A NOTE ON COMMUTATIVITY OF RINGS WITH UNITY

BY

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Throughout this paper, \( R \) represents an associative ring with centre \( Z(R) = \{ x \in R \mid xy = yx, \forall y \in R \} \), and as usual for any pair of elements \( x, y \in R \), the commutator \([x, y] = xy - yx\) and anticommutator \( x \circ y = xy + yx\). For a positive integer \( m \), an element \( x \in R \) is said to be \( m \)-torsion free if and only if \( mx = 0 \) implies \( x = 0 \).

As it is well known, \( R \) is said to be commutative or anticommutative accordingly as \([x, y] = 0\) or \( x \circ y = 0\) for all \( x, y \in R \). It is logically interesting to investigate how far a ring is commutative or anticommutative if \([xy, yx] = 0\) or \(xy \circ yx = 0\) instead. In this direction, long back GUPTA [2] established that a division ring \( R \) is commutative if and only if \([xy, yx] = 0\).

This result motivated many researchers who generalized the result in various directions (see for example [1], [3], [4] and [5]). One of these generalizations is due to the first author [5] which asserts that a semi prime ring \( R \) in which \([xy, yx] = xy^2x - yx^2y \in Z(R)\) or \(xy \circ yx = xy^2x + yx^2y \in Z(R)\) is necessarily commutative. In the same paper the possibility of extending the result for arbitrary rings has been ruled out in view of the readily available non–commutative ring of \(3 \times 3\) strictly upper triangular matrices over the ring \( Z \) of integers which satisfies both the above conditions. One may notice that the mentioned ring does not contain unity and so the natural question arises whether a ring with unity satisfying any of the above conditions turns to be commutative. The following example shows that the above theorem is not valid for arbitrary rings even if it contains unity.

Example. Let \( R = \left\{ \begin{pmatrix} a_1 & a_2 & a_2 \\ 0 & a_1 & a_4 \\ 0 & 0 & a_1 \end{pmatrix} \mid a_1 \in Z, \ i = 1, 2, 3, 4 \right\} \). Then
$R$ is a non–commutative ring with unity which satisfies $xy^2x - yx^2y \in Z(R)$. Also, if we replace $Z$ by $GF(2)$ in the above example, then $R$ satisfies both the conditions, namely $xy^2x - yx^2y \in Z(R)$ and $xy^2x + yx^2y \in Z(R)$, but $R$ is not commutative.

This time we observe that the latter ring in the above example is of characteristic 2 and some appropriate conditions on the characteristic of the rings may imply commutativity. In this direction we prove very easily the following:

**Proposition 1.** A ring $R$ with unity is commutative if $xy^2x - yx^2y \in Z(R)$, for all $x, y \in R$ and the commutators of $R$ are 2-torsion free.

**Proof.** Since $xy^2x - yx^2y \in Z(R)$, $[xy^2x, x] = 0$, for all $x, y \in R$, and thus we have

$$(1) \quad x[y^2, x] + [yx^2y, x] = 0.$$ 

Replacing $x$ by $1 + x$ in (1), we get

$$(2) \quad x[y^2, x] + [y^2, x] + x[y^2, x] + [y^2, x] + 2[yxy, x] + [yx^2y, x] = 0,$$

which in view of (1) becomes

$$(3) \quad x[y^2, x] + [y^2, x] + 2[y^2, x] + 2[yxy, x] = 0.$$ 

Again, replacing $x$ by $1 + x$ in (2) and using (2), we obtain $4[y^2, x] = 0$, which using the torsion condition on commutators yields

$$(4) \quad [y^2, x] = 0.$$ 

Now, replacing $y$ by $1 + y$ in (3) and using (3), we find that $2[y, x] = 0$. Thus $[y, x] = 0$ and $R$ is commutative.

Interestingly using the same simple computations, we can prove the following result in a general setting.

**Proposition 2.** Let $R$ be a ring with unity in which there exists a positive integer $n > 2$ such that $xy^n x - yx^n y$ or $xy^n x + yx^n y$ is central for every ring elements $x$ and $y$. Then $R$ must be commutative provided the commutators in $R$ are $n!$–torsion free.
Proof. We establish the theorem for the rings satisfying \( xy^n x - yx^n y \in Z(R) \) and for the other condition the proof will follow on the same lines.

Since \( xy^n x - yx^n y \in Z(R) \), we have for all \( x, y \in R \), \([xy^n x - yx^n y, x] = 0\), or

\[
(4) \quad x[y^n, x]x = [yx^n y, x]
\]

Replace \( x \) by \( 1 + x \) in \((4)\) to get,

\[
[y^n, x] + x[y^n, x] + [y^n, x]x + x[y^n, x]x = [y(1 + nC_1 x + nC_2 x^2 + \cdots + nC_n x^n)y, x].
\]

This in view of \((4)\) forces,

\[
(5) \quad [y^n, x] + x[y^n, x] + [y^n, x]x = [y(1 + nC_1 x + nC_2 x^2 + \cdots + nC_n x^n)y, x].
\]

Again replacing \( x \) by \( 1 + x \) in \((5)\) and using \((5)\), we get

\[
(6) \quad 2[y^n, x] = \{n + nC_2 (1 + 2C_1 x) + nC_3 (1 + 3C_1 x + 3C_2 x^2) + \\
+ nC_4 (1 + 4C_1 x + 4C_2 x^2 + 4C_3 x^3) + \cdots + \\
+ nC_{n-1} (1 + nC_1 x + nC_2 x^2 + \cdots + nC_{n-2} x^{n-2}) \} y, x].
\]

Repeating the same process third and fourth time, we get respectively,

\[
0 = [y\{nC_2^2 C_1 + nC_3^2 C_2 (1 + 2C_1 x) + nC_2^2 (1 + 2C_1 x) + \\
+ nC_3 (1 + 3C_1 x + 3C_2 x^2) + \cdots + nC_{n-2} (1 + nC_1 x + nC_2 x^2 + \cdots + nC_{n-3} x^{n-3}) \} y, x]
\]

and

\[
0 = [y\{nC_3^3 C_2^2 C_1 + nC_4^3 C_2^2 C_1 + nC_3^3 C_2 (1 + 2C_1 x) + \\
+ \cdots + nC_{n-1} (nC_2^2 C_1 + nC_3^2 C_2 (1 + 2C_1 x) + \\
+ \cdots + nC_{n-2} (nC_2 C_1 + nC_2 (1 + 2C_1 x) + \\
+ \cdots + nC_{n-3} (1 + nC_1 x + nC_2 x^2 + \cdots + nC_{n-4} x^{n-4}) \} y, x].
\]

Thus repeating the same process \( n \)-times of replacing \( x \) by \( 1 + x \) and using the previously obtained identity at each stage, we finally obtain

\[
0 = [y(nC_{n-1} n^{-1} C_{n-2} n^{-2} C_{n-3} \cdots 2C_1) y, x].
\]
This gives that $n![y^2, x] = 0$ and in view of the torsion condition on the commutators, we get $[y^2, x] = 0$.

As in the end of the proof of Proposition 1, this in view of the hypothesis that $n > 2$ renders $R$ commutative.

**Remark.** In fact, the end part of the above proof establishes the result for $n = 0$, provided commutators of $R$ are 2-torsion free. Similarly, the result for $n = 1$ can also be established.

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