included in Blanchard-Röckner-R.
Remark 8

(7) means

\[
\int_{\mathbb{R}^d} \phi(x) z(t, dx) = \int_{\mathbb{R}^d} \phi(x) z^0(dx) \\
+ \int_0^t ds \int_{\mathbb{R}^d} \Delta \phi(x) a(s, x) z(s, dx)
\]

for every \( t \in [0, T] \) and any \( \phi \in C_0^\infty(\mathbb{R}^d) \).
5 Probabilistic representation

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\) a filtered probability space, 
\(W_t = (W^1_t, \ldots, W^d_t)\) be a classical \(d\)-dimensional Wiener process.

5.1 Basic observation.

Let \(u_0\) belong to \(L^1(\mathbb{R}^d)\). Suppose for a moment \(\Phi : \mathbb{R} \to \mathbb{R}\) to be single-valued (i.e. continuous) on \(\mathbb{R}^*\). 
We denote by \(\Phi_d : \mathbb{R} \to \mathbb{R}^{d \times d}\) the matrix-valued function

\[
\Phi_d(u) = \text{diag}(\Phi(u), \ldots, \Phi(u)).
\]
Theorem 9  Let us assume the (weak) existence of a solution $Y$ for

\begin{align*}
\begin{cases}
    Y_t & = Y_0 + \int_0^t \Phi(u(s, Y_s))dW_s \\
    \text{Law density}(Y_t) & = u(t, \cdot), \\
    u(0, \cdot) & = u_0
\end{cases}
\end{align*}

Then $u : [0, T] \times \mathbb{R} \to \mathbb{R}_+$ provides a solution in the sense of distributions of (1).
Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^d)$. We apply Itô formula to $Y$, to obtain

\[
\varphi(Y_t) = \varphi(Y_0) + \int_0^t \nabla \varphi(Y_s)^T \Phi(u(s, Y_s)) \, dW_s \\
+ \frac{1}{2} \int_0^t \Delta \varphi(Y_s) \Phi^2(u(s, Y_s)) \, ds
\]

Taking the expectation we obtain

\[
\int_{\mathbb{R}^d} \varphi(y) u(t, y) \, dy = \int_{\mathbb{R}} \varphi(y) u_0(y) \, dy \\
+ \frac{1}{2} \int_0^t ds \int_{\mathbb{R}^d} \Delta \varphi(y) \Phi^2(u(s, y)) u(s, y) \, dy.
\]
Remark 10  Immediate consequences of the probabilistic representation:

1. The associated solution of (1) is positive at any time if the initial value is positive.

2. \[ \int_{\mathbb{R}^d} u(t, x) \, dx = \int_{\mathbb{R}^d} u_0(x) \, dx, \quad t \geq 0. \]

3. There is a multi-valued formulation of previous result.
5.2 Basic literature

An incomplete list of references:

McKean (1966-67). If the coefficients are smooth.

Later: Sznitman 1983-89, mainly for non-linearity in the first order (as Burgers).

Benachour-Chassaing-Roynette-Vallois, 1996. 
Case $\beta(u) = u^q, q > 1$.


Evolution in the space of probability distribution functions (Jourdain 2000).
6 The non-degenerate case

We assume here the following non degeneracy condition.

\[ \Phi \geq \text{const} > 0. \]

With respect to the initial motivation we aim at

\[ \Phi(u) = H(u - e_c) + \varepsilon. \]

Under the linear growth Assumption (G)
6.1 The representation result in dimension 1.

We start with the single-valued case.

**Theorem 11** We fix \(d = 1\). Let \(u_0 \in L^1 \cap L^\infty(\mathbb{R}^d)\) such that \(u_0 \geq 0\) and \(\int_{\mathbb{R}^d} u_0(x) dx = 1\). Suppose \(\Phi\) Borel non-degenerate bounded and continuous. There is a unique solution \(Y\) to problem

\[
\begin{cases}
Y_t & = Y_0 + \int_0^t \Phi(u(s, Y_s)) dW_s \\
\text{Law density of } Y_t & = u(t, \cdot), \\
u(0, \cdot) & = u_0
\end{cases}
\]

with \(u \in C^0([0, T]; L^1(\mathbb{R}^d)) \cap L^\infty([0, T] \times \mathbb{R}^d)\).
In the non-continuous case we have the following.

**Theorem 12**  Let \( u_0 \in L^1 \cap L^\infty \) such that \( u_0 \geq 0 \) and \( \int_{\mathbb{R}^d} u_0(x) \, dx = 1 \). Suppose the multi-valued map \( \Phi \) is non-degenerate and bounded. Then there is a process \( Y \), unique in law, such that there exists \( \chi \in (L^1 \cap L^\infty)([0, T] \times \mathbb{R}^d) \) with
\[ Y_t = Y_0 + \int_0^t \chi(s, Y_s) dW_s \]  
(weakly)  
\[ \chi(t, x) \in \Phi(u(t, x)), \]  
for \(dt \otimes dx\)-a.e.  
Law density of \(Y_t\)  
\[ u(0, \cdot) = u_0, \]  
with \(u \in C([0, T]; L^1(\mathbb{R}^d)) \cap L^\infty([0, T] \times \mathbb{R}^d)\).
6.2 The ingredients of the representation proof in the non-degenerate case

For simplicity we consider the single-valued case.

“Uniqueness” of equation (7)

\[
\begin{cases}
\partial_t z &= \partial^2_{xx}(az) \\
z(0, x) &= u_0(x),
\end{cases}
\]
Let $u$ be a solution to (1). Since $\beta$ is non-degenerate, Stroock-Varadhan criterion applies. We can construct a (weak) solution $Y$ to the SDE

$$Y_t = Y_0 + \int_0^t \Phi(u(s, Y_s))dW_s$$

where $Y_0$ is distributed as $u_0$.

Here dimension $d = 1$ is not crucial.
Let \( z(t, \cdot) \) be the law of \( Y_t, \ t \in [0, T] \).

Itô formula: \( z \) solves the linear equation (7) with

\[
a(t, x) = \frac{\Phi^2(u(t, x))}{2}.
\]

\( u \) solves the same linear equation.
In order to prove existence for (9) it is enough to prove $u = z$.

Theorem 5 allows to conclude if $u, z$ fulfill Assumption (B2).

Since $u \in (L^1 \cap L^\infty)([0, T] \times \mathbb{R})$, it is enough to show that $z \in L^2([0, T] \times \mathbb{R})$ or even $z \in L^2([t_0, T] \times \mathbb{R})$ for any $0 < t_0 < T$. 
a being strictly positive, \( z(t, \cdot) \) admits a density \( p(t, \cdot) \) for any \( t \in [0, T] \).

Since

\[
Y_t = y_0 + \int_0^t \sigma(s, Y_s) dW_s,
\]

\( \sigma \geq c_0 > 0, \sigma(s, y) = \Phi(u(s, y)) \). \textbf{Krylov estimates} imply

\[
\left| \int_0^T E(f(t, Y_t)) dt \right| \leq \text{const} \| f \|_{L^p([0, T] \times \mathbb{R})}, \quad p \geq d + 1.
\]
Since $d = 1$ we can choose $p = 2$.

So, by density arguments

$$\left| \int_{[0,T] \times \mathbb{R}} f(t, y)p(t, y)\,dt\,dy \right| \leq \text{const} \|f\|_{L^2([0,T] \times \mathbb{R})},$$

for any $f \in L^2([0, T] \times \mathbb{R})$.

Finally, by Riesz theorem $p \in L^2([0, T] \times \mathbb{R})$. Assumption (B) is verified.
6.3 Problems with dimension $d > 1$.

Essentially we only obtain that $p \in L^{d+1 \over d} ([0, T] \times \mathbb{R}^d)$.

In general $1 < {d+1 \over d} < 2$.

Our Fokker-Planck type equation does not apply.
6.4 Uniqueness

Let $Y^1$, $Y^2$ be two processes solving (9). Let $u^1$, $u^2$ such that $u^i$ is the law of $Y^i_t$, $t \in [0, T]$, $i = 1, 2$. Then $u^1$ and $u^2$ solve the PDE (5). Proposition 3 imply that $u^1 = u^2$.

Then $Y^1 = Y^2$ follows by Stroock-Varadhan arguments since they solve an SDE with measurable non-degenerate coefficients.

Here again, dimension $d \geq 2$ constitutes a problem since $a$ is, a priori, only measurable.
7 Considerations about the degenerate case.

7.1 About the validity of Assumption (B)

If Assumption (B) is not verified, then equation (7) may not be well-posed, even if \( a \) is time-homogeneous. Let

\[
 a : \mathbb{R} \to \mathbb{R}_+ \text{ such that }
\]
1. continuous,
2. \(a(0) = 0\),
3. \(a(x) \neq 0\) if \(x \neq 0\),
4. \(\frac{1}{a}\) is locally integrable.
Then

\[
\begin{cases}
\partial_t z = \partial^2_{xx} (az) \\
z(0, \cdot) = \delta_0,
\end{cases}
\]

has at least two solutions \( z : [0, T] \rightarrow \mathcal{M}^+(\mathbb{R}) \), i.e.
1. $z(t, \cdot) \equiv \delta_0$.

2. On a suitable enlarged filtered probability space, using Engelbert-Schmidt type arguments, it is possible to construct a process $Z$ such that
\[ Z_t = \int_0^t \sqrt{2a(Z_s)} dW_s, \]

Setting \( z : [0, T] \times \mathbb{R} \to \mathbb{R} \) such that \( z(t, \cdot) \) is the law of \( Z_t \), again it solves (11).
7.2 Existence in the degenerate case

(joint work with M. Röckner and V. Barbu.)

Definition 13  We say that $\beta$ (degenerate) is strictly increasing after some zero, if there is $e_c \geq 0$, such that

1. $\beta$ restricted at $[0, e_c]$ is zero.
2. $\beta$ is strictly increasing on $[e_c, \infty]$ (with possible jumps).
We will be able now to treat the probabilistic representation in the case.

\[ \Phi(u) = H(u - e_c), \]

since the corresponding \( \beta \) is strictly increasing after some zero.

We consider here two classes of supplementary assumptions when \( \beta \) is degenerate.

\( B1 \) \( u^0 \) is locally of bounded variation.

\( B2 \) \( \beta \) is strictly increasing after some zero.
Proposition 14  Suppose $\beta$ degenerate and that one of the two previous assumptions are verified. There is a weak (in the sense of probability laws) $Y$ such that there exists $\chi \in (L^1 \cap L^\infty)([0, T] \times \mathbb{R})$ with
\[
\begin{aligned}
Y_t &= Y_0 + \int_0^t \chi(s, Y_s) dW_s \\
\chi(t, x) &\in \Phi(u(t, x)), \\
\text{for } dt \otimes dx-\text{a.e.}
\end{aligned}
\]

Law density of \( Y_t \) = \( u(t, \cdot) \)

\( u(0, \cdot) = u_0, \)

with \( u \in C([0, T]; L^1(\mathbb{R})) \cap L^\infty([0, T] \times \mathbb{R}). \)
7.3 The fast diffusion equation

(N. Belaribi, F.R)

- Generalized porous media with $\beta(u) = u^m$, $0 < m < 1$. When $m \geq 1$, it is called classical porous media equation.

- Barenblatt solutions correspond to the solution in the case $u_0 = \delta_0$.

$$U(t, x) = t^{-\alpha} \left( D + \tilde{k} |x|^{2 \alpha} \right)^{-\frac{1}{1-m}},$$  (13)
where,

\[ \alpha = \frac{1}{m + 1}, \quad \tilde{k} = \frac{1 - m}{2(m + 1)m}, \quad D = \left( \frac{I}{\sqrt{\tilde{k}}} \right)^{\frac{2(1-m)}{m+1}}, \]

\[ I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [\cos(x)]^{\frac{2m}{1-m}} dx. \]

The Barenblatt solution admits a probabilistic representation.
8 Some numerical simulations

Joint work with

Nadia Belaribi and François Cuvelier.
Interacting particle system

\[
Y_{t,\varepsilon,n}^i = Y_{0}^i + \int_{0}^{t} \Phi \left( \frac{1}{n} \sum_{j=1}^{n} K_{\varepsilon}(Y_{s,\varepsilon,n}^i - Y_{s,\varepsilon,n}^j) \right) dW_{s}^i,
\]

\(i = 1, \ldots, n\)

6 \(W = (W^1, \ldots, W^n)\): n-dimensional Brownian motion,

6 \((Y_{0}^i)_{1 \leq i \leq n}\): a sequence of independent random variables with density law \(u_0\) and independent of the Brownian motion \(W\).
Here: \( \Phi(u) = H(u - e_c) \).

\[
K_\varepsilon(x) = \frac{1}{\varepsilon} K\left(\frac{x}{\varepsilon}\right)
\]

\( K \) : Gaussian \( N(0, 1) \) density

The technique supposes that \textbf{Chaos propagation} occurs for this equation. \textit{Difficult question}.

**Problem**: For fixed number of interacting particles \( n \), find a good value for \( \varepsilon \).
Classical techniques in non-parametric density estimation:

Choose $\varepsilon = \varepsilon(n)$ so that the quadratic error

$$E(p(x) - \hat{p}(x))^2$$

$$\hat{p}(x) = \frac{1}{n} \sum_{j=1}^{n} K_{\varepsilon}(x - X_j).$$
Figure 1: $e-c=0.2; t=0; n=15000; \Delta t=0.0001$. 
Figure 2: $e=c=0.2; t=0.5; n=15000; \Delta t=0.0001$. 

PROBABILISTIC REPRESENTATION OF A GENERALIZED NONLINEAR MEGA-TYPE EQUATION: THE DETERMINISTIC AND STOCHASTIC CASES.
Figure 3: e-c=0.2; t=1; n=15000; Delta t=0.0001
Figure 4: $e = c = 0.15; t = 0; n = 15000; \Delta t = 0.0001$
Figure 5: $e,c=0.15; t=0.5; n=15000; \Delta t=0.0001$
Figure 6: $e-c=0.15; t=1; n=15000; \Delta t=0.0001$. 

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Figure 7: $c-e=0.12; t=0; n=15000; \Delta t=0.0005$. 

Prob. Sol.  
Determ. Sol.

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Figure 8: e,c=0.12; t=2.5; n=15000; \Delta t=0.0005
Figure 9: e-c=0.12; t=5; n=15000; Delta t=0.0005
Figure 11: $c-c=0.08; t=3; n=15000; \Delta t=0.0006$
Figure 12: e-c=0.08; t=6; n=15000; Delta t=0.0006
Figure 13: \( e_c = 0.3; t = 0; n = 15000; \Delta t = 0.0001 \)
Figure 14: $e \cdot c=0.3; t=0.5; n=15000; \Delta t=0.0001$
Figure 15: \( e^{-c} = 0.3; \, t = 1; \, n = 15000; \, \Delta t = 0.0001 \)
Figure 16: Numerical (dashed line) and exact solutions (solid line) values at t=0 (a), t=0.5 (b), t=1 (c) and t=1.5 (d).
Numerical (dashed line) and exact solutions (solid line).
Values at $t=0$ (a), $t=0.5$ (b), $t=1$ (c) and $t=1.5$ (d).
The evolution of the $L^2$ error over the time interval $[0, 1.5]$ (e).
9 The SPDE case (multiplicative noise).

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A stochastic Fokker-Planck equation and double probabilistic representation for the stochastic porous media type equation.

Features

6 Prolongation of Theorem 9.
6 Probabilistic (double) representation in the stochastic case.
6 Stochastic Fokker-Planck equation.
Let $B^1, \ldots, B^n$ be $n$ independent Brownian motions, $e^1, \ldots, e^n$ be functions in $H^1(\mathbb{R})$ being $H^{-1}$-multiplier, i.e. the maps $\varphi \to \varphi e^i$ are continuous in $H^{-1}$. We define the random field

$$
\mu(t, dx) = \sum_{i=1}^{n} e^i(x) dB^i_t.
$$
Let $u_0 \in L^1(\mathbb{R})$. We obtain a (double) probabilistic representation of

\[
\begin{aligned}
\left\{
\begin{array}{l}
\partial_t u = \frac{1}{2} \Delta (\beta(u)) + \partial_t \mu(t, x) u(t, x) \quad \text{in} \quad L^1(\mathbb{R}) \\
u(0, x) = u_0(x)
\end{array}
\right.
\end{aligned}
\]  

(14)
Basic idea

Let $\xi : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ bounded Borel, which represents a *quenched* smoothed realization of $\mu$.

Let $u_0 \in L^1(\mathbb{R})$.

\[
\begin{aligned}
\partial_t u &= \frac{1}{2} \Delta(\beta(u)) + \xi(t, x)u(t, x) \quad \text{in} \quad L^1(\mathbb{R}) \\
u(0, x) &= u_0(x)
\end{aligned}
\]  

(15)
Let \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\) a filtered probability space, \((W_t)\) a classical \((\mathcal{F}_t)\)-Wiener process.

Let \((Y_t)\) be a stochastic process such that \(u_0\): law density of \(Y_0\).

We consider the Radon measure (\(\mu\)-law of \(Y_t\)):

\[
\mu_Y(t) : \varphi \rightarrow E \left( \varphi(Y_t) \exp\left( \int_0^t \xi(r, Y_r) dr \right) \right).
\]
Proposition 15  Let $Y$ be a solution to

\[
\begin{align*}
Y_t &= Y_0 + \int_0^t \Phi(u(s, Y_s)))dW_s \\
\text{Law density}(\mu_Y(t)) &= u(t, \cdot).
\end{align*}
\]  

Then $u$ is a solution to (15) in the sense of distributions.
Proof. Itô formula and integration by parts.

Natural generalization

\[ \xi(t, x) \text{ versus } \sum_{j=1}^{N} \xi^j(t, x) \dot{B}_t^j. \]

\[ \dot{B}_t^j : \text{independent Brownian motions.} \]