ON THE GEOMETRY OF TANGENT BUNDLE OF A
(PSEUDO-) RIEMANNIAN MANIFOLD

BY

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0. Introduction. In [5], [6] the authors have studied the properties of
a pseudo–Riemannian metric $G$ on the cotangent bundle $T^*M$ of a manifold
$M$ by using an arbitrary symmetric nonlinear connection on this bundle.
Remark that this pseudo–Riemannian metric $G$ on $T^*M$ is very much similar
to the Riemann extension considered in [7], [8], [9].

In [3] the first author has studied some aspects of the differential ge-
ometry of the tangent bundle of a Langange manifold when this bundle is
endowed with a pseudo–Riemannian metric obtained from the fundamen-
tal tensor field by a method similar to the obtaining of the complete lift
of a (pseudo– )Riemannian metric on a differentiable manifold. So, the
Levi Civita connection associated with the considered pseudo– Riemannian
metric has been obtained, next the local coordinate expressions of its curva-
ture tensor field and the corresponding Bianchi identities have been written
down.

In the present paper we study the properties of a pseudo– Riemannian
metric $G$ of type ”complete lift” on the tangent bundle $TM$ of a (pseudo–)
Riemannian manifold $(M, g)$. The considered pseudo–Riemannian metric
$G$ on $TM$ is defined by using the Levi Civita connection of the (pseudo–)
Riemannian metric $g$ on $M$ and an arbitrary $M$– tensor field of type $(0,2)$
on the tangent bundle $TM$. First we show that the geometric properties of
$(TM, G)$ to be either flat, or projectively flat, or conformally flat or locally
symmetric are expressed only in the terms of the Levi Civita connection
of $g$ on $M$ and of the symmetric part of the considered $M$–tensor field on
$TM$. Next, in the particular case where the considered $M$–tensor field is
independent of the tangential coordinates we get the necessary ans suffi-
cient conditions under which the pseudo–Riemannian manifold \((TM, G)\) is either flat, or projectively flat, or conformally flat, or locally symmetric. In section 2 we define an almost complex structure \(J\) on \(TM\) and we get the conditions under which the almost complex manifold with Norden metric \((TM, J, G)\) is a Kaehlerian manifold with Norden metric. In section 3, we consider a naturally defined almost product structure \(P\) on \(TM\) and study the conditions under which the almost parahermitian manifold \((TM, G, P)\) is a parakaehlerian manifold. Finally, some classes of manifolds whose tangent bundles carry parakaehlerian structures are also presented (Theorems 7, 8, 9 and 10).

The manifolds, tensor fields and geometric objects we consider in this paper, are assumed to be differentiable of class \(C^\infty\). We use the well known summation convention, the range for the indices \(i, j, k, h, l, s, t\) being always \(\{1, 2, \ldots, n\}\). We shall denote by \(\Gamma(TM)\) the module of smooth vector fields on \(TM\).

1. **The pseudo–Riemannian manifold** \((TM, G)\). Let \(M\) be an \(n\)–dimensional manifold and denote by \(\tau: TM \rightarrow M\) its tangent bundle. Then \(TM\) is a \(2n\)–dimensional manifold and some special local charts on \(TM\) induced from local charts on \(M\) may be used. Namely, if \((U, x^i); i = 1, \ldots, n\) is a local chart on \(M\), then the local chart \((\tau^{-1}(U), x^i \circ \tau, y^i); i = 1, \ldots, n\) is defined on \(TM\) where \(y^1, \ldots, y^n\) are the vector space coordinates of an element from \(\tau^{-1}(U)\) with respect to the natural frame \((\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})\) defined by the local chart \((U, x^i); i = 1, \ldots, n\). We shall denote, by an abuse of notation, \(x^i = x^i \circ \tau\), so \(x^i\) are considered simultaneously as local coordinates on \(M\) and on \(TM\). The \(M\)–tensor fields and the linear \(M\)–connections may be considered on \(TM\) and the usual tensor fields and linear connections on the base manifold \(M\) may be thought of naturally as \(M\)–tensor fields and linear \(M\)–connections on \(TM\) (sec [5], [8]). The tangent bundle \(TTM\) of \(TM\) has an integrable vector subbundle \(VTM = \ker \tau_*\), called the vertical distribution on \(TM\). A nonlinear connection on \(TM\) is defined by a distribution \(HTM\), complementary to \(VTM\) in \(TTM\) (called the horizontal distribution)

\[
TTM = VTM \oplus HTM.
\]

In the following we assume that \(M\) is a (pseudo–)Riemannian manifold with the (pseudo–)Riemannian metric

\[
g = g_{ij}(x)dx^i dx^j
\]
and denote by $g^{ij}$ the components of the inverse of the matrix $(g_{ij}); \ i, j = 1, \ldots, n, i.e. \ g_{ik}g^{jk} = \delta^i_j$.

Denote by $\Gamma^i_j$ the connection coefficients of the Levi Civita connection $\nabla$ of $g$ (i.e. the Christoffel symbols) and let $a_{ij}; \ i, j = 1, \ldots, n$ be the components of an arbitrary $M$–tensor field of type $(0,2)$ on $TM$. We think of $g_{ij}$ as the components of an $M$–tensor field of type $(0,2)$ on $TM$ and consider the functions

$$N^i_j = \Gamma^i_{jk}y^k + g^{ih}a_{hj}; \ i, j = 1, \ldots, n.$$  

It follows that these functions are the connection coefficients of a non-linear connection which defines a horizontal distribution denoted by $HTM$ on $TM$. The system of the local vector fields $(\partial_i = \frac{\partial}{\partial y^i}); \ i = 1, \ldots, n$ is a local frame in $VTM$ and the system of local vector fields $(\delta_i = \frac{\delta}{\delta x^i}); \ i = 1, \ldots, n$, where

$$\delta_i = \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^i_j \frac{\partial}{\partial y^j}$$

is a local frame in $HTM$. Then $(\partial_i, \delta_i); \ i = 1, \ldots, n$ is a local frame on $TM$ adapted to the direct sum decomposition (1).

The system of local 1–froms $(\delta y^i, dx^i); \ i = 1, \ldots, n$, where

$$\delta y^i = dy^i + N^i_j dx^j$$

is the dual local frame of the local frame $(\partial_i, \delta_i); \ i = 1, \ldots, n$.

Then we have, as usual:

$$\left[ \frac{\partial}{\partial y^i}, \delta \frac{\delta}{\delta x^j} \right] = -\Phi^k_{ji} \frac{\partial}{\partial y^k}; \ \left[ \delta \frac{\delta}{\delta x^i}, \delta \frac{\delta}{\delta x^j} \right] = -R^k_{ij} \frac{\partial}{\partial y^k},$$

where

$$\Phi^k_{ji} = \frac{\partial N^k_i}{\partial y^j}; \ R^k_{ij} = \frac{\delta N^k_i}{\delta x^j} - \frac{\delta N^k_j}{\delta x^i}$$

and the integrability of the system defined by $HTM$ on $TM$ is equivalent to the vanishing of the components $R^k_{ij}$ on $\tau^{-1}(U)$. 


Remark that the components $\Phi_{ji}^k$ define a linear $M-$connection and the components $R_{ij}^k$ define an $M-$tensor field of type $(1,2)$ on $TM$.

Consider the following pseudo–Riemannian metric $G$ of complete lift type on $TM$

$$G = 2g_{ij}dy^jdx^i + 2g_{ij}\Gamma^i_{hk}y^kdx^hdx^j + (a_{ij} + a_{ji})dx^idx^j.$$ 

Remark that the pseudo–Riemannian metric $G$ depends only on (the pseudo–)Riemannian metric $g$ on $M$ and on the symmetric part of the $M-$tensor field on $TM$ defined by the components $a_{ij}$.

Denote by $\tilde{\nabla}$ the Levi Civita connection of the considered pseudo–Riemannian metric $G$ on $TM$.

Then the following result is proved by a straightforward computation.

**Proposition 1.** The local coordinate expression of $\tilde{\nabla}$ in the local frame $(\partial_i, \delta_i)$ adapted to the direct sum decomposition (1) is:

$$\tilde{\nabla}_{\partial_i}\partial_j = 0; \quad \tilde{\nabla}_{\partial_i}\delta_j = g^{kh} \frac{\partial b_{jh}}{\partial y^l} \frac{\partial}{\partial y^k}; \quad \tilde{\nabla}_{\delta_i}\partial_j = \left(\Gamma^k_{ij} + g^{kh} \frac{\partial c_{ij}}{\partial y^h}\right) \frac{\partial}{\partial y^k};$$

$$\tilde{\nabla}_{\delta_i}\delta_j = \left(\Gamma^k_{ij} - g^{kh} \frac{\partial c_{ij}}{\partial y^h}\right) \frac{\delta}{\delta y^k} + \frac{1}{2}g^{kh}(R_{ijh} - R_{jhi} - R_{hij}) \frac{\partial}{\partial y^k},$$

where

$$R_{hij} = g_{hk}R_{ij}^k$$

and $c_{ij}$ (respectively $b_{ij}$) denotes the symmetric part (respectively the skew–symmetric part) of $a_{ij}$, i.e. $c_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$ and $b_{ij} = \frac{1}{2}(a_{ij} - a_{ji})$.

**Remark.** From Proposition 1 it follows that the essential coefficients of the local coordinate expression of $\tilde{\nabla}$ in the local adapted frame $(\partial_i, \delta_i)$; $i = 1, \ldots, n$ are expressed by using only the components of the (pseudo–)Riemannian metric $g$ on $M$, the coefficients of the Levi Civita connection $\nabla$ of $g$, the symmetric part and the skew–symmetric part of the $M-$tensor field defined by the components $a_{ij}$ on $TM$ and the $M-$tensor field on $TM$ defined by $\frac{1}{2}(R_{ijh} - R_{jhi} - R_{hij})$. On the other hand, taking into account that the pseudo–Riemannian metric $G$ depends only on the components of the
(pseudo-)Riemannian metric $g$ on $M$ and on the symmetric part of the $M$-tensor field defined by the components $a_{ij}$ on $TM$ we can consider the nonlinear connection $\mathbb{HTM}$ defined by the connection coefficients $\overline{N}_j^i$:

$$
\overline{N}_j^i = \nabla^i_j y^k + g^{ih} c_{hj}; \quad i,j = 1, \ldots, n,
$$

where $c_{ij}$ denotes the symmetric part of $a_{ij}$, i.e. $c_{ij} = \frac{1}{2} (a_{ij} + a_{ji})$. Then the vector fields $(\overline{\delta}_i = \frac{\delta}{\delta x^i}; \quad i = 1, \ldots, n$, where

$$
\overline{\delta}_i = \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_j^i \frac{\partial}{\partial y^j}
$$

define a local frame in the horizontal distribution $\mathbb{HTM}$ and the vector fields $(\partial_i, \overline{\delta}_i) \quad i = 1, \ldots, n$ define a local frame in $TM$ adapted to the direct sum decomposition

$$
TTM = VTM \oplus \mathbb{HTM}.
$$

Denote by $(\overline{\delta} y^i, dx^i); \quad i = 1, \ldots, n$ the dual local frame of the local frame $(\partial_i, \overline{\delta}_i); \quad i = 1, \ldots, n$. Then we have

$$
\overline{\delta} y^i = dy^i + \overline{N}_j^i dx^j
$$

and the pseudo–Riemannian metric $G$ defined by (7) on $TM$ becomes

$$
G = 2g_{ij} \overline{\delta} y^i dx^j = 2g_{ij} dy^i dx^j + \frac{\partial g_{ij}}{\partial x^k} y^k dx^i dx^j + 2c_{ij} dx^i dx^j.
$$

Since we have

$$
\delta_i = \overline{\delta}_i - g^{jh} b_{hi} \partial_j,
$$

where $b_{ij}$ denotes the skew–symmetric part of $a_{ij}$, from Proposition 1 we obtain by a straightforward computation.

**Proposition 2.** The local coordinate expression of the Levi Civita connection $\nabla$ of $G$ in the local frame $(\partial_i, \overline{\delta}_i); \quad i = 1, \ldots, n$ adapted to the direct sum decomposition (8) is:

$$
\tilde{\nabla}_{\partial_i} \partial_j = 0; \quad \tilde{\nabla}_{\overline{\delta}_i} \partial_j = 0; \quad \tilde{\nabla}_{\overline{\delta}_i} \overline{\delta}_j = \left( \Gamma^k_{ij} + g^{kh} \frac{\partial c_{hi}}{\partial y^j} \right) \frac{\partial}{\partial y^k};
$$
\[ \tilde{\nabla}_i \delta_j = \left( \Gamma^k_{ij} - g^{kh} \frac{\partial c_{ij}}{\partial y^h} \right) \frac{\partial}{\partial x^k} + g^{kh} \mathcal{R}_{ijk} \frac{\partial}{\partial y^k}, \]

where \( \mathcal{R}_{ijk} \) are defined by

\[ \mathcal{R}_{hij} = g_{hk} \mathcal{R}^k_{ij}; \quad \mathcal{R}^k_{ij} = \delta_i \mathcal{N}^k_j - \delta_j \mathcal{N}^k_i. \]

**Remark.** From Proposition 2 it follows that the essential coefficients of the local coordinate expression of \( \tilde{\nabla} \) in the local adapted frame \((\partial_i, \tilde{\delta}_i); \ i = 1, \ldots, n\) are expressed by using only the (pseudo-)Riemannian metric \( g \) on \( M \), the coefficients of the Levi Civita connection \( \nabla \) of \( g \), the components of the symmetric \( M \)-tensor field defined by \( c_{ij} \) and the components \( \mathcal{R}_{kij} \) defined above.

Since the geometric properties of the pseudo-Riemannian manifold \((TM, G)\) to be either flat, or projectively flat, or conformally flat or locally symmetric are independent of the choice of the horizontal distribution \( HTM \) or \( HTM^\perp \) on \( TM \), from Proposition 2 we have

**Theorem 3.** The geometric properties of the pseudo-Riemannian manifold \((TM, G)\) to be either flat, or projectively flat, or conformally flat or locally symmetric which depend only on the metric \( G \) and its Levi Civita connection \( \nabla \) are expressed only in the terms of the nonlinear connection defined by \( \mathcal{N}_j^i \) on \( TM \).

In the following we give the conditions under which the pseudo-Riemannian manifold \((TM, G)\) is respectively flat, projectively flat, conformally flat and locally symmetric, assuming that the \( M \)-tensor field defined by the components \( a_{ij} \) is independent of the tangential coordinates \( y^i \), i.e. the components \( a_{ij} \) define a tensor field of type \((0,2)\) on the base manifold \( M \) thought of as an \( M \)-tensor field on \( TM \).

We obtain by a straightforward computation the following result.

**Theorem 4.** Let \((M, g)\) be a (pseudo-)Riemannian manifold and denote by \( \Gamma^k_{ij} \) the coefficients of the Levi Civita connection \( \nabla \) of \( g \). Consider on \( TM \) the nonlinear connection defined by (2), where the components \( a_{ij} \) define a tensor field of type \((0,2)\) on \( M \) thought of as an \( M \)-tensor field on \( TM \). Denote by \((TM, G)\) the pseudo-Riemannian manifold \( TM \) endowed with the pseudo-Riemannian metric \( G \), where \( G \) is given by (7). Then we have:

(i) \((TM, G)\) is flat if and only if \((M, g)\) is flat and the symmetric part \( c_{ij} \) of \( a_{ij} \) satisfies the condition

\[ \nabla_i (\nabla_k c_{jh} - \nabla_h c_{jk}) - \nabla_j (\nabla_k c_{ih} - \nabla_h c_{ik}) = 0, \]
where \( \nabla_i c_{jk} \) are the local components of the covariant derivative of the tensor field on \( M \) defined by \( c_{jk} \) with respect to \( \nabla \).

(ii) \((TM, G)\) is projectively flat if and only if \((TM, G)\) is flat.

(iii) \((TM, G)\) is conformally flat if and only if \((M, g)\) has constant sectional curvature and the symmetric part \( c_{ij} \) of \( a_{ij} \) satisfies the condition

\[
\nabla_i (\nabla_k c_{jl} - \nabla_l c_{jk}) - \nabla_j (\nabla_k c_{il} - \nabla_l c_{ik}) = r n \left( c_{jk} g_{il} - c_{jl} g_{ik} - c_{ik} g_{jl} + c_{il} g_{jk} \right),
\]

where \( r \) denotes the scalar curvature of \((M, g)\) and \( n > 2 \) is the dimension of \( M \).

(iv) \((TM, G)\) is locally symmetric if and only if \((M, g)\) is locally symmetric and the symmetric part \( c_{ij} \) of \( a_{ij} \) satisfies the condition

\[
\nabla_h \nabla_i (\nabla_k c_{jl} - \nabla_l c_{jk}) - \nabla_h \nabla_j (\nabla_k c_{il} - \nabla_l c_{ik}) + \\
+ \nabla_h c_{is} R^s_{jkl} - \nabla_h c_{js} R^s_{ikl} + (\nabla_s c_{hs} - \nabla_l c_{sh}) R^s_{kij} - (\nabla_l c_{hs} - \nabla_s c_{sh}) R^s_{kij} = 0,
\]

where \( R^h_{klij} \) are the local coordinate components of the curvature field of the Levi Civita connection on \((M, g)\).

Remarks.

(i) If \( c_{ij} = 0 \) then the conditions (9), (10), (11) are identically fulfilled.

(ii) If \( c_{ij} \) is parallel with respect to \( \nabla \), then the conditions (9) and (11) are identically fulfilled too.

(iii) If \( c_{ij} \) satisfies the relation \( \nabla_i c_{jk} - \nabla_j c_{ik} = \nabla_k \omega_{ij} \), where the components \( \omega_{ij} \) define a 2–form on \( M \) and if \((M, g)\) is flat then the condition (9) is identically verified. In particular, if the 2–form defined by the components \( \omega_{ij} \) is parallel then the components \( c_{ij} \) define a Codazzi tensor field and (9) is identically verified.

2. An almost complex structure on \((TM, G)\). Let \((M, g)\) be a \((\text{pseudo–})\)Riemannian manifold and let \((TM, G)\) be the \(\text{pseudo–}\)Riemannian manifold with \( G \) defined by (7) where the considered nonlinear connection on \( TM \) is given by (2). We can define the following almost complex structure \( J \) on \( TM \):

\[
J \left( \frac{\delta}{\delta x^i} \right) = \frac{\partial}{\partial y^i}; \quad J \left( \frac{\partial}{\partial y^i} \right) = - \frac{\delta}{\delta x^i}.
\]
Then, according with the terminology from [2], we may verify that $G$ and $J$ define an almost complex structure with Norden metric on $TM$ (or equivalently, $(TM, J, G)$ is a hyperbolic almost Hermitian manifold), i.e.

\begin{equation}
G(JX, JY) = -G(X, Y); \quad X, Y \in \Gamma(TM).
\end{equation}

On $TM$ we consider the following tensor field $F$ of type $(0,3)$ defined by (see [2])

\begin{equation}
F(X, Y, Z) = G((\tilde{\nabla} X)J Y, Z); \quad X, Y, Z \in \Gamma(TM),
\end{equation}

where $\tilde{\nabla}$ denotes the Levi Civita connection on $(TM, G)$.

Then the almost complex manifold with Norden metric $(TM, J, G)$ is a Kaehlerian manifold with Norden metric (or an hyperbolic Kaehlerian manifold) if

\begin{equation}
F(X, Y, Z) = 0; \quad X, Y, Z \in \Gamma(TM),
\end{equation}

or, equivalently, if $\tilde{\nabla} J = 0$.

By using (12), (14) and Proposition 1 we get by a straightforward computation

\begin{equation}
\begin{aligned}
F \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k} \right) &= F \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right) = 0; \\
F \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right) &= F \left( \frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^i} \right) = \partial c_{ij}^k \frac{\partial}{\partial y^k}; \\
F \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^k} \right) &= F \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k} \right) = \frac{1}{2} (R_{kij} + R_{jki} - R_{ijk}).
\end{aligned}
\end{equation}

Examining the above relations it follows that the condition $F = 0$ which must be fulfilled for $(TM, G)$ to be a Kaehlerian manifold with Norden metric is reduced to

\begin{equation}
\begin{aligned}
(i) \frac{\partial b_{ij}}{\partial y^k} &= 0; \\
(ii) \frac{\partial c_{ij}}{\partial y^k} + \frac{\partial c_{ik}}{\partial y^j} &= 0; \\
(iii) R_{kij} + R_{jki} - R_{ijk} &= 0.
\end{aligned}
\end{equation}

From (16)(i) and (16)(ii) it follows that $a_{ij}$ must be independent of $y^k$, i.e. the components $a_{ij}$ define a tensor field of type $(0,2)$ on the base manifold $M$ thought of as an $M$–tensor field on $TM$. Next, using (2), (3)
and the second relation (6) we obtain by a straightforward computation that the condition (16)(iii) becomes

\[ y^h R_{hijk} + \nabla_k c_{ij} - \nabla_j c_{ik} - \nabla_i b_{jk} = 0, \]

where \( R_{hijk} \) denotes the local coordinate components of the Riemann–Christoffel tensor of \( \nabla \) on \( M \) and \( c_{ij} \) (respectively \( b_{ij} \)) is the symmetric part (respectively the skew–symmetric part) of \( a_{ij} \).

Hence we may state

**Theorem 5.** The almost complex manifold with Norden metric \((TM, J, G)\) is a Kaehlerian manifold with Norden metric if and only if the components \( a_{ij} \) are independent of \( y^k \), the Levi Civita connection \( \nabla \) of \( g \) is flat and the symmetric part \( c_{ij} \) and the skew–symmetric part \( b_{ij} \) of \( a_{ij} \) satisfy the condition

\[ \nabla_k c_{ij} - \nabla_j c_{ik} = \nabla_i b_{jk}. \]

**Remark.** The condition (18) implies that the 2–form defined by the components \( b_{ij} \) is closed. Then the condition (18) is equivalent to \( \nabla_i a_{jk} = \nabla_j a_{ik} \) and Theorem 5 may be formulated as follows: The almost complex manifold with Norden metric \((TM, J, G)\) is Kaehlerian with Norden metric if and only if the components \( a_{ij} \) define a tensor field on the base manifold \( M \), the 2–form defined by the components \( b_{ij} \) is closed and the horizontal distribution \( HT_M \) defined by the nonlinear connection coefficients \( N^1_i \) on \( TM \) is involutive. Remark also that if \( c_{ij} \) define a Codazzi tensor field with respect to \( \nabla \) and \( b_{ij} \) is parallel with respect to \( \nabla \), then the condition (18) is identically satisfied.

3. **An almost product structure on** \((TM, G)\). In this section we consider a nonlinear connection on \( TM \) defined by its connection coefficients of the form (2) and define an almost product structure \( P \) on \( TM \) determined by the distributions \( VT_M \) and \( HT_M \). Next we obtain the conditions under which the almost parahermitian manifold \((TM, G, P)\) is a parakaehlerian manifold (see [1]).

Consider on \( TM \) the almost product structure \( P \) naturally defined by the direct sum decomposition (1), i.e.

\[ P \left( \frac{\partial}{\partial y^i} \right) = \frac{\partial}{\partial y^i}; \quad P \left( \frac{\delta}{\delta x^i} \right) = - \frac{\delta}{\delta x^i}. \]
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It follows easily that \( G(PX, PY) = -G(X, Y); \quad X, Y \in \Gamma(TM), \) therefore \((TM, G, P)\) is an almost parahermitian manifold (see [1]). Define the 2–form \( \Omega \) associated with the almost parahermitian structure \((G, P)\) on \( TM \) by

\[
\Omega(X, Y) = G(PX, Y); \quad X, Y \in \Gamma(TM).
\]

According to the terminology from [1] we have that \((TM, G, P)\) is a para-kaehlerian manifold if \( \tilde{\nabla} \Omega \) vanishes identically on \( TM \). Using (19), (20) and Proposition 1 we obtain by a straightforward computation

\[
(\tilde{\nabla}\partial\Omega)(\partial_j, \partial_k) = (\tilde{\nabla}\delta\Omega)(\partial_j, \partial_k) = (\tilde{\nabla}\delta\Omega)(\partial_j, \partial_k) = 0
\]

\[
(\tilde{\nabla}\partial\Omega)(\delta_j, \delta_k) = 2 \frac{\partial b_{kj}}{\partial y^i}; \quad (\tilde{\nabla}\delta\Omega)(\delta_j, \delta_k) = R_{kij} + R_{jki} - R_{ijk}.
\]

Then the condition \( \tilde{\nabla} \Omega = 0 \) which must be fulfilled for \((TM, G, P)\) to be a para-kaehlerian manifold is reduced to

\[
(i) \quad \frac{\partial b_{kj}}{\partial y^i} = 0; \quad (ii) \quad R_{kij} + R_{jki} = R_{ijk}.
\]

From (22)(i) it follows that the skew–symmetric part \( b_{ij} \) of \( a_{ij} \) does not depend on the tangential components \( y^k \), i.e. the components \( a_{ij} \) are of the form

\[
a_{ij}(x, y) = c_{ij}(x, y) + b_{ij}(x),
\]

where \( c_{ij}(x, y) \) are the components of a symmetric \( M \)–tensor field of type \((0,2)\) on \( TM \) and \( b_{ij}(x) \) are the components of a skew–symmetric tensor field of type \((0,2)\) on \( M \) thought of as a skew–symmetric \( M \)–tensor field on \( TM \). Hence we have

\textbf{Theorem 6.} \quad \textit{The almost parahermitian manifold \((TM, G, P)\) where \( G \) is defined by (7) and the nonlinear connection is defined by (2) with \( a_{ij} \) of the form (23) is a para-kaehlerian manifold if and only if the components \( R_{kij} \) satisfy the condition (22)(ii).}

In order to obtain some classes of manifolds whose tangent bundles carry para-kaehlerian structures, we consider on \( TM \) some particular \( M \)–tensor fields of the form (23) and we study the conditions under which \((TM, G, P)\) is a para-kaehlerian manifold.
Firstly, we consider on $TM$ the $M$–tensor field defined by the components $a_{ij}$:

\begin{equation}
    a_{ij} = c_{ij} + b_{ij},
\end{equation}

where $c_{ij}$ (respectively $b_{ij}$) are the components of a symmetric (respectively skew–symmetric) tensor field on $M$ thought of as an $M$–tensor field on $TM$. Then, using (2), (6) and (24) we get by a straightforward computation

\[ R_{kij} = R_{khij}y^h + \nabla_i c_{kj} - \nabla_j c_{ki} + \nabla_k b_{ij}, \]

where $R_{khij}$ are the local coordinate components of the Riemann–Christoffel tensor field defined by $\nabla$ on $(M, g)$. Next, the condition (22)(ii) becomes

\begin{equation}
    R_{khij}y^h + \nabla_i c_{kj} - \nabla_j c_{ki} + \nabla_k b_{ij} = 0,
\end{equation}

hence we have

**Theorem 7.** Let the nonlinear connection $N^i_j$ on $TM$ be defined by (2) where $a_{ij}$ are given by (24). Then $(TM, G, P)$ is a parakaehlerian manifold if and only if the (peudo–)Riemannian manifold $(M, g)$ is flat and the components $c_{ij}$ and $b_{ij}$ satisfy the condition

\begin{equation}
    \nabla_i c_{kj} - \nabla_j c_{ki} = \nabla_k b_{ij}. \tag{26}
\end{equation}

**Remark.** The condition (25) is equivalent to

\[ R_{kij} + \sum_{(i,j,k)} \nabla_i b_{jk} = 0, \]

where $\sum_{(i,j,k)}$ denotes the sum consisting of three terms obtained by cyclic permutations of $i, j, k$. Thus, Theorem 7 becomes: Under the hypothesis of Theorem 7 we have that $(TM, G, P)$ is a parakaehlerian manifold if and only if the horizontal distribution $HTM$ on $TM$ is involutive and the $2$–form defined by the components $b_{ij}$ is closed.

Now, we consider on $TM$ the $M$–tensor field defined by the components $a_{ij}$ of the form (23), with
\[ a_{ij} = kg_{ik}g_{jl}y^h y^l + t_{ij}, \]

where \( k \) is a nonzero constant and the components \( t_{ij} \) define and arbitrary tensor field of type \((0,2)\) on \( M \) thought of as an \( M \)-tensor field on \( TM \). Using \((2), (6) \) and \((27)\) we obtain by a straightforward computation

\[ R_{ijk} = \left[ R_{iljk} + k(t_{ik} g_{jl} - t_{ij} g_{kl} + t_{jk} g_{il} - t_{kj} g_{il}) \right] y^l + \nabla_j t_{ik} - \nabla_k t_{ij}. \]

By using \((28)\) it follows that the condition \((22) \) becomes

\[ [R_{iljk} + k(t_{ik} g_{jl} - t_{ij} g_{kl} + t_{jk} g_{il} - t_{kj} g_{il})] y^l + \nabla_j h_{ik} - \nabla_k h_{ij} + \nabla_i b_{jk} = 0, \]

where \( h_{ij} \) (respectively \( b_{ij} \)) denotes the symmetric part (respectively the skew–symmetric part) of \( t_{ij} \). The condition \((29)\) is equivalent to

\[ (i) \quad R_{ijh} = k(t_{ik} \delta^h_j - t_{ij} \delta^h_k + t_{jk} \delta^h_i - t_{kj} \delta^h_i); \]

\[ (ii) \quad \nabla_j h_{ik} - \nabla_k h_{ij} = \nabla_i b_{kj}, \]

where \( R_{ijh} \) are the local coordinate components of the curvature tensor field of the Levi Civita connection \( \nabla \) an \( M \).

By taking into account that \( \nabla \) is the Levi Civita connection of \( g \), we have from \((30)(i)\)

\[ t_{ij} = \frac{1}{k(n-1)} R_{ij}, \]

where \( R_{ij} \) are the local coordinate components of the Ricci tensor defined by the curvature tensor of \( \nabla \), i.e. we have \( t_{ij} = h_{ij} \) and \( b_{ij} = 0 \). Thus, the condition \((30)(i)\) implies that the (pseudo–)Riemannian manifold \( (M, g) \) has constant sectional curvature, next replacing the expression of \( t_{ij} \) from \((31)\) in \((30)(i)\) and using the second Bianchi identity we get that the condition \((30)(ii)\) is identically verified.

Hence we may state
Theorem 8. Let \((M, g)\) be a \((\text{pseudo–})\)Riemannian manifold with \(\dim M = n > 2\). Consider on \(TM\) the nonlinear connection defined by

\[
N^k_i = \Gamma^k_{ij} y^j + k g_{ih} y^h y^k + \frac{1}{k(n - 1)} g^{kh} R_{hi},
\]

where \(\Gamma^k_{ij}\) are the connection coefficients of the Levi Civita connection \(\nabla\) of \(g\). \(R_{ij}\) are the local coordinate components of the Ricci tensor defined by the curvature tensor of \(\nabla\) and \(k\) is a nonzero constant. The the almost parahermitian manifold \((T M, G, P)\) where \(G\) is given by (7) and \(P\) is defined by (19) is a parakaehlerian manifold if and only if the \((\text{pseudo–})\)Riemannian manifold \((M, g)\) has constant sectional curvature.

More parakaehlerian structures on tangent bundles can be obtained in the cases of complex and quaternion manifolds.

Let \((M, g, F)\) be a Kaehler manifold with the almost complex structure defined by the tensor field \(F\) of type \((1,1)\) such that \(F^2 = -I\) and denote by \(\nabla\) the Levi Civita connection of \(g\) such that \(\nabla F = 0\) (see \([10]\)). Moreover, we have \(g(FX, FY) = g(X, Y); \ X, Y \in \Gamma(M)\). Consider on \(TM\) the \(M\)–tensor field defined by the components \(a_{ij}\) of the form (23):

\[
a_{ij} = k(g_{ih} g_{jk} - g_{is} F^s_{hi} g_{jt} F^t_k) y^h y^k + t_{ij},
\]

where \(F^h_i\) are the components of \(F\); \(k\) is a nonzero constant and \(t_{ij}\) are the components of an arbitrary tensor field of type \((0,2)\) on \(M\) thought of as an \(M\)–tensor field on \(TM\). In this case, using (2), (6) and (32) we obtain by a straightforward computation

\[
R_{ijk} = \{R_{iljk} + k [t_{ik} g_{jl} - t_{ij} g_{kl} + t_{kj} g_{il} - t_{kj} g_{il} + t_{hl} F^k_{hi} F^s_l g_{sl} - t_{hl} F^h_{ij} F^s_k g_{sl} - t_{lh} F^h_{ij} F^s_k g_{sl}] \} y^l + \nabla_j t_{ik} - \nabla_k t_{ij},
\]

where \(R_{ijk}\) are the local coordinate components of the Riemann–Christoffel tensor on \((M, g)\). Using the above expression of \(R_{ijk}\) we obtain by a straightforward computation that the condition (22)(ii) which must be fulfilled for \((TM, G, P)\) to be a parakaehlerian manifold is equivalent to the following two conditions

\[
\begin{align*}
(i) \quad R^h_{ki} & = k(t_{kj} \delta_i^h - t_{ki} \delta_j^h) + t_{ij} \delta_k^h + t_{ik} F^l_j F^h_k - t_{lj} F^l_k F^h_j - t_{ij} F^l_k F^h_j; \\
(ii) \quad \nabla_i h_{jk} - \nabla_j h_{ik} + \nabla_k b_{ij} & = 0,
\end{align*}
\]

(33)
where $R^h_{kij}$ are the local coordinate components of the curvature tensor field of $\nabla$ on $M$ and $h_{ij}$ (respectively $b_{ij}$) denotes the symmetric part (respectively the skew–symmetric part) of $t_{ij}$.

From (33)(i) it follows

$$t_{ij} = \frac{1}{k(n^2 - 4)} \left[ nR_{ij} - 2F^h_i F^k_j R_{hk} \right].$$

Replacing this expression of $t_{ij}$ in (33)(i), next using the second Bianchi identity we get that the condition (33)(ii) is identically verified.

Hence we may state

**Theorem 9.** Let $(M, g, F)$ be a Kähler manifold with real dimension $n > 2$ and consider on $TM$ the nonlinear connection defined by

$$N^k_i = \Gamma^k_{ij} y^j + g_{ik} y^h F^b_i g_{is} F^a_s y^h + \frac{1}{k(n^2 - 4)} g^{kh} (nR_{hi} - 2F^l_i F^s_l R_{ls}),$$

where $\Gamma^k_{ij}$ are the connection coefficients of the Levi Civita connection $\nabla$ of $g$, $R_{ij}$ are the components of the Ricci tensor defined by the curvature tensor of $\nabla$, $F^i_j$ are the components of $F$ and $k$ is a nonzero constant. Then the almost parahermitian manifold $(TM, G, P)$ where $G$ is defined by (7) and $P$ is defined by (19) is a parakaehlerian manifold if and only if $(M, g, F)$ has constant holomorphic sectional curvature.

Consider now $(M, g, S)$ a quaternion Kähler manifold. Then $M$ is a $4m$–dimensional manifold, $S$ is a subbundle with fibre dimension 3 of the vector bundle of tensors of type (1,1) on $M$ and, locally, $S$ has a canonical base $(F^1, F^2, F^3)$ such that

$$F^2_\alpha = -I; \quad F_\alpha \circ F_\beta = -F_\beta \circ F_\alpha = F_\gamma,$$

where $\alpha = 1, 2, 3$ and $(\alpha, \beta, \gamma)$ is any cyclic permutation of (1,2,3). Moreover, we have

$$g(F_\alpha X, F_\alpha Y) = g(X, Y); \quad X, Y \in \Gamma(TM), \quad \alpha = 1, 2, 3.$$

Denote by $\nabla$ the Levi Civita connection of $g$ which preserves $S$, i.e., locally

$$\nabla F_\alpha = \sum_{\beta=1,2,3} \eta_{\alpha\beta} \oplus F_\beta,$$
where \( \eta_{\alpha\beta} \) are locally defined 1–forms adapted with the Levi Civita connection \( \nabla \) of \( g \) (see \([4]\)) and \( \eta_{\alpha\beta} = -\eta_{\beta\alpha} \).

Consider on \( TM \) the \( M \)–tensor field defined by the components \( a_{ij} \) of the form (23):

\[
(35) \quad a_{ij} = k \{ g_{ik}g_{jk} y^h y^k - \sum_{\alpha=1,2,3} g_{is}(F_{\alpha})^h_k g_{jt}(F_{\alpha})^t_j y^h y^k \} + t_{ij},
\]

where \( (F_{\alpha})^h_j \) are the local coordinate components of \( F_{\alpha} \), \( k \) is a nonzero constant and \( t_{ij} \) are the components of an arbitrary tensor field of type (0,2) on \( M \) thought of as an \( M \)–tensor field on \( TM \). Then using (2), (6) and (35) we obtain by a straightforward computation

\[
R_{ijk} = \{ R_{iljk} + k [ t_{ik}g_{jl} - t_{ij}g_{kl} - t_{kj}g_{il} + t_{ji}g_{lk} + \]
\[
\sum_{\alpha=1,2,3} (t_{hj}(F_{\alpha})^h_i (F_{\alpha})^j_k g_{st} - t_{hk}(F_{\alpha})^h_i (F_{\alpha})^t_j g_{st} +
\]
\[
+ t_{hj}(F_{\alpha})^h_i (F_{\alpha})^t_j g_{st} - t_{hk}(F_{\alpha})^h_i (F_{\alpha})^t_j g_{st} + \}
\]
\[
= \{ - (F_{\alpha})^h_i (F_{\alpha})^j_k t_{lj} + (F_{\alpha})^j_k (F_{\alpha})^h_i t_{li} - (F_{\alpha})^j_k (F_{\alpha})^h_i t_{ij} \},
\]

By using (36) it follows by a straightforward computation that the condition (22)(ii) is equivalent to the following two relations

\[
(i) \quad R_{hijk} = \{ a_{ij} = \frac{1}{8km(m+2)} [(2m+3)R_{ij} - \sum_{\alpha=1,2,3} (F_{\alpha})^h_i (F_{\alpha})^j_k R_{hk}] \).
\]

Replacing this expression of \( t_{ij} \) in (37)(i), next using the second Bianchi identity we get that the condition (37)(ii) is identically verified.

Hence we may state
Theorem 10. Let \((M, g, S)\) be a quaternion Kaehler manifold with \(\dim M > 4\). Consider on \(TM\) the nonlinear conection defined by \(N^k_i\), where

\[
N^k_i = \Gamma^k_{ij} y^j + k [g_{ik} y^k - \sum_{\alpha=1,2,3} (F^k_\alpha g_{is} (F^s_\alpha)^k y^i y^h)] +
\]

\[
+ \frac{1}{8km(m+2)} g^{kh} [(2m + 3) R_{hi} - \sum_{\alpha=1,2,3} (F^k_\alpha)^h g^k_{is} R_{st}],
\]

where \(\Gamma^k_{ij}\) are the coefficients of the Levi Civita connection \(\nabla\) on \((M, g, S)\), \(k\) is a nonzero constant and \(R_{ij}\) are the components of the Ricci tensor defined by the curvature tensor of \(\nabla\). Then the almost parahermitian manifold \((TM, G, P)\) where \(G\) is given by (7) and \(P\) is defined by (19) is a parakehlerian manifold if and only if the quaternion Kaehler manifold \((M, g, S)\) has constant \(Q\)-sectional curvature (see [4]).

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Received: 31.V.1996

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