ON ISOHEDRAL TILINGS OF HYPERBOLIC MANIFOLDS

BY

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Abstract. Consider the hyperbolic manifold of genus 2 obtained from 8-gon with angles equal to $\pi/4$ by identification of opposite sides with translations. Consider its full isometry group of the order 96 and all its subgroups. The problem is to classify isohedral tilings of the manifold for these groups. The method of adjacency symbols analogous to one developed by Delone is applied.

Let us have a surface of constant curvature, a discrete subgroup $G$ of its full isometry group and a tiling $W$ of the surface with bodies. A tiling $W$ of the surface is called isohedral with respect to an isometry group $G$ if $G$ maps the tiling $W$ onto itself and acts transitively on the set of all bodies. The problem is to classify all isohedral tilings for a given surface.

As to the Euclidean plane and the sphere, the complete classifications were obtained (see [6] for references). The hyperbolic plane and manifolds are much more complicated to investigate. In [2, 3] the concepts and methods are developed which permitted to classify isohedral tilings on the Euclidean plane. We are going to apply analogous methods to the classification of isohedral tilings on hyperbolic manifolds (and the hyperbolic plane).

The works [8, 9] are devoted to tilings on the hyperbolic plane. In [7] some algorithms are developed to produce Delaney–Dress symbols corresponding to tilings on the hyperbolic plane (as well as the Euclidean plane and the sphere). There exists a one-to-one correspondence between the Delaney–Dress symbols and the types of isohedral tilings. The classification

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of tilings on the hyperbolic manifold of genus 2 with some transitivity properties is given in [4]. Remark that tilings on the manifolds are usually called maps (for references see [1]).

When classifying isohedral tilings of the Euclidean plane with disks, we obtain 93 types (Delone classes), which correspond to 17 Fedorov groups. For the hyperbolic plane, the number of different types of isohedral tilings is infinite, as well as the number of discrete isometry groups with compact fundamental domain. If considering manifolds with a fixed genus, the number both of groups and isohedral tilings is very large, although finite. In this work we restrict our attention to a manifold with a fixed metrics.

Let us have a tiling of the hyperbolic plane with regular 8-gons and 8-valent vertices (this means that the angle is equal to $\pi/4$). The manifold is obtained from an 8-gon by identification of opposite sides with translations. It is a hyperbolic manifold of genus 2 with fixed metrics. In [5] it is shown that the isometry group of the surface is finite and has the order 96. The manifold which we chose for our investigation is one of the richest among the manifolds of genus 2 (for the orders of automorphism groups of manifolds see [11]).

Now consider the isometry group of the manifold and all its subgroups. The problem is to classify all possible isohedral tilings of the surface for this group and its subgroups. To solve the problem we use the approach and technique analogous to the ones developed in [2, 3].

Fig. 1

Our first task is to classify all isometries of the surface from the geo-
metrical point of view. On Fig. 1 the surface is shown as 8-gon (the opposite sides are glued with translations). On the manifold 12 the shortest geodesics and 12 other geodesics of the equal length are drawn, which altogether divide the surface into 96 triangles being fundamental domains of the full isometry group.

48 direct isometries are the following. There are 6 rotations \( (r_k \text{ and } r_k^7, k = 1, 2, 3) \) of the order 8 with 2 invariant points each \( (r_k \text{ and } r_k^r \text{ on Fig. 1, a}) \) and their degrees. By the way, the inversion \( (i = r_1^i = r_2^i = r_3^i) \) has 6 invariant points. There are 12 rotations \( (s_k, k = 1, \ldots, 12) \) of the order 2 with 2 invariant points each \( (s_k \text{ on Fig. 1, a}) \). 8 isometries \( (t_k = iv_k, t_k^3 = iv_k^3, k = 1, 2, 3, 4) \) of the order 6 have no invariant points, but their degrees are the inversion \( i \) and 8 rotations \( (v_k \text{ and } v_k^2, k = 1, 2, 3, 4) \) of the order 3 with 4 invariant points each \( (v_k \text{ and } v_k^r \text{ on Fig. 1, a}) \).

There are 48 indirect or reflective isometries, and among them 18 are reflections: \( m_k (k = 1, \ldots, 6) \) is a reflection in 2 of the shortest geodesics and \( l_k (k = 1, \ldots, 12) \) is a reflection in a geodesic (see Fig. 1, b). Other indirect isometries have the order 4, 8 or 12.

The second task is to find all subgroups of the full isometry group of the surface. For this purpose the Cayley table of compositions of isometries was compiled. Using the Cayley table, 12 geometrically different groups of direct isometries and 24 groups with reflective isometries (among them 15 Coxeter groups) are determined. For example, there are 3 groups of the order 48: the group 48 of all direct isometries of the manifold, the group 24 \( M \) generated by the rotations \( r_k^2 \) of the order 4, the isometries \( t_k \) and the reflections \( m_k \) in the shortest geodesics, and the Coxeter group 24 \( L \) generated by the same direct isometries \( r_k^2, t_k \) as 24 \( M \) and the reflections \( l_k \).

Now we give the definitions of basic concepts concerning isohedral tilings on the manifold.

**Definition 1.** A tiling is a set of bodies on the surface such that each point of the surface belongs to at least one body and no two bodies have an inner point in common.

The concept of the tiling with bodies is too general. Taking into consideration some peculiarities of hyperbolic manifolds, the tilings investigated must satisfy the following conditions.

**Definition 2.** The manifold is divided into simply connected open regions by a finite number of arcs (and simply connected curves) called edges. Such a region, together with its boundary, is called a tile. Edges meet only at their endpoints called vertices, and each vertex is incident to at least 3 edges (where loops are counted twice) (cf. [4]).
Definition 3. Let $W$ be a tiling of the manifold as above and $G$ be a
discrete isometry group. The tiling $W$ is called isohedral with respect to
the group $G$ if $G$ maps the tiling $W$ onto itself and acts transitively on the
set of all tiles.

When enumerating isohedral tilings, we use the concept of Delone class.

Definition 4. Consider all possible pairs $(W, G)$ where a tiling $W$ of
the surface is isohedral with respect to a group $G$. Two pairs $(W, G)$ and
$(W', G')$ are of one Delone class if there exists a homeomorphism $\varphi$ of the
surface which maps the tiling $W$ onto the tiling $W'$ such that the relation
$G = \varphi^{-1} G' \varphi$ holds.

We distinguish between fundamental and non-fundamental Delone
classes depending on whether the group $G$ acts one time transitively on
the set of tiles or not.

Proceed to the classification of isohedral tilings of our manifold. When
investigating similar problems, the first step is to solve Diophantine equa-
tions obtained from Euler formula. For the tiling of the manifold of genus
$p$ which has $N_0$ vertices, $N_1$ edges and $N_2$ tiles, the Euler formula is the
following:

$$N_0 - N_1 + N_2 = 2(1 - p).$$

For an isohedral tiling, each tile has the same cycle of valences: $(\alpha_1, \alpha_2,$
$\ldots, \alpha_k)$. Denoting by $q_i$ the number of times the number $i$ occurs in this
cycle and using the relations between the numbers $N_0$, $N_1$ and $N_2$

$$N_1 = \frac{\sum q_i}{2} N_2,$$

$$N_0 = \sum \frac{q_i}{i} N_2,$$

we obtain the equation

$$\sum \frac{q_i}{i} = \sum \frac{q_i}{2} - 1 - \frac{2}{N_2}$$

where $N_2$ coincides with the order of the group $G$.

Taking into account the geometrical sense of the number $i$, we have the
additional condition that $q_i N_2$ is a multiple of $i$. From $i > 0$, $i \geq 3$ we also
obtain additional inequalities

$$2 + \frac{4}{N_2} < \sum q_i \leq 6 + \frac{12}{N_2}.$$

The full group of the manifold and all its subgroups are known, and
for each given order of the group we can solve the equation (1) using the
additional condition and inequalities (2). As a result we obtain the set of all possible cycles \((\alpha_1, \alpha_2, \ldots, \alpha_k)\) of isohedral tilings of the manifold for a given order of the group.

Remark that there are 15 Coxeter groups among the groups of our list. For some of them (not for all) there exists the only isohedral tiling quite determined by reflection geodesics, for example, for the full isometry group of the manifold. Therefore, there is no need to solve the equation (1) (and apply further technique) for \(N_2 = 96\).

Because the problem investigated is very vast, we have to limit ourself with description of the methods and some examples.

As an example we solve the equation (1) for \(N_2 = 48\) (it is the greatest number we must take in the case of our manifold). We have the equation

\[
\sum \frac{q_i}{i} = \frac{\sum q_i}{2} - 1 - \frac{1}{24},
\]

the condition that \(48q_i\) is a multiple of \(i\) and the inequalities \(3 \leq \sum q_i \leq 6\).

The solutions obtained are the following: \((3, 12, 24), (3, 16, 16), (4, 6, 24), (4, 8, 12), (6, 6, 8), (3, 3, 6, 8), (3, 4, 4, 8), (3, 3, 3, 3, 8)\). We must consider also the permutations of the numbers in cycles. It is easy to see that the cycles \((3, 12, 24), (3, 3, 6, 8), (3, 6, 3, 8)\) are not possible in a tiling. So, the remained cycles are the following: \((3, 16, 16), (4, 6, 24), (4, 8, 12), (6, 6, 8), (3, 4, 8, 4)\) and \((3, 3, 3, 3, 8)\).

Observe that for isohedral tilings on the Euclidean plane each of 11 good cycles determines only one isohedral tiling up to the combinatorial equivalence \([2, 3]\). For isohedral tilings on our manifold the situation is quite different and there exist combinatorially different tilings which have the same cycle of valences.

Let a tiling \(W\) of the manifold be isohedral with respect to a group \(G\). On the universal covering hyperbolic plane the corresponding tiling \(\tilde{W}\) with bodies is isohedral with respect to a group \(\tilde{G}\). Remark that on the plane of constant curvature (both the Euclidean and hyperbolic) in isohedral tiling the compact connected regions are simply connected and a tile, its boundary consisting of a finite number of arcs, is a closed topological disk.

So, we have a tiling \(\tilde{W}\) of the hyperbolic plane with disks which is isohedral with respect to a discrete isometry group \(\tilde{G}\). The local combinatorial structure of the tiling \(\tilde{W}\) is the same as one of the tiling \(W\) on the manifold, in particular, \(W\) and \(\tilde{W}\) have the same cycle of valences \((\alpha_1, \alpha_2, \ldots, \alpha_k)\). The group \(\tilde{G}\) acts locally in the same way as the group \(G\) locally acts. If the
Delone class of the pair \((\tilde{W}, \tilde{G})\) and the covering mapping are known, then the Delone class of the pair \((W, G)\) is known. Observe that to one Delone class on the hyperbolic plane several Delone classes on manifolds of different genera may correspond.

Now we apply the method of adjacency symbols similar to one developed in [2, 3] to obtain all fundamental Delone classes on the hyperbolic plane.

Let a tiling \(W\) of the hyperbolic plane be isohedral with respect to a group \(G\) with fundamental Delone class of the pair \((W, G)\). Choose a tile and ascribe letters \(a, b, c, \ldots\) to its edges in a cyclic order. Applying the isometry group \(G\) yields a quite definite arrangement of letters on all the tiles of \(W\). As an example consider Fig. 2, b. The symbol \((ab_3ba_4c\overline{c_8}d\overline{d_4})\) means that to the edge \(a\) of a tile \(T_0\) the edge \(b\) of neighbour tile \(T_1\) corresponds, the tiles \(T_0\) and \(T_1\) having the same orientation, to the edge \(c\) of \(T_0\) the edge \(c\) of neighbour tile \(T_3\) corresponds, the tiles \(T_0\) and \(T_3\) having the inverse orientation (as indicated with the line above), and so on. The indices below indicate the valences of vertices. Such an adjacency symbol giving the information about generators and relations of the group fully determines the Delone class of the pair \((W, G)\).

Fig. 2

Proceeding from the list of cycles, find corresponding adjacency symbols. The number of all possible adjacency symbols a priori is very large, but there are many reasons to reject the most part of them. As the first step we examine locally if each candidate in adjacency symbol is good or not going
round all the vertices of a tile. For our particular 6 cycles we obtain the following list of adjacency symbols: \((ab_3ba_4cc_16),(ab_3ba_4cc_16),(aa_4bb_6cc_24), (aa_4bb_6cc_12),(aa_6bb_6cc_8),(aa_6bb_6cc_8), (ac_6bb_6ca_8), (ac_6bb_6ca_8), (ab_3ba_4cc_8dd_4), (ab_3ba_4cd_8dc_4), (ae_3bb_3cd_3dc_3ea_8)\), which correspond to Delone classes of pairs \((W,G)\) on the hyperbolic plane.

Now turning to our 3 isometry groups of the order 48 on the manifold listed above, as the second step we check if the adjacency symbols are compatible with them. For the group 48 we obtain 4 Delone classes on the hyperbolic plane with adjacency symbols \((ab_3ba_4cc_16)\), \((ac_6bb_6ca_8)\), \((ab_3ba_4cd_8dc_4)\) and \((ae_3bb_3cd_3dc_3ea_8)\). Remark that the symbol \((ae_3bc_3cc_3dd_3ea_8)\) is equivalent to \((ae_3bb_3cd_3dc_3ea_8)\) corresponding to the same Delone class. For the group 24\(M\) we obtain 2 Delone classes on the hyperbolic plane with adjacency symbols \((ab_3ba_4cc_16)\) and \((ab_3ba_4cc_8dd_4)\) depicted on Fig. 2. And for the Coxeter group 24\(L\) we obtain one Delone class on the hyperbolic plane with adjacency symbol \((a\bar{a}_6bb_6cc_8)\) and \((ac_6bb_6ca_8)\). The symbols \((a\bar{a}_4bb_6cc_24)\) and \((aa_4bb_6cc_12)\) are not compatible with our groups because their corresponding groups must have the order 12 or 6. Neither are compatible with our groups the symbols \((aa_6bb_6cc_8)\) and \((ac_6bb_6ca_8)\).

Thus, the method described permits to find all fundamental Delone classes of isohedral tilings. The author already obtained complete results for many of the groups on this manifold and continues to work. The full classification of fundamental Delone classes of isohedral tilings on this manifold appears to be rather vast. The isohedral tilings for the groups of the order 48 (as for many other groups) have the advantage that the tiles are closed topological disks. But some difficulties arise when the order of the group is low and we have to deal with the tiles which are not disks (only their interiors are open disks).

The method of obtaining non-fundamental Delone classes from fundamental ones is developed in [3] for the case of isohedral tilings of the Euclidean plane. The analogous method can be applied to isohedral tilings of this manifold after all the fundamental Delone classes are known.

As other prospects we consider the investigation of \(k\)-isohedral tilings, i.e. tilings with \(k\) transitivity classes of bodies, on this manifold. The methods of obtaining \(k\)-isohedral \((k \geq 2)\) tilings from isohedral ones on the 2-dimensional space of constant curvature (i.e. the Euclidean plane, the sphere and the hyperbolic plane) are developed in [10]. Analogous methods are applicable to tilings on manifolds. The author already obtained 2-isohedral tilings of this manifold for some groups.
REFERENCES


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