GENERALIZED METRIC SPACES
AND TOPOLOGICAL STRUCTURE I

BY

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Abstract. In this paper some results in $D$–metric spaces are obtained and the notion of open and closed balls is introduced. The $D$-metric topology is defined which is further studied for its topological properties, completeness and compactness properties of $D$-metric spaces.

1. Introduction. The study of ordinary metric spaces is fundamental in topology and functional analysis. In the past two decades this structure has gained much attention of the mathematicians because of the development of the fixed point theory in ordinary metric spaces. During the sixties the notion of a 2-metric space is introduced by GAHLER [7], [8] in a series of papers which he claimed to be a generalization or ordinary metric spaces. This structure is as follows:

Let $X$ be a nonempty set and $\mathbb{R}$ denote the set of all real numbers. A function $d : X \times X \times X \to \mathbb{R}$ is said to be a 2-metric on $X$ if it satisfies the following properties.

(i) For distinct points $x, y \in X$, there is a point $z \in X$ such that $d(x, y, z) \neq 0$.
(ii) $d(x, y, z) = 0$ if any two elements of the triplet $x, y, z \in X$ are equal.
(iii) $d(x, y, z) = d(x, z, y) = \cdots$ (symmetry)
(iv) $d(x, y, z) \leq d(x, y, a) + d(x, a, z) + d(a, y, z)$ for all $x, y, z \in X$ (triangle inequality)

A nonempty set $X$ together with a 2-metric $d$ is called a 2-metric space. In [7], GAHLER claims that 2-metric function is a generalization of an ordinary metric function, but we do not see any relation among these two functions. Also ordinary metric is a continuous function, whereas 2-metric is not a continuous function (see HA et al. [12]). It is mentioned
in GAHLER [8] that the notion of a 2-metric is an extension of an idea of ordinary metric and geometrically \( d(x, y, z) \) represents the area of a triangle formed by the points \( x, y \) and \( z \) in \( X \) as its vertices. But this is not always true: see for example SHARMA [13]. Recently, 2-metric spaces are exploited for deducing fixed point theorems by several authors (for example see ISEKI [9], ROAHDES [11] and SHARMA [13] etc.) and at present a vast literature is available in this direction. It is worthwhile to mention that we do not find any relation among all important theorems and particularly between the contraction mapping theorems in complete ordinary metric and 2-metric spaces. Some reviewers, while commenting on some fixed point theorems in 2-metric spaces expressed their views that there is a need to improve the basic structure of a 2-metric space so that the fixed point theorems in 2-metric spaces could have some meaning in relation to other branches of mathematics. All these considerations and natural generalization of an ordinary metric led the present author to introduce a new structure of a generalized metric space called \( D \)-metric space in his Ph.D. thesis [2], see also, for example DHAGE [3], [4]. This structure of \( D \)-metric space is quite different from a 2-metric space and natural generalization of an ordinary metric space in some sense. Also we do find the relation between the contraction mapping theorem in these two spaces. Therefore it is of importance to study \( D \)-metric space for its other properties. In the present paper we discuss the topological properties of a \( D \)-metric space. the rest of the paper is organized as follows.

In section II, we give the definitions and examples of \( D \)-metric space. Section III deals with the open and the closed balls in \( D \)-metric spaces. Section IV deals with the \( D \)-metric topology and continuity of \( D \)-metric function. The topological separation properties of a \( D \)-metric space are discussed in section V. Finally the completeness and compactness properties of a \( D \)-metric space are given in section VI.

2. \( D \)-metric spaces. Throughout this paper, unless otherwise mentioned, we let \( X \) denote a nonempty set and \( \mathbb{R} \) the set of real numbers. A function \( D : X \times X \times X \rightarrow \mathbb{R} \) is said to be a \( D \)-metric on \( X \) if it satisfies the following properties.

\[
\begin{align*}
(i) & \quad D(x, y, z) \geq 0 \text{ for all } x, y, z \in X \text{ and equality holds if and only if } x = y = z \text{ (nonnegativity)} \\
(ii) & \quad D(x, y, z) = D(x, z, y) \quad \text{(symmetry)} \\
(iii) & \quad D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z) \text{ for all } x, y, z \in X \quad \text{(tetrahedral inequality)}
\end{align*}
\]
A nonempty set $X$ together with a $D$-metric $D$ is called a $D$-metric space and is denoted by $(X, D)$. The generalization of a $D$-metric space with $D$-metric as a function of $n$ variables is given in DHAGE [3]. Below we give few examples of $D$-metric spaces.

**Example 2.1.** Define the function $\sigma$ and $\rho$ on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ for $n \in \mathbb{N}$, $\mathbb{N}$ denote the set of natural numbers, by

$$\sigma(x, u, z) = k \max\{\|x - y\|, \|y - z\|, \|z - x\|\}, \quad k > 0,$$

$$\rho(x, y, z) = c\{\|x - y\| + \|y - z\| + \|z - x\|\}, \quad c > 0$$

for all $x, y, z \in \mathbb{R}^n$, where $\|\cdot\|$ is the usual norm in $\mathbb{R}^n$. Then $(\mathbb{R}^n, \sigma)$ and $(\mathbb{R}^n, \rho)$ are $D$-metric spaces.

**Example 2.2.** Define a function $D$ on $X^3$ by

$$D(x, y, z) = \begin{cases} 0 & \text{if } x = y = z \\ 1 & \text{otherwise.} \end{cases}$$

then $(X, D)$ is a $D$-metric space.

**Example 2.3.** Let $E$ denote the set of all ordered pairs $x = (x_1, x_2)$ of real numbers. Then the function $D$ on $E^3$ defined by

$$(x, y, z) = \max\{|x_1 - y_1|, |x_2 - y_2|, |y_1 - z_1|, |y_2 - z_2|, |z_1 - x_1|, |z_2 - x_2|\}$$

is a $D$-metric on $E$ and hence $(E, D)$ is a $D$-metric space.

**Remark 2.1.** If $d$ is a standard ordinary metric on $X$ then we define the functions $D_1$ and $D_2$ on $X^3$ by

$$(2.5) \quad D_1(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$$

and

$$(2.6) \quad D_2(x, y, z) = d(x, y) + d(y, z) + d(z, x)$$

for all $x, y, z \in Z$. Clearly $D_1$ and $D_2$ are $D$-metrics on $X$ (see DHAGE [3]) and are called the standard $D$-metrics on $X$.

Geometrically, the $D$-metric $D_1$ represents the diameter of a set consisting of three points $x, y$ and $z$ in $X$ and the $D$-metric $D_2(x, y, z)$ represents the perimeter of a triangle formed by three points $x, y, z$ in $X$ as its vertices.

**Theorem 2.1.** Let $(X_1, \rho_1)$ and $(X_2, \rho_2)$ be two $D$-metric spaces. Then $(X, \rho)$ is also a $D$-metric space, where $X = X_1 \times X_2$ and

$$
\rho(x, y, z) = \max\{\rho_1(x_1, y_1, z_1), \rho_2(x_2, y_2, z_2)\} \text{ for } x, y, z \in X
$$

**Proof.** Obviously first two conditions viz., nonnegativity and symmetry are satisfied. To prove the rectangle inequality, let $x, y, z, a \in X = X_1 \times X_2$ with $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2)$ and $a = (a_1, a_2)$. Then we have

$$
\rho(x, y, z) = \max\{\rho_1(x_1, y_1, z_1), \rho_2(x_2, y_2, z_2)\},
$$

$$
\leq \max\{\rho_1(a_1, y_1, z_1) + \rho_1(x_1, y_1, a_1) + \rho_2(x_1, a_1, z_1), \rho_2(a_2, y_2, z_2) + \rho_2(x_2, y_2, a_2) + \rho_2(x_2, a_2, z_2)\},
$$

$$
\leq \max\{\rho_1(x_1, y_1, a_1), \rho_2(x_2, y_2, a_2)\} + \max\{\rho_1(x_1, a_1, z_1), \rho_2(x_2, a_2, z_2)\} + \max\{\rho_1(a_1, y_1, z_1), \rho_2(a_2, y_2, z_2)\},
$$

$$
= \rho(x, y, a) + \rho(x, a, z) + \rho(a, y, z).
$$

Hence $(X, \rho)$ is a $D$-metric space.

Let $X$ be a $D$-metric space with $D$-metric $D$. Then the diameter $\delta(X)$ of $X$ is defined by

$$
\delta(X) = \sup\{D(x, y, z) \mid x, y, z \in X\}
$$

**Definition 2.1.** A $D$-metric space $X$ is said to be bounded if there exists a constant $M > 0$ such that $D(x, y, z) \leq M$ for all $x, y, z \in X$. A $D$-metric space $X$ is said to be unbounded if it is not bounded, in that case, $D(x, y, z)$ takes values as large as we please.

**Remark 2.2.** It is clear that $\delta(X) < \infty$ iff $X$ is a bounded $D$-metric space.

**Theorem 2.2.** Let $(X, D)$ be a $D$-metric space and let $M > 0$ be a fixed positive real number. Then $(X, \overline{D})$ is a bounded $D$-metric space with bound $M$, where $\overline{D}$ is given by

$$
\overline{D}(x, y, z) = \frac{MD(x, y, z)}{k + D(x, y, z)}, \ k > 0
$$

for all $x, y, z \in X$. 
Proof. We first show that $D$ is a $D$–metric on $X$. Obviously first two properties, viz., nonnegativity and symmetry of a $D$–metric are satisfied. We only prove the rectangle inequality.

Let $x, y, z, a \in X$. Then we have

$$D(x, y, z) = \frac{MD(x, y, z)}{k + D(x, y, z)} = M - \frac{Mk}{k + D(x, y, z)} \leq M - \frac{k + D(x, y, z) + D(x, a, z) + D(a, y, z)}{k + D(x, y, z)}$$

$$= \frac{M[D(x, y, a) + D(x, a, z) + D(a, y, z)]}{k + D(x, y, a) + D(x, a, z) + D(a, y, z)}$$

$$= \frac{MD(x, y, a)}{k + D(x, y, a) + D(x, a, z) + D(a, y, z)} + \frac{MD(x, a, z)}{k + D(x, a, z) + D(a, y, z)}$$

$$+ \frac{MD(a, y, z)}{k + D(a, y, z) + D(x, a, z) + D(x, y, z)}$$

$$\leq \frac{MD(x, y, a)}{k + D(x, y, a)} + \frac{MD(x, a, z)}{k + D(x, a, z)} + \frac{MD(a, y, z)}{k + D(a, y, z)}$$

$$= D(x, y, a) + D(x, a, z) + D(a, y, z).$$

Thus $D$ satisfies all the properties of a $D$-metric on $X$ and hence $D$ is a $D$-metric on $X$.

This further implies that $(X, D)$ is a $D$-metric space. Next we prove that $X$ is a bounded $D$-metric space w.r.t. $D$. Let $x, y, z \in X$. Then we have

$$D(x, y, z) = \frac{MD(x, y, z)}{k + D(x, y, z)} \leq \frac{MD(x, y, z)}{D(x, y, z)} = M.$$ 

This shows that $(X, D)$ is bounded with $D$-bound $M$. The proof is complete.

**Corollary 2.1.** If $(X, D)$ is any $D$-metric space then $(X, \overline{D})$ is a bounded $D$-metric space with $D$-bound 1, where $\overline{D}$ is given by

$$\overline{D}(x, y, z) = \frac{D(x, y, z)}{1 + D(x, y, z)}$$

for all $x, y, z \in X$. 
Theorem 2.3. Let $S$ denote the space of all real sequences $x = \{x_n\}$, and let $D$ be a function on defined by

$$D(x, y, z) = \sum_{n=1}^{\infty} A_n \max \left\{ \frac{|x_n - y_n|}{1 + |x_n - y_n|}, \frac{|y_n - z_n|}{1 + |y_n - z_n|}, \frac{|z_n - x_n|}{1 + |z_n - x_n|} \right\}$$

for all $x, y, z \in S$, where $\sum_{n}^{\infty} A_n$ is a convergent series of positive terms. Then $(S, D)$ is a bounded $D$-metric space.

Proof. Clear $D$ satisfies all the properties of a $D$-metric and hence $(S, D)$ is a $D$-metric space. Let $x, y, z \in S$, then we have $D(x, y, z) < \sum_{n=1}^{\infty} A_n$. Therefore, the $D$-metric space $(S, D)$ is bounded.

Theorem 2.4. Let $(X_1, \rho_1)$ and $(X_2, \rho_2)$ be two bounded $D$-metric spaces with $D$-bounds $M_1$ and $M_2$ respectively. Then the $D$-metric spaces $(X, \rho)$ is bounded with $D$-bound $M = \max\{M_1, M_2\}$, where $X = X_1 \times X_2$ and $\rho$ is defined as in Theorem 2.1.

Proof. Since $(X_1, \rho_1)$ and $(X_2, \rho_2)$ are bounded, we have

$$\rho_1(x_1, y_1, z_1) \leq M_1 \quad \text{for all } x_1, y_1, z_1 \in X_1; \quad \text{and}$$
$$\rho_2(x_2, y_2, z_2) \leq M_2 \quad \text{for all } x_2, y_2, z_2 \in X_2.$$

By definition of $\rho$, we obtain

$$\rho(x, y, z) = \max\{\rho_1(x_1, y_1, z_1), \rho_2(x_2, y_2, z_2)\} \leq \max\{M_1, M_2\} = M$$

for all $x, y, z \in X$.

This shows that $(X, \rho)$ is bounded with $D$-bound $M$. The proof is complete.

Theorem 2.5. Let $X$ denote the set of bounded sequences $x = \{x_n\}$ of real numbers and let a function $D$ on $X^3$ be defined by

$$D(x, y, z) = \max \left\{ \sup_n |x_n - y_n|, |y_n - z_n|, |z_n - x_n| \right\}$$

for all $x, y, z \in X$. Then $(X, D)$ is an unbounded $D$-metric space.
Proof. Obviously $D$ satisfies all the properties of a $D$-metric and hence $(X, D)$ is a $D$-metric space.
Since for every positive number $k$, one has
$$D(kx, ky, kz) = kD(x, y, z)$$
the $D$-metric space $(X, D)$ is unbounded.

3. Open and closed balls. Let $x_0 \in X$ be fixed and $r > 0$ given. The ball centered at $x_0$ and or radius $r$ in $X$ is the set $B^*(x_0, r)$ in $X$ given by

$$(3.1) \quad B^*(x_0, r) = \{y \in X \mid D(x_0, y, y) < r\}$$

Similarly by $\overline{B}^*(x_0, r)$ denote the closure of $B^*(x_0, r)$ in $X$ i.e.
$$\overline{B}^*(x_0, r) = \{y \in X \mid D(x_0, y, y) \leq r\}$$

It is shown in DHAGE [4] that $B^*(x_0, r)$ is an open set in $X$ i.e. it contains a ball of each of its point provided the $D$-metric $D$ satisfies the following condition
(iv) $D(x, y, z) \leq D(x, z, y) + D(z, y, y)$ for all $x, y, z \in X$.

There do exist $D$-metrics satisfying the condition (iv). Actually all the $D$-metrics defined in section 2 satisfy this condition. The details of this point is given in DHAGE [4].

Let us define another ball $B(x_0, r)$ in $X$ by

$$(3.3) \quad B(x_0, r) = \{y \in B^*(x_0, r) \mid \text{if } y, z \in B^*(x_0, r) \text{ are any two points then } D(x_0, y, z) < r\}$$

Remark 3.1. It is clear that $B(x_0, r) \subset B^*(x_0, r)$.

Remark 3.2. If $0 < r_1 < r_2$ then
(i) $B^*(x_0, r_1) \subset B^*(x_0, r_2)$ and
(ii) $B(x_0, r_1) \subset B(x_0, r_2)$

By $\overline{B}(x_0, r)$ we mean a set in $X$ given by
$$\overline{B}(x_0, r) = \{y \in B^*(x_0, r) \mid \text{if } y, z \in B^*(x_0, r) \text{ then } D(x_0, y, z) \leq r\}$$
\quad = \{y, z \in X \mid D(x_0, y, z) \leq r\}$$
It is clear that $B(x_0, r) \subseteq \overline{B}(x_0, r)$.

Below we give some results concerning the balls $B^*(x_0, r)$ and $B(x_0, r)$ in a $D$-metric space $X$.

**Theorem 3.1.** Let $(\mathbb{R}, D_1)$ be a $D$-metric space. Then for a fixed $x_0 \in \mathbb{R}$, the balls $B^*(x_0, r)$ and $B(x_0, r)$ are the sets in $\mathbb{R}$ given by

$B^*(x_0, r) = (x_0 - r, x_0 + r)$ and $B(x_0, r) = (x_0 - r/2, x_0 + r/2)$.

**Proof.** Let $x, y, z \in \mathbb{R}$ be arbitrary. By definition of $D_1$ on $\mathbb{R}$, we have $D_1(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}$.

Let $x_0 \in \mathbb{R}$ be fixed and $r > 0$ given. Then

\[ B^*(x_0, r) = \{ y \in \mathbb{R} \mid D_1(x_0, y, y) < r \} = \{ y \in \mathbb{R} \mid |x_0 - y| < r \} = (x_0 - r, x_0 + r). \]

Again

\begin{equation}
B(x_0, r) = \{ y \in \mathbb{R} \mid D_1(x_0, y, z) < r \text{ for all } z \in B(x_0, r) \} = \{ y \in \mathbb{R} \mid \max\{|x_0 - y|, |y - z|, |z - x_0|\} < r \}
\end{equation}

The relation (3.5) implies that the set $B(x_0, r)$ contains all the points $y, z \in \mathbb{R}$ for which one has

\begin{equation}
|x_0 - y| < r, \quad |x_0 - z| < r \text{ with } |y - z| < r
\end{equation}

In order to hold the inequalities in (3.6) we must have

\begin{equation}
|x_0 - y| + |x_0 - z| < r,
\end{equation}

since $|y - z| \leq |y - x_0| + |x_0 - z|$. Therefore if we take $|x_0 - y| < r/2$ and $|x_0 - z| < r/2$ then the inequalities in (3.6) are satisfied. Thus we have

$B(x_0, r) = \{ y \in \mathbb{R} \mid |x_0 - y| < r/2 \} = (x_0 - r/2, x_0 + r/2)$.\]

This completes the proof.

**Example 3.1.** The balls $B^*(0, 1), B^*(1, 2), B(0, 1)$ and $B(0, 2)$ in $(\mathbb{R}, D_1)$ are given by $B^*(0, 1) = (-1, 1)$, $B^*(1, 2) = (-1, 3)$, $B(0, 1) = (x_0 - 1/2, x_0 + 1/2)$ and $B(1, 2) = (0, 2)$.
Theorem 3.2. Let \((\mathbb{R}, D_2)\) be a \(D\)-metric space and let \(x_0 \in \mathbb{R}\) be fixed and \(\varepsilon > 0\) given. Then balls \(B^*(x_0, r)\) and \(B(x_0, r)\) are the sets in \(\mathbb{R}\) given by \(B^*(x_0, r) = (x_0 - r/2, x_0 + r/2)\) and \(B(x_0, r) = (x_0 - r/4, x_0 + r/4)\).

Proof. By definition of \(D_2\), we have

\[
D_2(x, y, z) = |x - y| + |y - z| + |z - x|
\]
for all \(x, y, z \in \mathbb{R}\). Now

\[
B^*(x_0, r) = \{ y \in \mathbb{R} \mid D_2(x_0, y, z) < r \} = \{ y \in \mathbb{R} \mid 2|x_0 - y| < r \} = \{ y \in \mathbb{R} \mid |x_0 - y| < r/2 \} = \{ x_0 - r/2, x_0 + r/2 \}
\]

Again

\[
B^*(x_0, r) = \{ y \in \mathbb{R} \mid D_2(x_0, y, z) < r, \text{ for all } z \in B(x_0, r) \} = \{ y \in \mathbb{R} \mid 2|x_0 - y| < r \} = \{ y \in \mathbb{R} \mid |x_0 - y| + |y - z| + |z - x_0| < r \}
\]

Now

\[
|y - z| \leq |y - x_0| + |x_0 - z|
\]
for all \(y, z \in \mathbb{R}\). If we take all those points \(y, z \in \mathbb{R}\) for which the inequality

\[
2|x_0 - y| + 2|x_0 - z| < r
\]
or

\[
|x_0 - y| + |x_0 - z| < r/2
\]
is satisfied, then the points \(y, z\) are in \(B(x_0, r)\), because

\[
|x_0 - y| + |y - z| + |z - x_0| \leq 2|x_0 - y| + 2|x_0 - z|.
\]

Therefore, in order to hold (3.10) for \(y, z \in \mathbb{R}\), we must have \(|x_0 - y| < r/4\) and \(|x_0 - z| < r/4\). Hence

\[
B(x_0, r) = \{ y \in \mathbb{R} : |x_0 - y| < r/4 \} = (x_0 - r/4, x_0 + r/4).
\]
The proof is complete.
Example 3.2. The balls $B^*(0, 1), B^*(1, 2), B(0, 1)$ and $B(1, 2)$ are the sets in $(\mathbb{R}, D_2)$ given by $B^*(0, 1) = (-1/2, 1/2), B^*(1, 2) = (0, 2), B(0, 1) = (-1/4, 1/4)$ and $B(1, 2) = (1 - 2/4, 1 + 2/4) = (1/2, 3/2)$.

Theorem 3.3. Let $(\mathbb{R}^n, D_1)$ be a $D$-metric space. Let $x_0 \in \mathbb{R}^n$ be fixed and $r > 0$ given. Then the balls $B^*(x_0, r)$ and $B(x_0, r)$ are the set in $\mathbb{R}^n$ given by
\[
B^*(x_0, r) = \{ y \in \mathbb{R}^n \mid \|x_0 - y\| < r \} \quad \text{and} \quad B(x_0, r) = \{ y \in \mathbb{R}^n \mid \|x - y\| < r/2 \}.
\]

Theorem 3.4. Let $(\mathbb{R}^n, D_2)$ be a $D$-metric space. Let $x_0 \in \mathbb{R}^n$ be fixed and $r > 0$ given. Then the balls $B^*(x_0, r)$ and $B(x_0, r)$ are the set in $\mathbb{R}^n$ given by
\[
B^*(x_0, r) = \{ y \in \mathbb{R}^n \mid \|x_0 - y\| < r/2 \} \quad \text{and} \quad B(x_0, r) = \{ y \in \mathbb{R}^n \mid \|x - y\| < r/4 \}.
\]

The proofs of Theorems 3.3 and 3.4 are similar to the Theorems 3.1 and 3.2 and hence we omit the details.

Next we define the balls $B^*(x_0, r)$ and $B(x_0, r)$ in a $D$-metric space $X$ by
\[
\overline{B}^*(x_0, r) = \{ y \in X \mid D(x_0, y, y) \leq r \} \quad \text{and} \quad \overline{B}(x_0, r) = \{ y \in X \mid D(x_0, y, z) \leq r \text{ for all } z \in B(x_0, r) \}.
\]

Remark 3.3. It is clear that
\[
B^*(x_0, r) \subset \overline{B}^*(x_0, r) \quad \text{and} \quad B(x_0, r) \subset \overline{B}(x_0, r).
\]

Lemma 3.1. If there is a point $a \in B(x_0, r)$ with $D(x_0, a, a) = r_1 < r$, then $\overline{B}(x_0, r_1) \subset B(x_0, r)$.

Proof. The proof is obvious.

Definition. A set $U$ in a $D$-metric space is said to be open if it contains a ball of each of its points.

Theorem 3.5. Every ball $B(x, r)$, $x \in X$, $r > 0$ is an open set in $X$ i.e. it contains a ball of each of its points.
Proof. Let $x_0 \in X$ be arbitrary and $r > 0$ given. Consider the ball $B(x_0, r)$ in $X$ and suppose that $a \in B(x_0, r)$. We show that there is an $r^* > 0$, $r^* < r$ such that $B(a, r^*) \subset B(x_0, r)$. Since $a \in B(x_0, r)$, there is a number $r_1 > 0$ such that $D(x_0, a, a) = r_1$ and $r_1 < r$. We may choose an arbitrary $\varepsilon > 0$ such that $B^*(x_0, r_1 + \varepsilon) \subset B(x_0, r)$ which is possible in view of $r_1 < r$. Since $B^*(x_0, r_1 + \varepsilon)$ is open (DHAGE [4]), there is an open ball $B^*(a, r^*)$, $r^* < 0$ such that $B^*(a, r^*) \subset B^*(x_0, r_1 + \varepsilon) \subset B(x_0, r)$. Again by Remark 3.1, $B(a, r^*) \subset B^*(a, r^*)$. Hence $B(a, r^*) \subset B(x_0, r^*)$. This proves that $B(x_0, r)$ is an open set in $X$.

Theorem 3.6. Arbitrary union and finite intersection of open balls $B(x, r)$, $x \in X$ is open.

Proof. The proof is similar to ordinary metric space case and hence we omit the details.

Definition 3.3. A set $V$ is a $D$-metric space. $X$ is said to be closed if its complement $X \setminus V$ in $X$ is $\tau$-open.

Obviously, the ball $B(x, r)$, $x \in X$, $r > 0$ is a closed set in a $D$-metric space $X$. In this case we say $B(x, r)$ is a closed ball in $X$. Below we give some examples of closed balls in $X$.

Example 3.3. Let $(\mathbb{R}, D_1)$ be a $D$-metric space. Then the closed balls $\overline{B}^*(0, 1)$, $\overline{B}^*(1, 4)$, $\overline{B}(0, 1)$ and $\overline{B}(1, 4)$ are the sets in $\mathbb{R}$ given by $\overline{B}^*(0, 1) = [-1, 1]$, $\overline{B}^*(1, 4) = [-3, 5]$, $\overline{B}(0, 1) = [-1/2, 1/2]$, $\overline{B}[1, 4] = [-1, 3]$.

Example 3.4. Let $(\mathbb{R}, D_2)$ be a $D$-metric space. Then the closed balls $\overline{B}^*(0, 2)$, $\overline{B}^*(2, 6)$, $\overline{B}(0, 2)$ and $\overline{B}(2, 6)$ are the sets in $\mathbb{R}$ given by $\overline{B}^*(0, 2) = [-1, 1]$, $\overline{B}^*(2, 6) = [-1, 5]$, $\overline{B}(0, 2) = [-1/2, 1/2]$, $\overline{B}(2, 6) = [1/2, 7/2]$.

Theorem 3.7. Finite union and arbitrary intersection of closed balls in a $D$-metric space is closed.

Proof. The proof is similar to ordinary metric space case.

Theorem 3.8. Every ball $\overline{B}(x_0, r)$ is $\tau$-closed.

Proof. To prove the conclusion, we prove that the complement $(\overline{B}(x_0, r))^\prime$ of $\overline{B}(x_0, r)$ in $X$ is open. Let $a \in (\overline{B}(x_0, r))^\prime$ be any point. Then there is a number $r_1 > 0$ such that $D(x_0, a, a) = r_1$. Without loss of generality, we may assume that $r_1 > r$. Consider an open ball $B(a, \rho)$
centered at $a$ of radius $\rho = r_1 - r > 0$. Then for any $y \in B(a, \rho)$, one has by (iv),
$$D(x_0, y, y) \geq D(x_0, a, a) - D(y, a, a) > r_1 - \rho = r.$$ 
This shows that $y \in (B^*(x_0, r))'$. Since $(B(x_0, r))' \supset (B^*(x_0, r))'$, we have $y \in (B(x_0, r))'$. As $y \in B(a, \rho)$ is arbitrary, so $B(a, \rho) \subset (B(x_0, r))'$. Hence $B(x_0, r)$ is a closed set in $X$ w.r.t. the topology $\tau$. The proof is complete.

4. $D$-metric topology. In this section we discuss the topology on a $D$-metric space $X$. We first show the collection $\mathcal{B} = \{B(x, \varepsilon) : x \in X\}$ of all $\varepsilon$-balls induces a topology on $X$ called the $D$-metric topology on $X$.

**Theorem 4.1.** The collection $\mathcal{B} = \{B(x, \varepsilon) : x \in X, \varepsilon > 0\}$ of all balls is a basis for a topology $\tau$ on $X$.

**Proof.** Let $\tau$ be a given topology on $X$. To show that the collection $\mathcal{B}$ is a basis for $\tau$ it is enough to prove that the collection $\mathcal{B}$ satisfies the following two conditions:

(i) $X(\subset \bigcup_{x \in X} B(x, \varepsilon))$, and

(ii) if $a \in B(x, \varepsilon) \cap B(y, \varepsilon)$, for some $x, y \in X$, is any point, then there is a ball $B(a, \varepsilon^*)$ for some $\varepsilon^* > 0$ such that $B(a, \varepsilon_1) \subset B(x, \varepsilon)$ and $B(a, \varepsilon_2) \subset B(y, \varepsilon)$. Choose $\varepsilon^* = \min\{\varepsilon_1, \varepsilon_2\}$ then by Remark, $B(a, \varepsilon^*) \subset B(x, \varepsilon) \cap B(y, \varepsilon)$.

This complete the proof.

Thus the $D$-metric space $X$ together with a topology $\tau$ generated by $D$-metric $D$ is called a $D$-metric topological space and $\tau$ is called a $D$-metric topology on $X$.

A topological space $X$ is said to be $D$-metrizable if there exists a $D$-metric $D$ on $X$ that induces the topology of $X$. A $D$-metric space $X$ is $D$-metrizable space together with the specific $D$-metric $D$ that induces the topology of $X$.

A set $U$ is $\tau$-open in $X$ in the $D$-metric topology $\tau$ induced by the $D$-metric $D$ if and only if for each $x \in U$, there is a $\delta > 0$ such that $B_D(x, \delta) \subset U$. Similarly, a set $V$ in $X$ is called $\tau$-closed if its complement $X \setminus V$ is $\tau$-open in $X$.

In [2], [3], the notion of the convergence of a sequence in a $D$-metric space $X$ is given. Below in the following we discuss the relation between the $D$-metric topology $\tau$ and the topology of $D$-metric convergence in $X$. For, we need the following definition in the sequel.
Definition 4.1. A sequence \( \{x_n\} \) in a \( D \)-metric space \( X \) is said to be convergent and converges to a point \( x_0 \in X \) if for \( \varepsilon > 0 \) there exists an \( n_0 \in N \) such that \( D(x_m, x_n, x_0) < \varepsilon \) for all \( m, n \geq n_0 \).

Theorem 4.2. The topology of \( D \)-metric convergence and the \( D \)-metric topology on a \( D \)-metric space are equivalent.

Proof. We prove the theorem by showing that a sequence in \( X \) converges in the topology of \( D \)-metric convergence if and only if it converges in the \( D \)-metric topology on \( X \).

Let \( \varepsilon > 0 \) be arbitrary and consider an \( \varepsilon \)-ball \( B(x_0, \varepsilon) \) in \( X \). Consider a sequence \( \{x_n\} \) in \( X \) converging to a point \( x_0 \in X \) in the topology of \( D \)-metric convergence. We show that for sufficiently large value of \( m, n, x_m \) and \( x_n \) are in \( B(x_0, \varepsilon) \). Since \( x_n \to x_0 \) by definition of convergence for \( \varepsilon > 0 \), there exists an \( n_0 \in N \) such that for all \( m, n \geq n_0 \), \( D(x_m, x_n, x_0) < \varepsilon \). By definition of an open ball \( B(x_0, \varepsilon) \) this implies that \( x_m, x_n \in B(x_0, \varepsilon) \) for all \( m, n \geq n_0 \).

Conversely, suppose that the sequence \( \{x_n\} \) in \( X \) converges to a point \( x_0 \in X \) in the \( D \)-metric topology \( \tau \) on \( X \). Then there is an \( n_0 \in N \) such that \( x_n \in B(x_0, \varepsilon) \) for all \( n \geq n_0 \). If \( m \geq n_0 \), \( x_m \in B(x_0, \varepsilon) \). Now by the definition of the ball \( B(x_0, \varepsilon) \) implies that \( D(x_m, x_n, x_0) < \varepsilon \) for all \( m, n \geq n_0 \). Thus \( x_n \to x_0 \) in the topology of \( D \)-metric convergence if and only if \( x_n \to x_0 \) in the \( D \)-metric topology \( \tau \) on \( X \). This completes the proof.

Next we prove the continuity of the \( D \)-metric function \( D \) on \( x^3 \) in the \( D \)-metric topology \( \tau \) on \( X \). For, we need the following lemma, the proof of which is simple and follows from the rectangle inequality of the \( D \)-metric \( D \) on \( X \).

Lemma 4.1. In a \( D \)-metric space \( X \),

(i) \( |D(x, y, z) - D(x, y, z')| \leq D(x, z, z') + D(y, z, z') \)

for all \( x, y, z, z' \in X \).

(ii) \( |D(x, y, z) - D(x', y', z)| \leq D(x', x, z) + D(x, y, y') + D(y', y, z) + D(x', y, z) \)

for all \( x, y, z, x', y', z' \in X \) and

(iii) \( |D(x, y, z) - D(x', y', z)| \leq D(x, x', z) + D(x', y', z') + D(y', y, z) + D(z, y', z') + D(z, z', z) \)

for all \( x, y, z, x', y', z' \in X \).

Theorem 4.3. The \( D \)-metric function \( D(x, y, z) \) is continuous in one variable.

Proof. Let \( \varepsilon > 0 \) be given and let \( x, y, z \in X \) be such that \( D(x, y, z) < \varepsilon/2 \). Now, for any \( x' \in X \), by Lemma 4.1 (i), we have

\[
|D(x, y, z) - D(x', y, z)| \leq D(x, y, x') + D(x, z, x')
\]
Take $x' \in B(x, \varepsilon/2)$, then from (4.1), we get $|D(x, y, z) - D(x', y, z)| < \varepsilon$. This proves that $D(x, y, z)$ is continuous in the variable $x$. Similarly, it can be proved that $D(x, y, z)$ is continuous in the variable $y$ or $z$. This proof is complete.

**Theorem 4.4.** The $D$-metric function $D(x, y, z)$ is continuous in all its three variables.

**Proof.** Let $\varepsilon > 0$ be given and let $x, y, z \in X$ be such that $D(x, y, z) < \varepsilon/b$. Then for any $x', y', z' \in X$, by Lemma 4.1 (iii), we get

$$|D(x, y, z) - D(x', y, z)| \leq D(x, y, x') + D(x, z, x') + D(y, z, y') + D(y, y', z) + D(z, z', x')$$

Take $(x', y', z') \to (x, y, z)$, that is

$$x', y', z' \in B(x, \varepsilon/2) \cap B(y, \varepsilon/2) \cap B(z, \varepsilon/2).$$

Then from inequality (4.3), we obtain

$$|D(x, y, z) - D(x', y', z')| < \varepsilon.$$  
This proves that $D(x, y, z)$ is continuous function in all its three variables.

**Remark 4.1.** In $x_n \to x$, then by continuity of $D$, we have

$$\lim_{m,n \to \infty} D(x_m, x_n, x) = \lim_{n \to \infty} D(x_n, x, x).$$

We have seen that every $D$-metric $D$ on a set $X$ induces a topology for $X$. Now the question is whether for a given topological space there exists a $D$-metric on $X$ or not. The following example shows that the answer is negative.

**Example 4.1.** Let $X = \{a, b\}$, $a \neq b$. Define a topology $\partial$ on $X$ by $\partial = \{\phi, \{a\}, X\}$. For, let $D$ be any $D$-metric for $X$ and let $D(b, a, a) = r$. Since $a \neq b$, $r > 0$. Then $B(b, r) = \{b\}$, because $B^*(b, r) = \{b\}$. Then $\{b\}$ is a $\tau^*$-open subset of $X$. Since $\tau^* \subset \tau$, $\{b\}$ is also a $\tau$-open set in $X$. But $\{b\}$ is not a $\partial$-open subset of $X$. Hence $(X, \partial)$ is not a $D$-metrizable topological space.

It is an important problem in general topology that whether or under what conditions the given topological space is metrizable and the problem to determine the metrizability of a topological space has been the most active
area of research work, see for example SINGAL [14] and the references therein. A result in this direction is the following.

**Theorem 4.5.** If the topological space $X$ is metrizable then it is $D$-metrizable.

**Proof.** Let $X$ be a metrizable space. Then there exists an ordinary metric $d$ on $X$ that induces the topology of $X$. Define a $D$-metric $D$ on $X$ by (2.5) or (2.6). Then this $D$-metric generate the same topology on that of $X$. Hence $X$ is $D$-metrizable. The proof is complete.

5. Topological properties. In this section we discuss the topological properties of a $D$-metric space $X$ equipped with the $D$-metric topology $\tau$.

The terminologies which are used in the following but not explained may be found in Dugundji [6] or in any standard reference book on general topology.

**Theorem 5.1.** A $D$-metric space $X$ is a $T_0$-space.

**Proof.** Let $x_0, y_0 \in X$ be such that $x_0 \neq y_0$. Then $D(x_0, y_0, y_0) = r$, for some $r > 0$. Consider an open ball $B(x_0, r)$ in $X$, then by definition $y_0 \notin B(x_0, r)$. Hence $X$ is a $T_0$-space.

**Theorem 5.2.** A $D$-metric space $X$ is $T_1$-space.

**Proof.** Let $x_0, y_0 \in X$ be such that $x_0 \neq y_0$. Suppose that $D(x_0, y_0, y_0) = r_1 > 0$, and consider a ball $B(x_0, r_1)$ in $X$. Clearly $y_0 \notin B(x_0, r_1)$. Similarly, suppose that $D(y_0, x_0, x_0) = r_2 > 0$ and consider the open ball $B(y_0, r_2)$ in $X$. Then $x_0 \notin B(y_0, r_2)$. This proves that $X$ is $T_1$-space.

**Theorem 5.3.** (Hausdorff property) A $D$-metric space $X$ is $T_2$-space.

**Proof.** Let $x_0, y_0 \in X$ be such that $x_0 \neq y_0$. We show that there exist open balls $B_1$ and $B_2$ containing $x_0$ and $y_0$ respectively such that $B_1 \cap B_2 = \emptyset$. Consider two $\tau^*$-open balls $B^*_1$ and $B^*_2$ of the points $x_0$ and $y_0$ respectively in $X$ defined by

\begin{align}
(5.1) & \quad B^*_1 = \{ x \in X \mid D(x_0, x, x) < D(y_0, x, x) \} \\
(5.2) & \quad B^*_2 = \{ x \in X \mid D(y_0, x, x) < D(x_0, x, x) \}
\end{align}
We show that $B_1^* \cap B_2^* = \emptyset$. Suppose not i.e. $B_1^* \cap B_2^* \neq \emptyset$, then there is a point $z \in B_1^* \cap B_2^*$. Since $z \in B_1^*$, we have

\begin{equation}
D(x_0, z, z) < D(y_0, z, z)
\end{equation}

Again since $z \in B_2^*$, we have

\begin{equation}
D(y_0, z, z) < D(x_0, z, z)
\end{equation}

Thus we obtain two contradictory statements (5.3) and (5.4). Hence $B_1^* \cap B_2^* = \emptyset$. By Remark we can find $\tau$-open balls $B_1$ and $B_2$ in $X$ of the points $x_0$ and $y_0$ respectively such that $B_1 \subset B_1^*$ and $B_2 \cap B_2^*$. Therefore $B_1 \cap B_2 = \emptyset$. This completes the proof.

Next we show that the $D$-metric spaces are normal and perfectly normal. Let $A, B$ and $C$ be $\tau$-closed subsets of $D$-metric space $X$. We define a function $d(A, B, C)$ by

\begin{equation}
d(A, B, C) = \inf \{D(a, b, c) \mid a \in A, b \in B, c \in C\}
\end{equation}

In particular, we have

\[d(x, x, A) = \inf \{D(x, x, a) \mid a \in A\}\]

It is clear that $d(x, x, A) = 0 \iff x \in A$. We need the following lemma in the sequel.

**Lemma 5.1.** $x \to d(x, x, A)$ is a continuous function on a $D$-metric space $X$.

**Proof.** Let $x, y \in X$ be such that $x \to y$. Then by rectangle inequality we have

\begin{equation}
D(x, x, a) \leq D(x, x, y) + D(x, y, a) + D(y, x, a)
\end{equation}

and

\begin{equation}
D(y, y, a) \leq D(y, y, x) + D(y, x, a) + D(x, y, a)
\end{equation}

Then from (5.6) and (5.7), we obtain

\[d(x, x, A) \leq D(x, x, y) + D(x, y, A) + D(y, x, A)\]
and
\[ d(y, y, A) \leq D(y, y, x) + d(y, x, A) + d(x, y, A) \]

Therefore
\[ d(x, x, A) - d(y, y, A) \leq D(x, x, y) + D(y, y, x) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon \]

This shows that \( x \rightarrow d(x, x, A) \) is a continuous function on \( X \).

**Lemma 5.2.** Let \( x_0 \in X \) be fixed and \( r > 0 \) given. If \( A^* = \{ x \in X \mid d(x, x, A) < r \} \) then \( A^* = B^*(A, r) = \bigcup_{a \in A} B^*(a, r) \), and \( A^* \) is a \( \tau^* \)-open set in \( X \).

**Proof.** The proof is obvious.

**Theorem 5.4.** Let \( A \) and \( B \) be two closed subsets of a \( D \)-metric space \( X \) such that \( A \cap B = \emptyset \). Then there exists a continuous real function \( f : X \rightarrow \mathbb{R} \) such that \( f(x) = 0 \) if \( x \in A \) and \( f(x) = 1 \) if \( x \in B \).

**Proof.** Define a function \( f : X \rightarrow \mathbb{R} \) by
\[ f(x) = \frac{d(x, x, A)}{d(x, x, A) + d(x, x, B)} \]

Since the function \( x \rightarrow d(x, x, A) \) is continuous and denominator is continuous and positive, the function \( f \) is continuous on \( X \). Obviously \( f \) satisfies the properties stated in the statement of the theorem. The proof is complete.

**Theorem 5.5.** A \( D \)-metric space \( X \) is normal.

**Proof.** Let \( A \) and \( B \) be two closed and disjoint sets in \( X \). Then by Theorem 5.4, there exists a continuous real function \( f : X \rightarrow \mathbb{R} \) such that \( f(x) = 0 \) if \( x \in A \) and \( f(x) = 1 \) if \( x \in B \). Define the open sets \( U \) and \( V \) in \( X \) by
\[ U = \{ x \in X \mid f(x) < 1/2 \} \]
and
\[ V = \{ x \in X \mid f(x) > 1/2 \} \]

Clearly, \( A \subset U \) and \( B \subset V \) and \( U \cap V = \emptyset \). This proves that \( X \) is normal.
Theorem 5.7. A D-metric space $X$ is perfectly normal.

Proof. We show that every $\tau$-closed set $A$ in $X$ is $G_\delta$, that is, $A$ can be expressed as the intersection of countable $\tau$-open sets in $X$. Consider the function $g : X \to \mathbb{R}$ defined by $g(x) = d(x, x, A)$, $x \in X$. Clearly $g$ is a continuous real function on $X$ and $g(x) = 0$, for all $x \in A$. Define the sets $A_n^*$ in $X$ by

$$(5.11) \quad A_n^* = \{ x \in X \mid g(x) < 1/n \}, \quad n \in \mathbb{N}$$

Then for each $n \in \mathbb{N}$, $A_n^*$ is an $\tau^*$-open set in $X$. Similarly by Lemma 5.2, $A_n$ is a $\tau$-open set in $X$ such that $A_n \subset A_n^*$ and $A \subset A_n$, for all $n \in \mathbb{N}$. Therefore $A = \bigcap_{n=1}^{\infty} A_n$. This shows that $A$ is a $G_\delta$ set in $X$. hence $X$ is perfectly normal. This completes the proof.

6. Completeness and compactness.


Definition 6.1.1. A sequence $\{x_n\}$ in a D-metric space $X$ is said to be D-Cauchy if for $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $D(x_m, x_n, x_p) < \varepsilon$ for all $m > n, p \geq n_0$.

Definition 6.1.2. A complete D-metric space $X$ is one in which every D-Cauchy sequence converges to a point in $X$.

Examples. $(\mathbb{R}^n, D_1)$ and $(\mathbb{R}^n, D_2)$ are complete D-metric spaces.

Theorem 6.1.1. Every convergent sequence $\{x_n\}$ in a D-metric space $X$ is D-Cauchy.

Proof. Suppose $\{x_n\}$ converges to a point $x \in X$. Then for $\varepsilon > 0$, we can find $n_0 \in \mathbb{N}$ such that $D(x_m, x_n, x) < \varepsilon/3$ for all $m, n \geq n_0$. Hence if $m > n, p \geq n_0$

$$0 < D(x_m, x_n, x_p) \leq D(x_m, x_n, x) + D(x_m, x, x_p) + D(x, x_n, x_p) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$  

This proves that $\{x_n\}$ is a D-Cauchy sequence in $X$.

Theorem 6.1.2. If a D-Cauchy sequence of points in a D-metric space contains a convergent subsequence, then the sequence is convergent.
Proof. Suppose \( \{x_n\} \) is a \( D \)-Cauchy sequence in \( D \)-metric space \( X \). Then for \( \varepsilon > 0 \), there exists an integer \( n_0 \in \mathbb{N} \) such that \( D(x_m, x_n, x_p) < \varepsilon \) for all \( m > n, \ p \geq n_0 \). Since the sequence \( \{x_n\} \) contains a convergent subsequence \( \{x_{k_n}\} \) converging to a point \( x \in X \), we have \( D(x_{k_n}, x_{k_n}, x) < \varepsilon \) for all \( m, n \geq n_0 \). As \( \{k_m\} \) is strictly increasing sequence of positive integers, we obtain, \( D(x_m, x_n, x) < \varepsilon \) for all \( m, n \geq n_0 \), which shows that \( x_n \to x \). The proof is complete.

Theorem 6.1.3. Let \( X_1 \) and \( X_2 \) be two \( D \)-metric spaces with \( D \)-metrics \( \rho_1 \) and \( \rho_2 \) respectively. Define a \( D \)-metric \( \rho \) on \( X = X_1 \times X_2 \) by

\[
\rho(x, y, z) = \max\{\rho_1(x_1, y_1, z_1), \rho_2(x_2, y_2, z_2)\}
\]

for \( x, y, z \in X \). Then \( (X, \rho) \) is complete if and only if \( (X, \rho_1) \) and \( (X, \rho_2) \) are complete.

Proof. The proof is obvious.

Theorem 6.1.4. Let \( d \) be an ordinary metric on \( X \) and let \( D_1 \) and \( D_2 \) be corresponding associated \( D \)-metrics on \( X \). Then \( (X, D_1) \) and \( (X, D_2) \) are complete if and only if \( (X, d) \) is complete.

Proof. The proof is simple and follows from the definitions of \( D_1 \) and \( D_2 \).

Definition 6.1.3. A sequence \( \{F_n\} \) of closed sets in a \( D \)-metric space \( X \) is said to be nested if \( F_1 \supset F_2 \supset \cdots \supset F_n \supset \cdots \).

Theorem 6.1.5. (Intersection Theorem) Let \( X \) be a \( D \)-metric space and let \( \{F_n\} \) be a nested sequence of non–empty subsets of \( X \) such that \( \delta(F_n) \to 0 \) as \( n \to \infty \). Then \( X \) is complete if and only if \( \bigcap_{n=1}^{\infty} F_n \) consists of exactly one point.

Proof. Let \( X \) be complete. For each \( n \in \mathbb{N} \), we choose \( x_n \in F_n \). Since \( \delta(F_n) \to 0 \) as \( n \to \infty \), for given \( \varepsilon > 0 \), there exists an \( n_0 \in \mathbb{N} \) such that \( \delta(F_{n_0}) < \varepsilon \). Again since \( \{F_n\} \) is nested, we have \( m > n, \ p \geq n_0, \ F_m, F_n, F_p \supset F_{n_0} \). This implies \( x_m, x_n, x_p \in F_{n_0} \to D(x_m, x_n, x_p) < \varepsilon \), for all \( m > n, \ p \geq n_0 \). Thus \( \{x_n\} \) is a \( D \)-Cauchy sequence in \( X \). Since \( X \) is complete, \( x_n \to x \) for some \( x \in X \). We assert that \( x \in \bigcap_{n=1}^{\infty} F_n \). To prove this, let \( m \in \mathbb{N} \) be arbitrary. Then \( m > n \to x_m \in F_n \). Since \( x_n \to x \), the sequence \( \{x_n\} \) is eventually in every neighbourhood of \( x \) and so every
neighbourhood of $x$ contains an infinite number of points of $F_n$. So $x$ is a limit point of $F_n$. As $F_n$ is closed, $x \in F_n$. Since $F_n$ is arbitrary, $x \in \bigcap_{n=1}^{\infty} F_n$.

Now suppose that there is another point $y \in \bigcap_{n=1}^{\infty} F$. Then $D(x, y, y) \leq \delta(F_n)$, for all $n \in \mathbb{N}$.

Therefore $D(x, y, y) = 0$, because $\delta(F_n) \to 0$ as $n \to \infty$. Hence $x = y$.

The converse part can be proved by using the arguments similar to ordinary metric space case with appropriate modifications. This completes the proof.

**Theorem 6.1.6.** D-metric space $X$ is of second category.

**Proof.** The proof is similar to ordinary metric space case and we omit the details.

**Theorem 6.1.7.** (Fixed point theorem) Let $f$ be a self-map of a complete and bounded D-metric space $X$ satisfying

$$D(fx, fy, fz) \leq \alpha D(x, y, z)$$

for all $x, y, z \in X$ and $0 \leq \alpha < 1$. Then $f$ has a unique fixed point.

**Proof.** The proof is given in DHAGE [3].

### 6.2. Compactness

**Definition 6.2.1.** Let $X$ be a D-metric space and let $\varepsilon > 0$ be given. A finite subset $A$ of $X$ is said to be an $\varepsilon$-net for $X$ if and only if for every $x \in X$, there exists a point $a \in A$ such that $x \in B(a, \varepsilon)$. In other words, $A$ is an $\varepsilon$-net for $X$ if and only if $A$ is finite and $X = \bigcup \{B(a, \varepsilon) : a \in A\}$.

A D-metric space $X$ is said to be totally bounded if and only if $X$ has an $\varepsilon$-net for every $\varepsilon > 0$, and $X$ is said to be compact if every $\tau$-open cover of $X$ has a finite subcover.

**Definition 6.2.2.** Let $\zeta = \{G_\lambda : \lambda \in \Lambda\}$ be a $\tau$-open cover of a D-metric space $X$. Then a real number $1 > 0$ is called a Lebesgue number for $\zeta$ if and only if every subset of $X$ with diameter less than 1 is contained in at least one of $G_\lambda$’s.

**Theorem 6.2.1.** Every sequentially compact D-metric space $X$ is totally bounded.

**Proof.** Suppose $X$ is not totally bounded. Then there exists $\varepsilon > 0$ such that $X$ has no $\varepsilon$-net. Let $x_1 \in X$. Then there must exists the points
Let $x_2, x_3 \in X$, not necessary distinct, such that $D(x_1, x_2, x_3) \geq \varepsilon$, for otherwise, \{x_1\}, would be an $\varepsilon$-net for $X$. Again there exists a point $x_4 \in X$ such that $D(x_2, x_3, x_4) \geq \varepsilon$, for otherwise \{x_1, x_2\} would be an $\varepsilon$-net for $X$. Continuing this process, we get a sequence \{x_1, x_2, \ldots\} having the property that $D(x_i, x_j, x_k) \geq \varepsilon$, $i \neq j$ or $j \neq k$ or $k \neq 1$. It follows that the sequence \{x_n\} cannot contain any convergent subsequence. Hence $X$ is not sequentially compact. This completes the proof.

We note that several results of a complete or compact ordinary metric spaces are true in a complete or compact $D$-metric space. Below we state some results in a compact $D$-metric space without proofs, since their proofs are similar to ordinary metric space case with appropriate modifications.

**Theorem 6.2.2.** In a $D$-metric space $X$, the following statement are equivalent.

(a) $X$ is compact,
(b) $X$ is countably compact,
(c) $X$ has Bolzano–Weierstrass property,
(d) $X$ is sequentially compact.

**Theorem 6.2.3.** Every open cover of a sequentially compact $D$-metric space $X$ has a Lebesgue number.

**Theorem 6.2.4.** In a $D$-metric space $X$,

(a) a compact subset of a $D$-metric space is closed and bounded,
(b) a $D$-metric space $X$ is a compact if and only if it is complete and totally bounded,
(c) a subset $S$ of a complete $D$-metric space is compact if and only if is closed and totally bounded.

**Theorem 6.2.5.** Let $f$ be a continuous mapping of a compact $D$-metric space $X$ into a $D$-metric space $Y$. Then $f(X)$ is compact. In other words, continuous image of a compact $D$-metric space is compact.

**Corollary 6.2.1.** Every real–valued continuous function on a compact $D$-metric space $X$ is bounded and attains its supremum and infimum on $X$.

Finally we mention that the further research work can be carried out in the following directions.

1. Find the conditions for a topological space to be $D$-metrizable.
2. Find the necessary and sufficient conditions for a self-mapping of a $D$-metric space to have a fixed point.
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