COMMUTATIVITY OF LEFT $s-$UNITAL RINGS WITH POLYNOMIAL CONSTRAINTS

BY

MOHARRAM A. KHAN

Abstract. Let $m,n,r,s$ and $t$ be fixed non–negative integers. If $r$ and $n$ are not equal to 1 and $R$ is left $s$-unital rings satisfying $[(x^m y^r x^s)^n - x^t y, x] = 0$, then $R$ is commutative. Commutativity of left $s$-unital ring subject to the condition $x^t[x^n , y] = \pm y^r[x, f(y)]x^s$, where $f(\lambda)$ is a polynomial in $\lambda^2 \mathbb{Z}[\lambda]$ and $n>0, r, s$ and $t$ fixed non–negative integers, has been studied. Further, the results have been extended to the case when integral exponents in the underlying conditions are no longer fixed, rather they depend on the pair of ring’s elements $x$ and $y$ for their values. Finally, our results generalize some well–known commutativity theorems (see [1], [2], [7] and [8]).

1. Introduction. Throughout the paper, $R$ will denote an associative ring (maybe without unity 1), $N(R)$ the set of nilpotent elements of $R$ and $U(R)$ the group of units of $R$.

Following [3], a ring $R$ is said to be a right (resp. left) $s$-unital ring if for each $x$ in $R$, $x \in xR$ (resp. $x \in Rx$) and $R$ is called $s$-unital in case it is right as well as left $s$-unital. As usual $\mathbb{Z}[X]$ is the totality of all polynomials in $X$ over $\mathbb{Z}$, the ring of integers. Consider the following ring properties:

(P) Let $m, n, r, s$ and $t$ be non-negative integers with $r$ and $n$ not simultaneously equal to 1 at $s = 0$ such that $[(x^m y^r x^s)^n - x^t y, x] = 0$, for all $x, y \in R$.

(P$^*$) For each $x, y \in R$ there exist non-negative integers $m, n, r, s$ and $t$ with $r$ and $n$ not simultaneously equal to 1 at $s = 0$ such that

$$[(x^m y^r x^s)^n - x^t y, x] = 0.$$
For each \( y \in R \) there exists a polynomial \( f(\lambda) \) in \( \lambda^{2}\mathbb{Z}[\lambda] \) such that 
\[
x^{t}[x^{n}, y] = \pm y^{r}[x, f(y)]x^{s} \quad \text{and} \quad x^{t}[x^{m}, y] = \pm y^{r}[x, f(y)]x^{s}
\]
for all \( x \in R \), where \( m \geq 1, n \geq 1, r, s \) and \( t \) are fixed non-negative integers with \((m, n) = 1\) and at least one of \( s \) and \( t \) is non-zero.

For each \( x, y \in R \) there exist a polynomial \( f(\lambda) \) in \( \lambda^{2}\mathbb{Z}[\lambda] \) and non-negative integers \( m \geq 1, n \geq 1, r, s, t \) with \((m, n) = 1\), such that 
\[
x^{t}[x^{n}, y] = \pm y^{r}[x, f(y)]x^{s} \quad \text{and} \quad x^{t}[x^{m}, y] = \pm y^{r}[x, f(y)]x^{s}.
\]

For each \( x, y \in R \), there exist \( f(\lambda), g(\lambda) \) in \( \lambda^{2}\mathbb{Z}[\lambda] \) such that 
\[
[x - f(x), y - g(y)] = 0.
\]

A theorem of Zhongxuan [11] has extended the result of the author et. al. [5] as follows: Let \( R \) be a semiprime ring in which either \((x^{m}y)^{2} - x^{n}y\) or \((x^{m}y)^{2} - yx^{t}\) is central for all \( x, y \in R \) and \( m, n, t \) are fixed integers. Then \( R \) is commutative.

The non-commutative ring \( R \) of \( 3 \times 3 \) strictly upper triangular matrices over \( \mathbb{Z} \), ring of integers, satisfying \( xyz = 0 \) for all \( x, y, z \) in \( R \) rules out the possible generalization of these results for rings without unity.

Recently, PsomoPoulos [8, Theorem] has obtained the following result: If \( R \) is an \( s \)-unital ring and there exist fixed integers \( m > 1 \) and \( n \geq 1 \) such that \([x^{n}y - y^{m}x, x] = 0\), for all \( x, y \in R \), then \( R \) is commutative.

Komatsu [6] proved the following result: Let \( m \) and \( n \) be fixed non-negative integers. Suppose that \( R \) satisfies the polynomial identity \( x^{n}[x, y] = [x, y^{m}] \) for all \( x, y \) in \( R \). If \( R \) is a left \( s \)-unital, then \( R \) is commutative provided that \((m, n) \neq (1, 0)\). Further, if \( R \) is a right \( s \)-unital, then \( R \) is commutative except in the case \((m, n) = (1, 0) \) and \( m = 0, n > 0\).

The aim of the present work is to generalise the above results for \( s \)-unital rings, further other commutativity theorems for one–sided \( s \)-unital rings are obtained under different set of conditions. Also, commutativity of rings satisfying Chacron’s criterion, namely (CH), together with one of the properties (P*) and (P*1) has been established.

2. Preliminary results. In order to facilitate our discussion, we consider the following types of rings.

\[
(1)_{\ell} \begin{pmatrix} GF(p) & GF(p) \\ 0 & 0 \end{pmatrix}, \ p \text{ a prime.}
\]

\[
(1)_{r} \begin{pmatrix} 0 & GF(p) \\ 0 & GF(p) \end{pmatrix}, \ p \text{ a prime.}
\]

\[
(1) \begin{pmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{pmatrix}, \ p \text{ a prime.}
\]
(2) \( M_\sigma(F) = \left\{ \begin{pmatrix} a & b \\ 0 & \sigma(a) \end{pmatrix} \mid a, b \in F \right\}, \)
where \( F \) is a finite field with a non–trivial automorphism \( \sigma \).

(3) A non–commutative ring with no non–zero divisors of zero.

(4) \( S = \langle 1 \rangle + T; T \) is non–commutative subring of \( S \) such that
\( T[T, T] = [T, T]T = 0. \)

In a recent paper [10], STREB gave a classification for non–commutative rings. The classification is effective in the present paper as a tool to establish several commutativity theorems (see [1], [2], [6], [7], [8]). As is easily seen from the proof of [8, Corollary 1], if \( R \) is a non–commutative left \( s \)-unital ring, then there exists a factor subring of \( R \) which is of the type (1)\(_l\), (2), (3) or (4). This gives a result which plays a vital role in our subsequent discussion.

Proposition 2.1. Let \( P \) be a ring property which is inherited by factor subrings. If no rings of type (1)\(_l\), (2), (3) or (4) satisfy \((P)\), then every left \( s \)-unital ring satisfying \( P \) is commutative.

3. Main results. The main results of the present paper are as follows.

Theorem 3.1. Let \( R \) be a left \( s \)-unital ring satisfy \((P)\). Then \( R \) is commutative.

Theorem 3.2. Let \( R \) be left \( s \)-unital ring with \((P_1)\). Then \( R \) is commutative.

For the proof of our results, we need the following known results.

Lemma 3.3. [4, Theorem] Let \( f \) be a polynomial in \( n \) non–commuting indeterminates \( x_1, x_2, \ldots, x_n \) with relatively prime integer coefficients. Then the following statements are equivalent:

(I) For any ring \( R \) satisfying \( f = 0 \), the commutator ideal of \( R \) is nil ideal.

(II) Every semiprime ring satisfying \( f = 0 \) is commutative.

(III) For every prime \( p \), the ring \( (GF(p))^2 \) of \( 2 \times 2 \) matrices over \( GF(p) \) fails to satisfy \( f = 0 \).

Lemma 3.4. [7, Lemma 1] Let \( R \) be a left \( s \)-unital ring and not right \( s \)-unital. Then \( R \) has a factor subring of type (1)\(_l\).

Lemma 3.5. [8, Corollary 1] Let \( R \) be a non–commutative ring satisfying (CH). Then there exists a factor subring of \( R \) which is of type (1)\(_l\) or (2).
Proof of Theorem 3.1. Let \( R \) be any ring of type (i). Then

\[
[(e_{11}^m e_{12} e_{11}^*)^n - e_{11} e_{12} e_{11}] = e_{12} \neq 0
\]

hence \( R \) does not satisfy (P). It follows by Lemma 3.4, that if \( R \) is any left \( s \)-unital ring satisfying \( (P) \), then \( R \) is right \( s \)-unital as well. Thus in view of Proposition 1 of [3], we may assume that \( R \) has unity 1.

Suppose that \( R \) is the ring of type (2). Taking \( x = \begin{pmatrix} a & 0 \\ 0 & \sigma(a) \end{pmatrix} \) (\( \sigma(a) \neq a \)), \( y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) in \( R = M_\sigma(F) \), one observes that

\[
[(x^m y^* x^*)^n - x^t y, x] = a^t (a - \sigma(a)) e_{12} \neq 0.
\]

This shows that \( R \) does not satisfy \( (P) \).

Since \( x = e_{22} \) and \( y = e_{21} \) do not satisfy (P), by Lemma 3.3, we see that the commutator ideal of \( R \) is nil and hence no ring of type (3) satisfies (P).

Finally, we put \( S = < 1 > + T, T \) is a non–commutative subring of \( S \) such that \( T[T,T] = [T,T]T = 0 \). Assume \( a, b \in T \) such that \([a,b] \neq 0 \). Then by hypothesis, we have

\[
(1 + a)^t [a,b] = [1 + a, ((1 + a)^m b^*(1 + a)^*)^n] = 0.
\]

This implies that \([a,b] = 0 \). This leads to a contradiction and so \( R \) is not type (4).

Hence we have seen that no ring of type (1), (2), (3) or (4) satisfies (P) and by Proposition 2.1, \( R \) is commutative.

Proof of Theorem 3.2. Putting \( x = e_{11} + e_{22}, y = e_{12} \in (GF(p))_2 \) in \( (P) \), we get

\[
x^t [x^m, y] \pm y^t [x, f(y)] x^* = e_{22} \neq 0.
\]

Hence, \( R \) is not type (1), by Lemma 3.4, \( R \) is also right \( s \)-unital and hence it is \( s \)-unital. In view of, Proposition 1 of [3], we may assume that the ring \( R \) has unity 1.

Consider the ring \( M_\sigma(F) \), a ring of type (2). Notice that \( N = Fe_{12} \). Hence for \( b \in N \) and arbitrary unit \( u \in U(R) \), we obtain that there exists polynomial \( f(\lambda) \in \lambda^2 \mathbb{Z}[\lambda] \) such that

\[
u^t [u^m, b] \pm b^*(u, f(b)) u^* = 0 \text{ and } u^t[u^n, b] \pm b^*(u, f(b)) u^* = 0.
\]
Since $b^2 = 0$ and $u$ is a unit of $R$, the last two equations yield that $[u^m, b] = 0$ and $[u^n, b] = 0$. Now for non-central element $b = e_{12}, [u, e_{12}] = 0$ gives that $e_{12}$ is central, a contradiction. Hence $R$ cannot be of type (2).

By hypothesis, we have
\[ x^t [x^m, y] = \pm y^r [x, f(y)] x^s. \] (1)

Replacing $x$ by $x + 1$ in (1), we get
\[ (x + 1)^t [(x + 1)^m, y] = y^r [x, f(y)] (x + 1)^s. \] (2)

Multiplying (1) by $(x + 1)^s$ on the left and (2) by $y^r$ and comparing the equations (1) and (2), we get
\[ (x + 1)^s [x^m, y] = x^s [(x + 1)^m, y] (x + 1)^s. \] (3)

Equation (3) is a polynomial identity and $x = e_{11} + e_{12}, y = e_{12} \in (GF(p))^2$ fail to satisfy (3). By Lemma 3.3, the commutator ideal of $R$ is nil and hence no ring of type (3) satisfies the property $(P_1)$.

Finally, let $S = < 1 > + T$, where $T$ is a non-commutative subring of $S$ such that $T[T, T] = [T, T]T = 0$. Let $[a, b] = 0$, where $a, b \in T$. There exists $f(\lambda)$ in $\lambda^2 \mathbb{Z}[\lambda]$ such that
\[ m[a, b] = (1 + a)^t [(1 + a)^m, b] = \pm b^r [a, f(b)] (1 + a)^s = 0 \]
and
\[ n[a, b] = (1 + a)^t [(1 + a)^n, b] = \pm b^r [a, f(b)] (1 + a)^s = 0. \]

Since $(m, n) = 1$, we get $[a, b] = 0$, a contradiction. Hence there is no ring of type (4) satisfies $(P_1)$.

No ring of type (1) or (2) satisfies $(P_1^*)$ and $(P_1^*)$. Thus by Proposition 2.1, $R$ is commutative.

From Theorems 3.1 and 3.2, it can be easily seen that no ring of type (1) or (2) satisfies $(P_1^*)$ and $(P_1^*)$. Combining this fact with Lemma 3.5, we obtain the following:

**Theorem 3.3.** Suppose that $R$ satisfies (CH). Then the following conditions are equivalent:

(i) $R$ is commutative.

(ii) $R$ is a left $s$-unital ring satisfying either $(P_1^*)$ or $(P_1^*)$.

Remark 3.1. The following example demonstrates that in the hypothesis of Theorem 3.3 (ii), the existence of both the conditions in $(P_1^*)$ and $(P_1^*)$ are not superfluous (even if $R$ has unity 1).
Example 3.1. Consider the ring $R = \{\alpha = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \gamma = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \delta = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \text{ where } \alpha, \beta, \gamma, \delta \in GF(2)\}$. Then $R$ is a non-commutative ring with unity 1 satisfying $[x^m, y^n]x^m = y^r[x, f(y)]$ for a fixed polynomial $f(y) = y^4$ and $m = 4$, where $r, s, t$ maybe any non-negative integers.

REFERENCES

2. ABUJABAL, H.A.S. and PERIC, V. – Commutativity of $s$-unital rings through a Streb result, Rad. Mat. 7 (1991), 73 - 92.
7. KOMATSU, H. and TOMINAGA, H. – On commutativity of rings, Rad. Mat. 6 (1990), 303-311.

Received: 15.III.1999