QUASI-CONSTANT HOLOMORPHIC SECTIONAL CURVATURES OF TANGENT BUNDLES WITH GENERAL NATURAL KÄHLER STRUCTURES

BY

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Dedicated to Professor V. Oproiu on his 70th birthday

Abstract. We study the general natural Kählerian tangent bundles \((TM, G, J)\) of an \(n\)-dimensional Riemannian manifold \((M, g)\), which have the property that the curvature of any holomorphic plane depends only on the point and on its angle with the Liouville vector field. We prove that the tangent bundles with this property, called tangent bundles of quasi-constant holomorphic sectional curvature, are only those of constant holomorphic sectional curvature.

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1. Introduction

Some new geometric structures on the tangent bundle \(TM\) of a Riemannian manifold \((M, g)\), obtained by considering the natural lifts of \(g\) to \(TM\), are studied in several recent works, like [1], [2], [6], [14]-[21], [23], [24].

Briefly speaking, a natural operator (in the sense of [10], [11], [12], [25]) is a fibred manifold mapping, which is invariant with respect to the group of local diffeomorphisms of the base manifold.

In 1958, Sasaki constructed in [22], his well known Riemannian metric on the tangent bundle \(TM\). The study of some geometric properties of the tangent bundle endowed with the Sasaki metric, leads to the flatness of the base manifold.
The results from [10] and [11] concerning the natural lifts, and the classification of the natural vector fields on the tangent bundle of a pseudo-Riemannian manifold, made by Janyska in [9], allowed Oproiu to introduce on the tangent bundle $TM$, a natural almost complex structure $J$ of diagonal type, as well as a natural metric $G$ of diagonal type, obtained from the Riemannian metric $g$ on the base manifold (see [20]). The new metric, slightly surmounted the rigidity of the Sasaki metric, in the integrability problem for the almost complex structure $J$. In particular, if the base manifold is a space of constant sectional negative curvature, then the tangent bundle endowed with the new metric is a space of constant sectional positive curvature (contrasting to the situations of the Sasaki and Cheeger-Gromoll metrics, studied in articles such as [14], [15]).

In the papers [18], [19], Oproiu obtained some new properties of the diagonal natural 1-st order almost Hermitian lifts of $g$ to $TM$, and in [17], he generalized the expressions of these lifts. In the definition of the natural almost complex structure $J$ of general type there are involved eight parameters (smooth functions of the density energy on $TM$). However, from the condition for $J$ to define an almost complex structure, four of the above parameters may be expressed as (rational) functions of the other four parameters. A Riemannian metric $G$ which is a natural lift of general type of the metric $g$ depends on other six parameters.

In [6], Oproiu and the present author have found the conditions under which the Kählerian manifold $(TM, G, J)$ has constant holomorphic sectional curvature.

In the present paper, we study the Kählerian tangent bundles $(TM, G, J)$, with the property that any holomorphic plane making a certain angle with the Liouville vector field, have the same curvature, where the metric $G$ and the integrable almost complex structure $J$ are 1-st order general natural lifts of $g$ to $TM$. So, the author is interested in finding the conditions under which $(TM, G, J)$ is of quasi-constant holomorphic sectional curvature.

2. Preliminary results

Consider a smooth $n$-dimensional Riemannian manifold $(M, g)$ and denote its tangent bundle by $\tau : TM \to M$. Recall that $TM$ has a structure of a $2n$-dimensional smooth manifold, induced from the smooth manifold structure of $M$. This structure is obtained by using local charts on $TM$ induced from usual local charts on $M$. If $(U, \varphi) = (U, x^1, \ldots, x^n)$ is a
local chart on $M$, then the corresponding induced local chart on $TM$ is
$(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \ldots, x^n, y^1, \ldots, y^n)$, where the local coordinates
$x^i, y^j, i, j = 1, \ldots, n$, are defined as follows. The first $n$ local coordinates
of a tangent vector $y \in \tau^{-1}(U)$ are the local coordinates in the local chart
$(U, \varphi)$ of its base point, i.e. $x^i = x^i \circ \tau$, by an abuse of notation. The last
$n$ local coordinates $y^j, j = 1, \ldots, n$, of $y \in \tau^{-1}(U)$ are the vector space
coordinates of $y$ with respect to the natural basis in $T_{\tau(y)}M$ defined by the
local chart $(U, \varphi)$. Due to this special structure of differentiable manifold
for $TM$, it is possible to introduce the concept of $M$-tensor field on it (see
[13]).

Some useful $M$-tensor fields on $TM$ may be obtained as follows. Let $u : [0, \infty) \to \mathbb{R}$ be a smooth function and let $\|y\|^2 = g_{\tau(y)}(y, y)$ be the square of the norm of the tangent vector $y \in \tau^{-1}(U)$. If $\delta^i_j$ are the Kronecker symbols (in fact, they are the local coordinate components of the identity tensor field $I$ on $M$), then the components $u(\|y\|^2)\delta^i_j$ define an $M$-tensor field of type $(1, 1)$ on $TM$. Similarly, if $g_{ij}(x)$ are the local coordinate components of the metric tensor field $g$ on $M$ in the local chart $(U, \varphi)$, then the components $u(\|y\|^2)g_{ij}$ define a symmetric $M$-tensor field of type $(0, 2)$ on $TM$. The components $g_{0i} = y^kg_{ki}$ define an $M$-tensor field of type $(0, 1)$ on $TM$.

Denote by $\nabla$ the Levi Civita connection of the Riemannian metric $g$ on
$M$. Then we have the direct sum decomposition

$$TTM = VTM \oplus HTM$$

of the tangent bundle to $TM$ into the vertical distribution $VTM = \text{Ker } \tau_*$ and the horizontal distribution $HTM$ defined by $\nabla$ (see [26]). The set of vector fields $(\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n})$ on $\tau^{-1}(U)$ defines a local frame field for $VTM$ and for $HTM$ we have the local frame field $(\frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n})$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{0i} \frac{\partial}{\partial y^h}, \quad \Gamma^h_{0i} = y^k \Gamma^h_{ki},$$

and $\Gamma^h_{ki}(x)$ are the Christoffel symbols of $g$.

The set $(\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}, \frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n})$ defines a local frame on $TM$, adapted to the direct sum decomposition (2.1).

Consider the energy density of the tangent vector $y$ with respect to the
Riemannian metric $g$

$$t = \frac{1}{2} \|y\|^2 = \frac{1}{2} g_{\tau(y)}(y, y) = \frac{1}{2} g_{ik}(x)y^iy^k, \quad y \in \tau^{-1}(U).$$
Obviously, we have $t \in [0, \infty)$ for all $y \in TM$.

Denote by $C = y^i \frac{\partial}{\partial y^i}$ the Liouville vector field on $TM$ and by $\tilde{C} = y^i \frac{\delta}{\delta x^i}$ the similar horizontal vector field on $TM$, named the geodesic spray. Consider the real valued smooth functions $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ defined on $[0, \infty) \subset \mathbb{R}$.

OPROIU introduced in [17] a natural 1-st order almost complex structure on $TM$, just like the natural 1-st order lifts of $g$ to $TM$ are obtained in [10], [11]:

\begin{align}
J_{\frac{\delta}{\delta x^i}} &= (J_1)_i^j \frac{\partial}{\partial y^j} + (J_4)_i^j \frac{\delta}{\delta x^j}, \\
J_{\frac{\partial}{\partial y^j}} &= (J_3)_i^j \frac{\partial}{\partial y^i} - (J_2)_i^j \frac{\delta}{\delta x^i},
\end{align}

where the $M$-tensor fields $(J_\alpha)_i^j$ are defined as

\begin{align}
(J_\alpha)_i^j &= a_\alpha(t) \delta_i^j + b_\alpha(t) g_{b_i b_j}, \quad \forall \alpha = 1, 4.
\end{align}

We get easily the invariant expression of $J$:

\begin{align}
J X^H_y &= a_1(t) X^V_y + b_1(t) g_{r(y)}(X, y) y^V_y + a_4(t) X^H_y + b_4(t) g_{r(y)}(X, y) y^H_y, \\
J X^V_y &= a_3(t) X^V_y + b_3(t) g_{r(y)}(X, y) y^V_y - a_2(t) X^H_y - b_2(t) g_{r(y)}(X, y) y^H_y,
\end{align}

$\forall X \in T_0^1(TM), \forall y \in TM$.

In the mentioned paper, OPROIU proved the following results.

**Theorem 2.1.** The natural tensor field $J$ of type $(1, 1)$ on $TM$, given by (2.3) defines an almost complex structure on $TM$, if and only if $a_4 = -a_3$, $b_4 = -b_3$, and the coefficients $a_1, a_2, a_3, b_1, b_2$ and $b_3$ are related by

\begin{align}
a_1 a_2 &= 1 + a_2^2, \\
(a_1 + 2tb_1)(a_2 + 2tb_2) &= 1 + (a_3 + 2tb_3)^2.
\end{align}

**Theorem 2.2.** Let $(M, g)$ be an $n(>2)$-dimensional connected Riemannian manifold. The almost complex structure $J$ defined by (2.3) on $TM$ is integrable if and only if $(M, g)$ has constant sectional curvature $c$, and the coefficients $b_1, b_2, b_3$ are given by:

\begin{align}
b_1 &= \frac{2c^2 a_2^2 + 2c a_3 a_2 + a_1 a_4' - c + 3ca_3^2}{a_1 - 2ta_1' - 2cta_2 - 4ct^2a_2'}, \\
b_2 &= \frac{2ta_2' - 2ta_1 a_2' + ca_2^2 + 2cta_2 a_2' + a_1 a_3'}{a_1 - 2ta_1' - 2cta_2 - 4ct^2a_2'}, \\
b_3 &= \frac{a_1 a_3' + 2ca_2 a_3 + 4cta_2' a_3' - 2cta_2 a_3'}{a_1 - 2ta_1' - 2cta_2 - 4ct^2a_2'}.
\end{align}
Let $G$ be the general natural lifted metric on $TM$, defined by Oproiu in [17]:

$$
\begin{align*}
G \left( \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right) &= c_1 g_{ij} + d_1 g_0 g_{00} = G^{(1)}_{ij}, \\
G \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) &= c_2 g_{ij} + d_2 g_0 g_{00} = G^{(2)}_{ij}, \\
G \left( \frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j} \right) &= c_3 g_{ij} + d_3 g_0 g_{00} = G^{(3)}_{ij},
\end{align*}
$$

(2.7)

where $c_1$, $c_2$, $c_3$, $d_1$, $d_2$, $d_3$ are six smooth functions of the density energy on $TM$.

The invariant form of the metric $G$ is

$$
\begin{align*}
G(X^H_y, Y^H_y) &= c_1(t)g_{\tau(y)}(X, Y) + d_1(t)g_{\tau(y)}(X, y)g_{\tau(y)}(Y, y), \\
G(X^V_y, Y^V_y) &= c_2(t)g_{\tau(y)}(X, Y) + d_2(t)g_{\tau(y)}(X, y)g_{\tau(y)}(Y, y), \\
G(X^V_y, Y^H_y) &= c_3(t)g_{\tau(y)}(X, Y) + d_3(t)g_{\tau(y)}(X, y)g_{\tau(y)}(Y, y),
\end{align*}
$$

$\forall X, Y \in T_0^1(TM), \forall y \in TM$.

The study of the conditions under which the metric $G$ is almost Hermitian with respect to the almost complex structure $J$, led to the next result from [17]:

**Theorem 2.3.** The family of the Riemannian metrics of general natural lifted type, $G$ on $TM$, such as $(TM, G, J)$ is an almost Hermitian manifold, is given by (2.7), provided that $c_1$, $c_2$, $c_3$, $d_1$, $d_2$, and $d_3$ be related with $a_1$, $a_2$, $a_3$, $b_1$, $b_2$, and $b_3$ by:

$$
\frac{c_1}{a_1} = \frac{c_2}{a_2} = \frac{c_3}{a_3} = \lambda, \quad \frac{c_1 + 2td_1}{a_1 + 2tb_1} = \frac{c_2 + 2td_2}{a_2 + 2tb_2} = \frac{c_3 + 2td_3}{a_3 + 2tb_3} = \lambda + 2t\mu,
$$

where the proportionality coefficients $\lambda > 0$ and $\lambda + 2t\mu > 0$ are functions depending on the energy density $t$.

**Theorem 2.4.** The almost Hermitian manifold $(TM, G, J)$ is almost Kählerian if and only if

$$
\mu = \lambda'.
$$

Thus the family of the almost general natural Kählerian structures on $TM$ depends on five essential coefficients, $a_1$, $a_3$, $b_1$, $b_3$, $\lambda$. Finally, one obtains that a Kählerian structure of general natural lifted type $(G, J)$ on $TM$ is defined by three essential coefficients: $a_1$, $a_3$, $\lambda$, provided that: $a_1 > 0$, $a_1 + 2tb_1 > 0$, $\lambda > 0$, $\lambda + 2t\mu > 0$. 


3. The quasi-constant holomorphic sectional curvatures of the tangent bundles with general natural lifted metrics

The notion of Kählerian manifold of quasi-constant holomorphic sectional curvatures (see [3], [8]) is the Kählerian correspondent to the notion of Riemannian manifold of quasi-constant sectional curvatures (see [5], [7]).

A Kählerian manifold \((M, g, J, \xi)\), endowed with a unit vector field \(\xi\), is said to be of quasi-constant holomorphic sectional curvatures if for any holomorphic section \(\text{span}\{X, JX\}\), generated by the unit vector \(X \in T_p M, p \in M\), the sectional curvature \(R(X, JX, JX, X)\) depends only on the point \(p\) and on the angle \(\varphi\) between the holomorphic plane and the unit vector field \(\xi\), i.e.

\[ R(X, JX, JX, X) = f(p, \varphi), \ p \in M, \ \varphi \in [0, \pi/2]. \]

In the papers [3], [8] it was shown that a Kählerian manifold \((M, g, J, \xi)\) is of quasi-constant holomorphic sectional curvatures if and only if the curvature tensor field \(R\) of the Levi-Civita connection \(\nabla\), is given as

\[ R = k_0 R_0 + k_1 R_1 + k_2 R_2, \]

where \(k_0, k_1, k_2\) are smooth functions on \(M\) and \(R_0, R_1, R_2\) are certain tensor fields of curvature type on \(M\), given by

\[ R_0(X, Y)Z = \frac{1}{4}\{g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY + 2g(X, JY)JZ\}, \]

\[ R_1(X, Y)Z = P(X, Y, Z) - P(Y, X, Z) + P(JX, JY, Z) - P(JY, JX, Z), \]

\[ R_2(X, Y)Z = \{\eta(X)\eta(JY) - \eta(JX)\eta(Y)\}\{\eta(JZ)\xi + \eta(Z)J\xi\}, \]

where \(\eta\) is a 1-form defined by \(\eta(X) = g(X, \xi)\), and \(P\) is an auxiliary \((1,3)\)-tensor field, defined by

\[ P(X, Y, Z) = \frac{1}{8}\{\eta(Y)\eta(Z)X + \eta(X)\eta(JZ)JY + \eta(X)\eta(JY)JZ \]

\[ + g(Y, Z)\eta(X)\xi + g(X, JZ)\eta(Y)\xi \]

\[ + \frac{1}{2}g(X, JY)\eta(JZ)\xi + \frac{1}{2}g(X, JY)\eta(Z)J\xi\}. \]
In the case of the general natural Kählerian manifold \((TM, G, J)\), we may work with the Liouville vector field \(C = y^i \partial_i\), instead of the unitary vector field \(\xi\), since \(\|\xi\|^2 C\) is unitary, and the scalar factors may be incorporated into \(k_1, k_2\).

The Liouville vector field is non-null only on the bundle \(TM\setminus\{0\}\) of non-zero vector fields, so we may study the property of \((TM\setminus\{0\}, G, J, C)\) to have quasi-constant holomorphic sectional curvatures (i.e., we find the conditions under which the curvature tensor field of \((TM\setminus\{0\}, G, J, C)\) may be written in the form \((3.1)\)).

The curvature tensor field \(R\) of the Levi-Civita connection \(\nabla\) of the Riemannian manifold \((TM, G)\) is defined by the well-known formula

\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z, \quad X, Y, Z \in \chi(TM),
\]

where \(\nabla\) is obtained from the Koszul formula, and it is characterized by the conditions \(\nabla G = 0, T = 0\), where \(T\) is the torsion tensor of \(\nabla\).

The inverse \(H\) of the matrix \(G\) associated to the general natural lifted metric defined by \((2.7)\) has the entries

\[
(3.6) \quad H_{(1)}^{kl} = p_1 g_{kl} + q_1 y^k y^l, \quad H_{(2)}^{kl} = p_2 g_{kl} + q_2 y^k y^l, \quad H_{(3)}^{kl} = p_3 g_{kl} + q_3 y^k y^l,
\]

where \(g_{kl}\) are the components of the inverse of the matrix \((g_{ij})\) and \(p_1, q_1, p_2, q_2, p_3, q_3 : [0, \infty) \to \mathbb{R}\), some real smooth functions.

In [6] we got \(p_1, p_2, p_3\) as functions of \(c_1, c_2, c_3\)

\[
(3.7) \quad p_1 = \frac{c_2}{c_1 c_2 - c_3^2}, \quad p_2 = \frac{c_1}{c_1 c_2 - c_3^2}, \quad p_3 = -\frac{c_3}{c_1 c_2 - c_3^2},
\]

and \(q_1, q_2, q_3\) as functions of \(c_1, c_2, c_3, d_1, d_2, d_3, p_1, p_2, p_3\)

\[
q_1 = -\frac{c_2 d_1 p_1 - c_3 d_3 p_1 - c_3 d_2 p_3 + c_2 d_3 p_3 + 2 d_1 d_2 p_1 t - 2 d_3^2 p_1 t}{c_1 c_2 - c_3^2 + 2 c_2 d_1 t + 2 c_1 d_2 t - 4 c_3 d_3 t + 4 d_1 d_2 t^2 - 4 d_3^2 t^2},
\]

\[
(3.8) \quad q_2 = -\frac{d_2 p_2 + d_3 p_3}{c_2 + 2 d_2 t} + \frac{(c_3 + 2 d_3 t)[(d_3 p_1 + d_3 p_3)(c_1 + 2 d_1 t) - (d_1 p_1 + d_3 p_3)(c_3 + 2 d_3 t)]}{(c_2 + 2 d_2 t)[(c_1 + 2 d_1 t)(c_2 + 2 d_2 t) - (c_3 + 2 d_3 t)^2]},
\]

\[
q_3 = -\frac{(d_3 p_1 + d_3 p_3)(c_1 + 2 d_1 t) - (d_1 p_1 + d_3 p_3)(c_3 + 2 d_3 t)}{(c_1 + 2 d_1 t)(c_2 + 2 d_2 t) - (c_3 + 2 d_3 t)^2},
\]

and we proved the next theorem.
The Levi-Civita connection $\nabla$ of $G$ has the following expression in the local adapted frame $(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^m})$

$$\nabla_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^j} = Q^{h}_{ij} \frac{\partial}{\partial y^j} + \tilde{Q}^{h}_{ij} \frac{\partial}{\partial x^k}, \quad \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial y^j} = (\Gamma^{h}_{ij})_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial x^l} + P^{h}_{ij} \frac{\partial}{\partial x^l} + S^{h}_{ij} \frac{\partial}{\partial x^l},$$

where $\Gamma^{h}_{ij}$ are the Christoffel symbols of the Levi-Civita connection $\tilde{\nabla}$ of $g$, and the coefficients $Q^{h}_{ij}, \tilde{Q}^{h}_{ij}, P^{h}_{ij}, \tilde{P}^{h}_{ij}, S^{h}_{ij}, \tilde{S}^{h}_{ij}$ are some $M$-tensor fields on $TM$, whose explicit expressions may be obtained from the Koszul formula for $\nabla$.

Taking into account the expressions (2.7), (3.6) and by using the formulas (3.7), (3.8) we may obtain the detailed expressions of $P^{h}_{ij}, Q^{h}_{ij}, S^{h}_{ij}, \tilde{P}^{h}_{ij}, \tilde{Q}^{h}_{ij}, \tilde{S}^{h}_{ij}$ (see [6]).

By using the local adapted frame $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}_{i,j=1,\ldots,n}$, which may be denote by $\{\delta_i, \delta_j\}_{i,j=1,\ldots,n}$, we obtained in [6], after a standard straightforward computation the horizontal and vertical components of the curvature tensor field $R$. For example we have

$R(\delta_i, \delta_j)\delta_k = XXXX^{h}_{kij} \delta_h + XXXY^{h}_{kij} \delta_h,$

$R(\delta_i, \delta_j)\delta_k = YXXX^{h}_{kij} \delta_h + YXXY^{h}_{kij} \delta_h,$

where the $M$-tensor fields appearing as coefficients are denoted by sequences of $X$ and $Y$. We mention that we use the character $X$ on a certain position to indicate that the argument on that position is a horizontal vector field and, similarly, we use the character $Y$ for vertical vector fields.

From (3.2), (3.3), and (3.4), we obtain after standard straightforward computations, the horizontal and vertical components of the three curvature type tensors, $R_0$, $R_1$, $R_2$, which are denoted in the same manner like the components of the curvature tensor field $R$, and they are followed by the index 0, 1, or 2, corresponding to the curvature tensors type $R_0$, $R_1$, $R_2$, respectively.

The $M$-tensor fields appearing as coefficients of the horizontal and vertical components of $R_0$ are given by some expressions of the next type:

$$XXXX^{h}_{0kij} = \frac{1}{4}\left\{G^{(1)}_{jk} \delta_i - G^{(1)}_{ik} \delta_j - (J_3)^h_{ij} [ (J_1)^h_{ik} G^{(3)}_{lk} - (J_3)^h_{ik} G^{(1)}_{lk} ] + (J_3)^h_{ij} [ (J_1)^h_{ik} G^{(3)}_{ik} - (J_3)^h_{ik} G^{(1)}_{ik} ] - 2(J_3)^h_{ik} [ (J_1)^h_{ik} G^{(3)}_{ul} - (J_3)^h_{ik} G^{(1)}_{ul} ] \right\},$$
constant holomorphic sectional curvatures, the next lemma will be useful. 

\[ R(3.9) \]

and only if all the components of the following difference vanish 

\[ (J_3)^h_j \{ (J_1)^{3h}_{ik} - (J_3)^{1h}_{ik} \} \].

The components of the second curvature tensor type, \( R_1 \), are obtained from (3.3), by a straightforward computation. They depend on the components of the auxiliary (1,3)-tensor field, \( P \). Thus we have, for example:

\[ XXX \ominus^{n}_{kj} = PXXX \ominus^{h}_{kj} - PXXX \ominus^{h}_{kji} \]

\[ + ((J_3)^{h}_{i}(J_3)^{m}_{j} - (J_3)^{m}_{i}(J_3)^{h}_{j}) PXXX \ominus^{h}_{klm} \]

\[ + ((J_1)^{h}_{i}(J_3)^{m}_{j} - (J_1)^{m}_{i}(J_3)^{h}_{j}) PYXX \ominus^{h}_{klm} \]

\[ + ((J_3)^{h}_{i}(J_1)^{m}_{j} - (J_3)^{m}_{i}(J_1)^{h}_{j}) PXXY \ominus^{h}_{klm} \]

\[ + ((J_1)^{h}_{i}(J_1)^{m}_{j} - (J_1)^{m}_{i}(J_1)^{h}_{j}) PYYX \ominus^{h}_{klm} \]

\[ YXX \ominus^{h}_{1kij} = PYXX \ominus^{h}_{kij} - PXYX \ominus^{h}_{kji} \]

\[ + ((J_3)^{h}_{i}(J_1)^{m}_{j} - (J_3)^{m}_{i}(J_1)^{h}_{j}) PYXX \ominus^{h}_{kij} \]

\[ + ((J_1)^{h}_{i}(J_3)^{m}_{j} - (J_1)^{m}_{i}(J_3)^{h}_{j}) PYYY \ominus^{h}_{kij} \]

\[ + ((J_3)^{h}_{i}(J_3)^{m}_{j} - (J_3)^{m}_{i}(J_3)^{h}_{j}) PXYX \ominus^{h}_{kij} \]

where the square is replaced by \( X \) for horizontal components, and by \( Y \) for vertical components.

In the case of the curvature type tensor \( R_2 \), given by (3.4), four of the components, namely \( XXXX X_{2kij}^{h}, XXYX_{2kij}^{h}, XYYX_{2kij}^{h}, XXYY_{2kij}^{h} \), vanish after replacing (2.4) and (2.7). The nonzero components of \( R_2 \) have expressions of the next type:

\[ YXXY_{2kij}^{h} = \left\{ \frac{G^{(2)}_{0m}(J_1)^{m}_{k} - G^{(3)}_{0m}(J_3)^{m}_{k}}{y^{h}} + G^{(3)}_{0k}(J_3)^{h}_{0} \right\} \]

\[ \left\{ G^{(2)}_{0i} \left[ G^{(2)}_{0j} (J_1)^{j}_{i} - G^{(3)}_{0j} (J_3)^{j}_{i} \right] - G^{(3)}_{0j} \left[ G^{(2)}_{0i} (J_3)^{i}_{j} - G^{(3)}_{0i} (J_2)^{i}_{j} \right] \right\} \].

Taking into account the relation (3.1), we have that \( (TM \setminus \{0\}, G, J, \xi) \) is a Kähler manifold of quasi-constant holomorphic sectional curvatures if and only if all the components of the following difference vanish 

\[ (3.9) \]

\[ R - k_0 R_0 - k_1 R_1 - k_2 R_2 = 0. \]

In the study of the conditions under which \( (TM \setminus \{0\}, G, J, C) \) has quasi-constant holomorphic sectional curvatures, the next lemma will be useful.
Lemma 3.2. If $\alpha_1, \ldots, \alpha_{10}$ are smooth functions on $TM$ such that
\begin{equation}
\alpha_1 \delta_i^j g_{jk} + \alpha_2 \delta_i^j g_{ik} + \alpha_3 \delta_i^j g_{0j} + \alpha_4 \delta_i^j g_{00} g_{0j} + \alpha_5 \delta_i^j g_{0k} g_{0j} + \alpha_6 \delta_i^j g_{0j} g_{0k} + \alpha_7 g_{jk} g_{00} h + \alpha_8 g_{ik} g_{00} h + \alpha_9 g_{ij} g_{00} h + \alpha_{10} g_{0j} g_{0k} g_{00} h = 0, \tag{3.10}
\end{equation}
then $\alpha_1 = \cdots = \alpha_{10} = 0$.

In the following we shall prove the main result of this paper.

Theorem 3.3. The bundle $TM \setminus \{0\}$ of non-zero tangent vectors to $M$, endowed with a general natural Kähler structure $(G, J)$, and with the Liouville vector field $C$, is a Kähler structure of quasi-constant holomorphic sectional curvatures, if and only if the tangent bundle $TM$ endowed with the same structure $(G, J)$, has constant holomorphic sectional curvature.

Proof. Let us consider the two components of the difference (3.9):
\begin{equation}
XXX h_{kij} - k_0 XXX h_{kij} - k_1 XXX h_{kij} - k_2 XXX h_{kij} = 0, \tag{3.11}
\end{equation}

\begin{equation}
YYXY_{kij} - k_0 YYXY_{kij} - k_1 YYXY_{kij} - k_2 YYXY_{kij} = 0.
\end{equation}

After replacing the detailed expressions of the $M$-tensor fields, the expressions (3.11) become of the form (3.10).

Using lemma 3.2, we have that the coefficients of $\delta_i^j g_{jk}$ from the two components in (3.11) must be zero. From the vanishing condition for the mentioned coefficients, in which we impose the conditions for $(TM, G, J)$ to be a Kähler manifold ((3.8), (3.7), theorem 2.3, (2.6), (2.5)), we get two distinct values for $k_0$, which are quite long, and we shall not present here. The equality between the obtained expressions of $k_0$ leads to the following differential equation
\begin{equation}
\frac{\lambda'}{\lambda} = - \frac{2c(a_1 - a_1' t)(1 + a_3^2) + a_1 a_1' a_3' t}{a_1[a_1^2 + 2ct(1 + a_3^2)]},
\end{equation}
with the solution
\begin{equation}
\lambda = A \frac{a_1}{a_1^2 + 2ct(1 + a_3^2)}, \quad A = \text{constant.} \tag{3.12}
\end{equation}

By substituting (3.12) in the coefficient of $\delta_i^j g_{ij}$ from the second difference in (3.11), we get
\begin{equation}
\frac{a_1^2(-4c + Ak_0)}{2[a_1^2 + 2ct(1 + a_3^2)]} = 0,
\end{equation}
from where, the value of $k_0$ is

\begin{equation}
(3.13) \quad k_0 = \frac{4c}{A}.
\end{equation}

Using (3.12), (3.13), and the conditions for $(TM, G, J)$ to be a Kähler manifold, we obtain that the coefficient of $g_{ij}g_{0k}y^h$ in the same difference from (3.11) has the numerator of the form

$$-4A^2a_1k_1[(a_1 - 2a_1' t)^2 + (a_1a_3 - 2a_1'a_3t + 2a_1a_3't)^2] \cdot [(a_1^2)^2 - 4a_1^2ct(1 - a_2^2) + 4c^2t^2(1 + a_2^2)^2],$$

in which the first parenthesis is always positive, and the second parenthesis may be seen as a function of second degree in $(a_1)^2$. The associated equation has negative discriminant, and the coefficient of $(a_1^2)^2$ is 1, so the function is always positive. Consequently, the coefficient of $g_{ij}g_{0k}y^h$ contains $k_1$ as the only factor which may vanish, so

\begin{equation}
(3.14) \quad k_1 = 0.
\end{equation}

From the vanishing condition for the coefficient of $g_{0i}g_{0j}g_{0k}y^h$ in the same component of the difference (3.9), we obtain a quite long expression for $k_2$. Using RICCI, we may prove that this expression vanish when we take into account the conditions for $(TM, G, J)$ to be a Kähler manifold, and we replace $k_0$ by (3.13), $k_1$ by 0, $\lambda$ by (3.12), and the corresponding expressions for $\lambda'$, $\lambda''$, $\lambda'''$. Hence, finally, $k_2 = 0$.

In the paper [6] we proved that if $\lambda$ has the expression (3.12), then all the components of the difference $R - k_0R_0$ vanish (i.e. $(TM, G, J)$ has constant sectional curvature). Thus the main result of the present paper is proved.

**Remark.** The theorem 3.3 is a generalisation of Shur’s theorem: if the sectional curvature of any holomorphic plane generated by the unit nonzero vector field $X \in T_pM$ depends only on the angle with the Liouville vector field and on the point $p \in M$, then $TM$ has constant holomorphic sectional curvature.

**Remark.** The result from [4] may by obtained as a particular case of that studied in the present paper, namely, considering the natural lifted Kählerian structure of diagonal type on $TM$, which may be obtained from
the general natural Kählerian structure \((G, J)\) on \(TM\), by equalizing to zero the coefficients \(a_3, b_3\) appearing in the definition of the almost complex structure \(J\), and \(c_3, d_3\) in the definition of the general natural lifted metric \(G\).

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