COMMON FIXED POINT THEOREMS
FOR COINCIDENTALLY COMMUTING SINGLE
AND SET–VALUED MAPPINGS IN METRIC SPACES

BY

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Abstract. In this paper some new notions concerning the commuting mappings are introduced and some common fixed point theorem for coincidentally commuting mappings on a complete metric space satisfying certain contractive condition are proved which include the known fixed point theorems of IMDAD and AHMED [8] and SESSA et.al. [14] as special cases under weaker conditions.

1. Introduction. The fixed point theory for the set–valued mappings is a major branch of set–valued analysis and at present a very extensive literature is available in this direction. All these results are extensions and generalizations of the celebrated fixed point theorem for set–maps first established by NADLER [12]. The common fixed point theorems for the pairs of point map and set–map have been studied by several authors in the literature, see for example, CHEN and SHIN [2], FISHER [2], [3], GAREGNANI and ZANCO [7], KAUGULD and PAI [10], KHAN [11] and SESSA et.al. [14] etc. under the weaker versions of commutativity condition.

In this paper we prove some common fixed point theorems for the pairs of point and set maps on a metric space satisfying more general contraction condition than existing in the literature under some weaker commutativity versions of coincidentally commuting mappings introduced by JUNGCK and RHOADES [9] and the present author [3]. Our results of this paper include the common fixed point theorems of above cited references as special cases under weaker conditions.

2. Preliminaries. Let $X$ be a metric space with metric $d$, and let $B(X)$ denote the collection of all non-empty bounded subsets of $X$. For any $A, B \in B(X)$ define by
(2.1) \[ D(A, B) = \inf\{d(a, b) : a \in A, b \in B\} \]

(2.2) \[ H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\} . \]

(2.3) \[ \delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\} \]

Observe that \( \delta(A, A) = \sup\{d(x, y) : x, y \in A\} = \text{diam } A \), and let \( \delta(A) := \delta(A, A) \).

A sequence \( \{A_n\} \) of bounded sets in \( B(X) \) is said to converge to a set \( A \in B(X) \) if \( \lim_{n \to \infty} H(A_n, A) = 0 \). We need the following lemma.

**Lemma 2.1.** (FISHER [5]) If \( \{A_n\} \) and \( \{B_n\} \) are sequences of bounded subsets of \( (X, d) \) converging to the bounded subsets \( A \) and \( B \) respectively, then the sequence \( \{\delta(A_n, B_n)\} \) converge to \( \delta(A, B) \).

In view of (2.4) we have the following lemma.

**Lemma 2.2.** Let \( \{A_n\} \) be a sequence of non-empty bounded subsets of \( (X, d) \) converging to a point set \( \{y\} \), then \( \lim_{n \to \infty} D(A_n, y) = 0 \).

A correspondence \( F : X \to B(X) \) is called a set-valued or simply set map on \( X \) and a point \( z \in X \) is called a fixed point of \( F \) if \( z \in Fz \). A point \( z \) is called a common fixed point of a point map \( g : X \to X \) and a set map \( F : X \to B(X) \) if \( g(z) = z \in F(z) \).

Let \( \Psi \) denote the class of all functions \( \psi : \mathbb{R}^+ \to \mathbb{R}^+ \) satisfying

(i) \( \psi \) is upper semi-continuous.

(ii) \( \psi \) is nondecreasing, and

(iii) \( \psi(t) < t \) for each \( t > 0 \).

A function \( \psi \) is called a control or growth or contraction function and a commonly used control function is \( \psi(t) = qt, 0 \leq q < 1 \). We need the following lemma.

**Lemma 2.3** [SESSA et al. [15]] If \( \psi \in \Psi \), then \( \lim_{n \to \infty} \psi^n(t) = 0 \) for each \( t > 0 \), where \( \psi^n \) denotes the \( n \) times composition of \( \psi \).

In the following section we prove the common fixed point theorems for a pair of point and set maps on a complete metric space under certain contraction and commutativity conditions.
3. Limit coincidentally commuting mappings. Before going to the limit coincidentally commuting mappings and main results, we review some of the earlier versions of weak commutativity.

**Definition 3.1.** (SESSA et al. [15]) let \( F : X \to B(X) \) be a set map and \( g : X \to X \) a point map. Then the pair \( \{F, g\} \) is

(a) weakly commuting on \( X \) if for any \( x \in X \),
\[ \delta(Fgx, gFx) \leq \max\{\delta(gx, Fx), \text{diam } gFx\}, \]
(b) commuting on \( X \) if for any \( x \in X, gFx \subseteq Fgx \), and
(c) slightly commuting on \( X \) if for any \( x \in X \),
\[ \delta(Fgx, gFx) \leq \max\{\delta(gx, Fx), \text{diam } Fx\}. \]

It has been shown in IMDAD [8] that the above three concepts of weak commutativity are mutually independent and none of them implies the other two.

**Definition 3.2.** The pair \( \{F, g\} \) of set and point maps on a metric space \( X \) is called limit coincident if \( \lim \delta(Fx_n, gx_n) = 0 \) and called limit coincidentally commuting if \( \delta(gFx_n, Fgx_n) \to 0 \), whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \delta(gx_n, Fx_n) \to 0 \).

The term "\( \delta \)-compatible" is used for the pair of limit coincidentally commuting maps on a metric space by JUNGCK and RHOADES [9]. But we note that our terminology is more informative and meaningful than given in [9].

It has been noted in JUNGCK and RHOADES [9] that the weakly commuting and slightly commuting imply \( \delta \)-compatibility.

The following common fixed point theorem is proved in SESSA et.al. [14].

**Theorem 3.1.** Let \( X \) be a complete metric space. Let \( g : X \to X \) and \( F : X \to B(X) \) be two mappings satisfying

\[ \delta(Fx, Fy) \leq \psi(\max\{\delta(gx, gy), \delta(gx, Fx), \delta(gy, Fy), \delta(gx, Fy), \delta(gy, Fx)\}) \]

for all \( x, y \in X \), where \( \psi \in \Psi \). Further suppose that

(i) \( F(X) \subseteq g(X) \).
(ii) \( \{F, g\} \) is weakly commuting, and
(iii) any one of \( F \) and \( g \) is continuous.

Then \( F \) and \( g \) have a unique common fixed point \( z \) such that \( \{gz\} = \{z\} = Fz \).
Below we generalize Theorem 3.1 by extending the class of maps given by the inequality (SKI) and weakening the hypothesis (iii) thereof. In this chapter we consider the class of maps \{F, g\} satisfying the contraction condition

\[
\delta^r(Fx, Fy) \leq \phi(d^r(gx, gy), \delta^r(gx, Fx), \delta^r(gy, Fy), \delta^r(gx, Fy), \delta^r(gy, Fx))
\]

for all \(x, y \in X, \ r \in \mathbb{N}\) and where \(\phi : (\mathbb{R}^+)^5 \to \mathbb{R}^+\) is upper semi-continuous, and nondecreasing in each co-ordinate variable, satisfying

\[
(3.2) \quad \psi(t) = \max\{\phi(t, t, t, \alpha t, 2^r t), \phi(t, 0, 0, t, t), \phi(0, 0, t, t, 0), \phi(0, t, 0, 0, t)\} < t
\]

for \(t > 0\), where \(\alpha = \left(\begin{array}{c} r \\ 0 \end{array}\right) + \left(\begin{array}{c} r \\ 1 \end{array}\right) + \cdots + \left(\begin{array}{c} r \\ r \end{array}\right) = 2^r\).

From the above condition (3.1) one is tempted to take \(r^{th}\) root of both sides of the inequality (3.1), but \(\psi(t) < t, t > 0\) does not necessarily imply \([\psi(t)]^{1/r} < t, t > 0\). A simple counter example, define \(\phi : (\mathbb{R}^+)^5 \to \mathbb{R}^+\) by

\[
\phi(t_1, t_2, t_3, t_4, t_5) = \frac{\sum_{i=1}^{5} t_i}{1 + \sum_{i=1}^{5} t_i}, \quad t_i > 0.
\]

This shows that our condition (3.1) is more general than the condition obtained by letting \(r = 1\) and thereby the condition (3.1) of SESSA et.al. [15].

**Theorem 3.2.** Let \(X\) be a complete metric space, and let \(F : X \to B(X)\) and \(g : X \to X\) two mappings satisfying (3.1) and (3.2). Further suppose that

(i) \(F(X) \subseteq g(X)\),

(ii) \(\{F, g\}\) are limit coincidentally commuting, and

(iii) any one of \(F\) and \(g\) is continuous.

Then \(F\) and \(g\) have a unique common fixed point \(z \in X\) such that \(\{g\} = \{z\} = Fz\) and \(F\) is continuous at \(z\).

**Proof.** Let \(x \in X\) be arbitrary and we define a sequence \(\{y_n\}\) in \(X\) as follows.

Let \(y_0 = gx_0\), choose a point \(y_1\) from \(Fx_0\). As \(Fx_0 \subseteq g(X)\), there is a point say \(x_1 \in X\) such that \(y_1 = gx_1\). In general choose \(x_{n-1} \in X\), then for any point \(y_n \in Fx_{n-1}\), there exists an \(x_n \in X\) such that \(y_n = gx_n\). Thus we have a well defined sequence \(\{y_n\}\) given by
\[ (3.3) \quad x_0 = x, \quad y_0 = gx_0, \quad gx_n = y_n \in Fx_{n-1}, \quad n \in \mathbb{N}, \]

for some sequence \( \{x_n\} \) in \( X \). Denote by \( X_n = Fx_{n-1}, \quad n = 1, 2, \ldots \), and \( \beta_n = \delta(X_n, X_{n+1}), \quad n \geq 0. \)

Now there are two cases:

**Case 1.** Suppose \( \beta_s = 0 \) for some \( s \in \mathbb{N} \). Then \( \delta(X_s, S_{s+1}) = 0 \Rightarrow X_s = X_{s+1} \) is a singleton set, and

\[ (3.4) \quad \{y_s\} = Fx_{s-1} = \{gx_s\} = Fx_s = \{gx_{s-1}\} = \{y_{s+1}\}. \]

From Proposition 3.1 of JUNGCK and RHOADES [9] it follows that

\[ (3.5) \quad Fgx_s = gFgx_s = FFx_s. \]

We shall show that \( z = gx_s \) is a common fixed point of \( F \) and \( g \). We have

\[ \delta^r(Fx_s, FFx_s) \leq \leq \phi(d^r(gx_s, gFx_s), \delta^r(ggx_s, FXx_s), \delta^r(gFgx_s, FFx_s), \delta^r(ggx_s, FFx_s), \delta^r(gFgx_s, FFx_s)) \]

and so

\[ \delta^r(z, Fz) \leq \phi(d^r(z, Fz), \delta^r(z, z), \delta^r(Fz, Fz), \delta^r(z, Fz), \delta^r(Fz, z)) = \phi(\delta^r(z, Fz), 0, 0, \delta^r(z, Fz), \delta^r(Fz, z)) \leq \psi(\delta^r(z, Fz)) \]

which proves that \( \delta^r(z, Fz) \leq 0 \) and hence \( \delta(z, Fz) = 0 \), i.e. \( \{z\} \subset Fz. \) Again from (3.5) we get \( g_z = z \) as claimed.

**Case II.** Suppose that \( \beta_n > 0 \) for all \( n \in \mathbb{N} \). Then

\[ \beta_n = \delta^r(X_n, X_{n+1}) = \delta^r(gx_n, Fx_n), \delta^r(gx_{n+1}, Fx_{n+1}), \delta^r(gx_n, Fx_{n+1}), \delta^r(gx_{n+1}, Fx_{n+1}) = \phi(\delta^r(y_n, y_{n+1}), \delta^r(y_n, X_{n+1}), \delta^r(y_{n+1}, y_{n+1}), \delta^r(y_n, X_{n+2}), \delta^r(y_{n+1}, X_{n+2})) \leq \phi(\delta^r(X_n, X_{n+1}), \delta^r(X_n, X_{n+1}), \delta^r(X_{n+1}, X_{n+2}), \delta^r(X_{n+1}, X_{n+1})) = \phi(\beta_n, \beta_{n+1}, \beta_{n+1}, \beta_n + \beta_{n+1}, 2 \cdot \beta_n) = \phi(\beta_n, \beta_{n+1}, \beta_{n+1}, \beta_n + 2 \cdot \beta_{n+1}, \beta_{n+1}) \]

If \( \beta_{n+1} > \beta_n \), then we get

\[ \beta^r_{n+1} \leq \phi(\beta^r_{n+1}, \beta^r_{n+1}, \beta^r_{n+1}, \alpha \beta^r_{n+1}, 2 \cdot \beta^r_{n+1}) \leq \psi(\beta^r_{n+1}) \]
which is a contradiction and hence $\beta_{n+1} \leq \beta_n$ for all $n \in \mathbb{N}$. Thus $\{\beta_n\}$ is a decreasing sequence of positive real numbers. Since $\beta_{r+1} \leq \phi(\beta_r, \beta_r, \alpha \beta_r, 2^r \beta_r) \leq \psi(\beta_r)$, it follows by induction that $\beta \leq \psi^n(\beta_r)$ and hence an application of Lemma 2.2 gives that $\lim_{n} \beta_{n} = 0 \Rightarrow \lim_{n} \beta_{n} = 0$.

We now show that $\{y_n\}$ is a Cauchy sequence. Suppose that $\{y_n\}$ is not Cauchy. Then there is an $\varepsilon > 0$ such that for given integer $k > 0$ there exists integers $m(k) > n(k) > k$ such that

(3.6) $d(y_{m(k)}, y_{n(k)}) > \varepsilon$.

Let $m(k)$ be the least positive integer exceeding $n(k)$ and satisfying (3.6) and so

(3.7) $d(y_{n(k)}, y_{m(k)-1}) < \varepsilon$.

Now

$$
\varepsilon \leq \lim_{k \to \infty} d(y_{n(k)}, y_{m(k)}) \leq \lim_{k \to \infty} [d(y_{n(k)}, y_{m(k)-1}) + d(y_{n(k)}, y_{m(k)})] \leq \lim_{k \to \infty} [\varepsilon + \delta(X_{m(k)-1}, X_{m(k)})] \leq \lim_{k \to \infty} [\varepsilon + \beta_{m(k)-1}] = \varepsilon
$$

which implies that

(3.8) $\lim_{k \to \infty} d(y_{n(k)}, y_{m(k)}) = \varepsilon$.

Again

$$
\varepsilon \leq \lim_{k \to \infty} d(y_{n(k)}, y_{m(k)}) \leq \lim_{k \to \infty} d(y_{n(k)}, y_{m(k)+1}) + \lim_{k \to \infty} d(y_{n(k)+1}, y_{m(k)}) \leq \lim_{k \to \infty} \beta_{n(k)} + \lim_{k \to \infty} \delta(X_{n(k)+1}, X_{m(k)}) = \lim_{k \to \infty} \delta(Fx_{n(k)}, Fx_{m(k)-1}).
$$

Therefore
\[ \varepsilon^r \leq \lim_{k \to \infty} \delta^r(Fx_{n(k)}, Fx_{m(k)-1}) \leq \lim_{k \to \infty} \phi(d^r(gx_{n(k)}, gx_{m(k)-1}), \delta^r(gx_{n(k)}, Fx_{n(k)})) \]

\[ = \lim_{k \to \infty} \phi(\delta^r(y_{n(k)}, y_{m(k)-1}), \delta^r(y_{n(k)}, X_{n(k)+1})) \]

\[ \leq \lim_{k \to \infty} \phi(\varepsilon^r, \delta^r(1, X_{n(k)}), \delta^r(1, X_{n(k)+1})) \]

\[ = \lim_{k \to \infty} \phi(\varepsilon^r, \delta^r(y_{n(k)}, y_{m(k)-1}), \delta^r(y_{n(k)}, X_{n(k)+1})) \cdot \delta^r(y_{n(k)}, y_{m(k)-1}) \]

\[ \leq \lim_{k \to \infty} \phi(\varepsilon^r, \delta^r(y_{n(k)}, y_{m(k)-1})), \delta^r(y_{n(k)}, y_{m(k)-1}) \]

which is possible only when \( \varepsilon = 0 \), since \( \psi \in \Psi \). Hence \( \{y_n\} \) is a Cauchy sequence. The metric space \( X \) being complete, there is a point \( z \) such that

\[ (3.9) \quad \lim_{n \to \infty} Fx_n = \{ z \} = \left\{ \lim_{n \to \infty} gx_n \right\} \]

As \( F \) and \( g \) are limit coincidentally commuting, we have

\[ (3.10) \quad \lim_{n} Fgx_n = \lim_{n} gFx_n. \]

Let us assume that \( g \) is continuous, then from (3.8), it follows that

\[ \lim_{n} gFx_n = g \left( \lim_{n} Fx_n \right) = gz. \]

We show that \( z \) is a common fixed point of \( F \) and \( g \). Now

\[ d^r(z, gz) = \lim_{n} d^r(y_{n+1}, gz) \leq \lim_{n} \delta^r(Fx_n, gFz) = \lim_{n} \delta^r(Fx_n, gFx_n) = \]

\[ \lim_{n} \delta^r(Fx_n, Fgx_n) \leq \lim_{n} \delta^r(Fx_n, Fgx_n) \leq \lim_{n} \delta^r(y_{n+1}, gz) \]

\[ \leq \phi(d^r(z, gz), \delta^r(z, z), \delta^r(z, gz), \delta^r(gz, z), \delta^r(gz, z)) = \phi(d^r(z, gz), 0, 0, \delta^r(z, gz), 0, \delta^r(z, gz)) \]

which is possible only when \( z = gz \) as \( \psi \in \Psi \). Similarly
\[ \delta^r(z, Fz) = \lim_{n} \delta^r(Fx_n, Fz) \leq \lim_{n} \delta^r(gx_n, gFz) = \delta^r(gz, Fz) = \delta^r(gz, Fx_n) = \delta^r(gz, Fz) = \delta^r(gz, Fx_n) =\]

\[ = \phi(0, 0, \delta^r(z, Fz), 0) \leq \psi(\delta^r(z, Fz)) \]

i.e.,

\[ \delta^r(z, Fz) = r \Rightarrow Fz = \{z\}. \]

Thus \( z \) is a common fixed point of \( F \) and \( G \).

Again suppose that \( F \) is continuous on \( X \). Then

\[ \lim_{n} Fgx_n = F \left( \lim_{n} gx_n \right) = Fz. \]

Then

\[ \delta^r(z, Fz) = \lim_{n} \delta^r(Fx_n, FFx_n) = \]

\[ = \lim_{n} \phi(d^r(gx_n, gFx_n), \delta^r(gx_n, Fx_n), \delta^r(gFx_n, FFx_n), \delta^r(gFx_n, Fx_n)) = \]

\[ = \phi(\delta^r(z, Fz), \delta^r(z, z), \delta^r(Fz, Fz), \delta^r(z, Fz)) = \]

\[ = \phi(\delta^r(z, Fz), 0, 0, \delta^r(z, Fz), \delta^r(z, Fz)) \leq \psi(\delta^r(z, Fz)) \]

and so \( Fz = \{z\} \) as \( \psi \in \Psi \).

Since \( F(X) \subseteq g(X) \), there is a point \( z' \in X \) such that \( gz' = z \in Fz \).

We show that \( Fz' = \{gz'\} \). Suppose not. Then by (3.1), we have

\[ \delta^r(gz', Fz') = \delta^r(z, Fz') = \lim_{n} \delta^r(Fx_n, Fz') \leq \]

\[ \leq \lim_{n} \phi(d^r(gx_n, gz'), \delta^r(gx_n, Fx_n), \delta^r(gz', Fz'), \delta^r(gx_n, Fz'), \delta^r(gz', Fx_n)) = \]

\[ = \phi(0, 0, \delta^r(gz', Fz'), \delta^r(gz', Fz'), 0) \leq \psi(\delta^r(gz', Fz')) \]

which is a contradiction. Hence \( \delta^r(gz', Fz') = 0 \), i.e. \( \{gz'\} = Fz' \).

Now, following the arguments similar to those in Case I, it is proved that \( z = gz' = Fz' \) is a fixed point of \( g \). Thus \( z \) is a common fixed point of \( F \) and \( g \).

To prove the uniqueness let \( z^* (\neq z) \) be another common fixed point \( F \) and \( g \), i.e., it is point such that \( Fz^* = \{z^*\} = gz^* \). Then by (3.1), we obtain

\[ d^r(z, z^*) = \delta^r(Fz, Fz^*) \leq \]

\[ \leq \phi(d^r(gz, gz^*), d^r(gz, Fz), d^r(gz^*, Fz^*), d^r(gz, Fz^*), d^r(gz^*, Fz)) = \]

\[ = \phi(d^r(z, z^*), 0, 0, d^r(z, z^*), d^r(z, z^*)) \leq \psi(d^r(z, z^*)) \]

which is a contradiction since \( \psi \in \Psi \). Hence \( z = z^* \).
Finally, we prove the continuity of $F$ at $z$, in the Hausdorff topology on $X$. Let $\{z_n\}$ be any sequence in $X$ converging to the unique common fixed point of $F$ and $g$. Then $\lim_{n} d(z_n, z) = 0$. As $g$ is continuous on $X$, it is continuous at $z$ and so $\lim_{n} d(gz_n, gz) = 0$. To conclude, it is enough to prove that $\lim_{n} H(Fz_n, Fz) = 0$. We know that

$$H(Fz_n, Fz) \leq \delta(Fz_n, Fz).$$

Now

$$\delta(Fz_n, Fz) \leq \phi(d(gz_n, gz), \delta(gz_n, Fz_n), \delta'(gz, Fz), \delta'(gz_n, Fz),$$

$$\delta'(gz_n, Fz), \delta'(gz, Fz))_. \delta(n)$$

Therefore,

$$\lim_{n} \delta(Fz_n, Fz) \leq \phi(0, \lim_{n} \delta'(z, Fz_n), 0, 0, \lim_{n} \delta'(z, Fz_n)) = \phi(0, \lim_{n} \delta'(Fz, Fz_n), 0, 0, \lim_{n} \delta'(Fz, Fz_n)) \leq \psi(\lim_{n} \delta'(Fz_n, Fz))$$

which is possible only when $\lim_{n} \delta'(Fz_n, Fz) = 0$, since $\psi \in \Psi$. Consequently, $\lim_{n} \delta(Fz_n, Fz) = 0$ and from (3.9) it follows that $\lim_{n} H(Fz_n, Fz) = 0$. This completes the proof.

**Corollary 3.1.** Let $X$ be a complete metric space and let $F : X \to B(X)$ satisfy

$$(3.12) \quad \delta'(Fz_n, Fz) \leq \phi(d'(x, y), \delta'(x, Fx), \delta'(y, Fy), \delta'(x, Fy), \delta'(y, Fx))$$

for all $x, y \in X$, where $\phi : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ is an upper semicontinuous, and nondecreasing function in each co–ordinate variable, satisfying (3.2). Then $F$ has a unique fixed point $z \in X$ and $F$ is continuous at $z$ in the Hausdorff topology on $X$.

**Corollary 3.2.** Let $X$ be a complete metric space and let $f, g : X \to X$ satisfy

$$(3.13) \quad d'(fx, fy) \leq \phi(d'(gx, gy), d'(gx, fx), d'(gy, fy), d'(gx, fy), d'(gy, fx))$$

for all $x, y \in X$, where $\phi : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ is an upper semicontinuous, and nondecreasing function in each co–ordinate variable, satisfying (3.2). Further suppose that
Then $f$ and $g$ have a unique common fixed point $z \in X$ and if $g$ is continuous at $z$, then $f$ is also continuous at $z$.

**Theorem 3.2.** Let $X$ be a complete metric space and $F : X \to B(X)$ and $g : X \to X$ be two mappings satisfying

\[
\delta(Fx, Fy) < \max\{d(gx, gy), \delta(gx, Fx), \delta(gy, Fy), \frac{1}{2}[\delta(gx, Fy) + \delta(gy, Fx)]\}
\]

for all $x, y \in X$, with right hand side being not zero. Further suppose that

(i) $F(X) \subseteq g(X)$,
(ii) $f$ and $g$ are limit coincidentally commuting, and
(iii) $F$ and $g$ are continuous on $X$.

Then $F$ and $g$ have a unique common fixed point.

**Proof.** First we note that if the set map $F$ and point map $g$ have a common fixed point, then from (3.12) it follows that the common fixed point is unique.

Since $X$ is compact, both sides of the inequality (3.14) are bounded on $X$. Now there are two cases:

**Case I.** Suppose that the right hand side of the inequality (3.14) is zero for some $x, y \in X$. Then we have $gx = Fx$, $x \in X$. Now following the arguments similar to Case I of Theorem 3.1, it is proved that $\{z\} = gx = Fx$ is a common fixed point of $F$ and $g$ and so, it is unique.

**Case II.** Suppose that the right hand side of the inequality (3.14) is positive for all $x, y \in X$. Denote for brevity,

\[
M(x, y) = \max\{d(gx, gy), \delta(gx, Fx), \delta(gy, Fy), \frac{1}{2}[\delta(gx, Fy) + \delta(gy, Fx)]\}
\]

for $x, y \in X$. Define a function $T : X \times X \to \mathbb{R}^+$ by

\[
T(x, y) = \frac{\delta(Fx, Fy)}{M(x, y)} \quad x, y \in X.
\]

Clearly the function $T$ is well defined since $M(x, y) \neq 0$ for all $x, y \in X$. Since $F$ and $g$ are continuous, from the compactness of $X$, it follows that
the function $T$ attains its maximum at some point $(u, v) \in X^2$. Call the value $c$. From (3.12), it follows that $0 < c < 1$. By the definition of $c$, we have $T(x, y) \leq c$ for all $x, y \in X$.

Now from (3.14), we obtain

$$\delta(Fx, Fy) \leq cM(x, y) = c \max\{d(g, gy), \delta(g, Fx), \delta(gy, Fy), 1/2[\delta(gx, Fy) + \delta(gy, Fx)]\}$$

for all $x, y \in X$.

As $X$ is compact, it is a complete metric space. Now the desired conclusion follows by an application of Theorem 3.1 with $r = 1$ and $\phi(t_1, t_2, t_3, t_4) = c \max\{t_1, t_2, t_3, 1/2(t_4 + t_5)\}$ for $t_i \in \mathbb{R}^+$, $i = 1, 2, 3, 4, 5$. The proof is complete.

**Corollary 3.3.** Let $X$ be a complete metric space and let $F : X \rightarrow B(X)$ be continuous set mapping satisfying

$$\delta(Fx, Fy) < \max\{d(x, y), \delta(x, Fx), \delta(y, Fy), 1/2[\delta(x, Fy) + \delta(y, Fx)]\}$$

for all $x, y \in X$ with right hand side not zero. Then $F$ has a unique fixed point.

**Corollary 3.4.** Let $X$ be a complete metric space, and let $F, g : X \rightarrow X$ be two continuous mapping satisfying

$$d(Fx, Fy) < \max\{d(gx, gy), d(gx, Fx), d(gy, Fy), 1/2[d(gx, Fy) + d(gy, Fx)]\}$$

for all $x, y \in X$ with right hand side not zero. Further suppose that

(i) $F(X) \subseteq g(X)$,

(ii) $F$ and $g$ are limit coincidentally commuting.

Then $F$ and $g$ have a unique common fixed point.

**4. Coincidentally commuting mappings.** In this section we prove the existence of the common fixed point of pairs of set and point maps without the requirement of the continuity of any of the maps and under the weaker version of the commutativity condition than that used in the previous section. But in this case one needs the stronger assumption that the image of the set under any one of the maps is complete.

**Definition 4.1.** Two maps $F : X \rightarrow B(X)$ and $g : X \rightarrow X$ are called **coincidentally commuting** if they commute at coincidence points i.e. $Fz = \{gz\}$ implies $gFz = Fgz$. 

It is clear that every pair of limit coincidentally commuting maps is coincidentally commuting, but the following example shows that the converse may not be true.

**Example 4.1.** Let \( X = [0, 2] \) and define two mappings \( F : [0, 1] \to 2^{[0,2]} \setminus \{ \emptyset \} \) and \( g : [0, 1] \to [0, 1] \) by

\[
Fx = \begin{cases} 
[x, 2 - x] & \text{if } x \in [0, 1) \\
2 & \text{if } x \in [1, 2]
\end{cases}
\]

and

\[
gx = \begin{cases} 
x & \text{if } x \in [0, 1) \\
2 & \text{if } x \in [1, 2].
\end{cases}
\]

We note that the mappings \( F \) and \( g \) are not continuous on \([0, 2]\). Now consider the sequence \( \{x_n\} \) in \([0, 2]\) such that \( x_n < 1 \) and \( x_n \to 1 \) and \( n \to \infty \). Then we have

\[
\lim_{n} \delta(Fx_n, gx_n) = \lim_{n} \delta([x_n, 2 - x_n], \{x_n\}) = 0.
\]

But

\[
\lim_{n} \delta(gFx_n, Fgx_n) = \lim_{n} \delta(g[x_n, 2 - x_n], Fx_n) = \lim_{n} \delta(\{x_n, 2\}, [x_n, 2 - x_n]).
\]

This shows that the pair \( \{F, g\} \) is not limit coincidentally commuting on \([0, 2]\). Now the set \( C \) of all coincidence points of \( F \) and \( g \) is \( C = [1, 2] \), i.e. \( gFu = Fgu \) for all \( u \in [1, 2] \). Hence \( \{F, g\} \) is a coincidentally commuting pair of maps on \([0, 2]\).

**Theorem 4.1.** Let \( F : X \to B(X) \) and \( g : X \to X \) be two mappings satisfying (3.1) and (3.2). Suppose that

(i) \( F(X) \subseteq g(X) \),
(ii) one of \( F(X) \) or \( g(X) \) is complete, and
(iii) \( \{F, g\} \) are coincidentally commuting.

Then \( F \) and \( g \) have a unique common fixed \( z \in X \). Further if \( g \) is continuous at \( z \), then so is also \( F \).

**Proof.** Let \( x \in X \) be arbitrary and define a sequence \( \{y_n\} \subseteq X \) by \( x_0 = x, y_0 = gx_0, gx_n = y_n \in Fx_{n-1}, n \in \mathbb{N}, \) for some sequence \( \{x_n\} \subseteq X \).

Clearly the sequence \( \{y_n\} \) is well defined in view hypothesis (i). Moreover \( \{y_n\} \subseteq F(X) \subseteq g(X) \).
Let $X_{n+1} = Fx_n$, $n = 0, 1, 2, \ldots$. Denote by $\beta_n = \delta(X_n, X_{n+1})$. Then there are two cases:

**Case I.** Suppose that $\beta_s = \delta(X_s, X_{s+1}) = 0$ for some $s \in \mathbb{N}$. Then one has

$$\{y_s\} = Fx_{s-1} = \{gx_s\} = Fx_s = \{gx_{s+1}\} = \{y_{s+1}\}.$$  \hspace{1cm} (4.1)

Since $F$ and $g$ are coincidentally commuting, one has $gF x_s = Fgx_s$. Now following the arguments similar to those in Case I of Theorem 3.1, it is proved that $z = gx_s$ is a common fixed point of $F$ and $g$.

**Case II.** Assume that $\beta_n = \delta(X_n, X_{n+1}) \neq 0$, for each $n \in \mathbb{N}$. Then proceeding with the arguments similar to that in Case II of Theorem 3.1, it is proved that $\{y_n\}$ is a Cauchy sequence in $F(X) \subseteq g(X)$. Suppose that $g(X)$ is complete, then there is a point $z \in g(X)$ such that

$$\lim_n Fx_n = \lim_n gx_n.$$  \hspace{1cm} (4.2)

As $z \in g(X)$, there is a point $u \in X$, such that $z = gu$. We show that $Fu = \{gu\}$. By (3.1), one gets

$$\delta^r(Fu, gu) = \delta^r(Fu, z) = \lim_n \delta^r(Fu, Fx_n) = \lim_n \phi(d^r(gu, gx_n), \delta^r(gu, Fu), \delta^r(gx_n, Fx_n),$$

$$\delta^r(gu, Fx_n), \delta^r(gx_n, Fu)) = \phi(0, \delta^r(gu, Fu), 0, 0, \delta^r(gu, Fu)) \leq \psi(\delta^r(gu, Fu))$$

which is possible only when $\delta^r(Fu, gu) = 0$, i.e. $Fu = gu$, since $\psi \in \Psi$.

Now following the arguments of Case I yields that $z = Fu = gu$ is a common fixed point of $F$ and $g$. Similarly if $F(X)$ is complete, then there is a point $z \in F(X) \subseteq g(X)$ such that (4.2) holds. Again repeating the above arguments yields that $F$ and $g$ have a common fixed point. The condition (3.1) implies the uniqueness of $z$. Finally if $g$ is continuous at $z$, then condition (3.1) implies the continuity of $F$ and $z$ in the Hausdorff topology on $X$.

The proof is complete.

When $F = \{f\}$, as a consequence of Theorem 4.1, we obtain

**Corollary 4.1.** Let $f, g : X \to X$ be two mappings satisfying (3.1) and (3.2). Suppose that

(i) $f(X) \subseteq g(X)$,
(ii) anyone of \( f(X) \) or \( g(X) \) is complete, and 
(iii) \( \{f,g\} \) are coincidentally commuting.

Then \( f \) and \( g \) have a unique common fixed point \( z \in X \). If \( g \) is continuous at \( z \), then \( f \) is also continuous at \( z \).

Again taking \( r = 1 \) in Corollary 4.1 we obtain Theorem 2.1 of DHAGE and KAKDE [4] which further contains some known results in the fixed point theory for the contraction mappings in metric spaces.

**Theorem 4.2.** Let \( F : X \to B(X) \) and \( g : X \to X \) be mappings satisfying (3.1) and (3.2). Suppose that 

(i) \( F(X) \subseteq g(X) \),  
(ii) \( g(X) \) is compact,  
(iii) \( F \) and \( g \) are coincidentally commuting, and  
(iv) \( F \) and \( g \) are continuous.

Then \( F \) and \( g \) have a unique common fixed point.

**Proof.** The proof is similar to Theorem 3.2 by replacing \( X \) with \( g(X) \), and now the desired conclusion of the theorem follows by an application of Theorem 4.1.

5. **Coincidentally pseudo–commuting mappings.** In this section we introduce some new notions concerning the commutativity of a pair of point and set mappings on a metric space which are weaker than that of previous section and prove some common fixed point theorems under suitable contraction condition.

**Definition 5.1.** Two mappings \( F : X \to B(X) \) and \( g : X \to X \) are called pseudo-commuting if \( gF x \cap Fgx \neq \emptyset \) for all \( x \in X \) and coincidentally pseudo-commuting if they pseudo-commute at coincidence points, i.e. if \( Fu = \{gu\} \) for some \( u \in X \), then \( gF u \cap F gu \neq \emptyset \).

**Definition 5.2.** Two mappings \( F : X \to B(X) \) and \( g : X \to X \) are called limit pseudo-commuting if \( \lim_{n} D(Fx_{n}, Fgx_{n}) = 0 \), where \( \{x_{n}\} \) is a sequence in \( X \), and limit coincidentally pseudo-commuting if \( \lim_{n} \delta(Fx_{n}, gx_{n}) = 0 \) implies \( \lim_{n} D(gFx_{n}, ggx_{n}) = 0 \), where \( \{x_{n}\} \) is a sequence in \( X \).

In the following we shall deal only with coincidentally pseudo-commuting and limit coincidentally pseudo-commuting mappings and establish
the existence of the common fixed point under the condition of generalized nonlinear connection.

We need the following lemma in the sequel.

**Lemma 5.1.** For all \( x, y \in X \) and \( A, B \in B(X) \)

\[
\begin{align*}
|D(x, A) - D(y, A)| & \leq D(x, y) \\
|D(x, A) - D(x, B)| & \leq H(A, B),
\end{align*}
\]

(5.1)

**Lemma 5.2.** If \( x_n \to x \) and \( \{A_n\} \) is a sequence in \( B(X) \) converging to a set \( A \) in \( B(X) \), then \( D(x_n, A_n) \to D(x, A) \).

**Proof.** Now

\[
|D(x_n, A_n) - D(x, A)| = |D(x_n, A_n) - D(x, A_n) + D(x, A_n) - D(x, A)| \leq
\]

\[
\leq |D(x_n, A_n) - D(x, A_n)| + |D(x, A_n) - D(x, A)| \leq
\]

\[
\leq D(x_n, x) + D(A_n, A) \leq D(x_n, x) + H(A_n, A) \to 0 \text{ as } n \to \infty.
\]

This shows that \( D(x_n, A_n) \to D(x, A) \) and the proof of the lemma is complete.

If the following we formulate the main results of this section.

**Theorem 5.1.** Let \( X \) be a complete metric space and let \( F : X \to B(X) \) and \( g : X \to X \) be two mappings satisfying

\[
\delta^r(Fx, Fy) \leq \phi(d^r(gx, gy), D^r(gx, Fx), D^r(gy, Fy), D^r(gx, Fy), D^r(gy, Fx))
\]

(5.2)

for all \( x, y \in X \), where \( \phi : (\mathbb{R}^+)^5 \to \mathbb{R}^+ \) is upper semi-continuous and nondecreasing in each co-ordinate variable satisfying (3.2). Further suppose that

(i) \( F(X) \subseteq g(X) \),

(ii) \( \{F, g\} \) are limit coincidentally pseudo-commuting and

(iii) anyone of \( F \) or \( g \) is continuous.

Then \( F \) and \( g \) have a unique common fixed point \( z \in X \) and if \( g \) is continuous at \( z \), then \( F \) is also continuous at \( z \).

**Proof.** Let \( x \in X \) be arbitrary and define a sequence \( \{y_n\} \subseteq X \) by (3.3). Clearly the sequence \( \{y_n\} \) is well defined in view of hypothesis (i). As in the proof of Theorem 3.1, denote by \( \beta_n = \delta(X_n, X_{n+1}) \), where \( X_n = Fx_{n+1}, n \in \mathbb{N} \). Then there are two cases:
Case I. Suppose that \( \beta_s = \delta(X_s, X_{s+1}) = 0 \) for some \( s \in \mathbb{N} \). Then one has
\[
y_s = Fx_{s-1} = gx_s = Fx_s = gx_{s+1} = y_{s+1}.
\]
Since \( F \) and \( g \) are limit coincidentally pseudo-commuting, by Lemma 5.2, we get \( gFx_s \cap Fgx_s \neq \emptyset \) i.e., \( D(gFx_s, Fgx_s) = 0 \).

We show that \( z = gx_s \) is a common fixed point of \( F \) and \( g \). By (5.1)
\[
\delta(z, Fz) = \delta^r(Fx_s, Fz) \leq \phi(d^r(gx_s, gz), D^r(gx_s, Fx_s), D^r(gz, Fz), D^r(gx_s, Fz), D^r(gz, Fx_s)) \leq \phi(d^r(gx_s, gz), 0, 0, \delta^r(Fgx_s, z)) = \phi(\delta^r(z, Fz), 0, 0, \delta^r(z, Fz)) \leq \psi(\delta^r(z, Fz))
\]
which gives \( \{z\} = Fz \) some \( \psi \in \Psi \). Again
\[
\delta(z, gz) = \delta(Fx_s, gFx_s) \leq \delta(Fx_s, Fgx_s) \leq \delta(Fx_s, Fz) \leq \delta(z, Fz) = 0
\]
which implies that \( z = gz \). Thus \( z \) is a common fixed point of \( F \) and \( g \).

Case II. Now we assume that \( \beta_n = \delta(X_n, X_{n+1}) \neq 0 \) for each \( n \in \mathbb{N} \). By (5.1),
\[
d^r(Fx, Fy) \leq \phi(d^r(gx, gy), D^r(gx, Fx), D^r(gy, Fy), D^r(gx, Fy), D^r(gy, Fx)) \leq \phi(d^r(gx, gy), \delta^r(gx, Fx), \delta^r(gy, Fy), \delta^r(gx, Fy), \delta^r(gy, Fx))
\]
for all \( x, y \in X \). This shows that the inequality (3.1) is satisfied. Hence proceeding with the arguments similar to Case II of the proof of Theorem 3.1, it is proved that \( \{y_n\} \) is a Cauchy sequence in \( X \). Also \( \lim_n \beta_n = 0 \). Since \( X \) is complete, there is a point \( z \in X \) such that \( \lim_n y_n = z \). By definition of \( \{y_n\} \) this implies that
\[
\lim_n Fx_n = y = \lim_n gx_n \quad \text{i.e.}
\]
\[
\lim_n \delta(Fx_n, gx_n) = 0.
\]
Since \( F \) and \( g \) are limit coincidentally pseudo-commuting, we have
\[
\lim_n D(gFx_n, Fgx_n) = 0.
\]
Suppose first that \( g \) is continuous, then
\[
(5.4) \quad gu = g(\lim_n Fx_n) = \lim_n gFx_n \in \lim_n Fgx_n.
\]

We show that \( Fu = \{gu\} \). If not, then by (5.1),
\[
\delta^r(Fu, gu) = \lim_n \delta^r(Fu, gFx_n) \leq \lim_n \phi(d(gu, ggx_n), D^r(gu, Fu), D^r(ggx_n, Fgx_n)) \leq \phi(0, \delta^r(Fu, gu), 0, d^r(gu, Fu)) \leq \psi(d^r(gu, Fu))
\]
which is a contradiction. Hence \( Fu = \{gu\} \). Now repeating the arguments of Case I, it is proved that \( z = Fu = gu \) is a common fixed point of \( F \) and \( g \).

Now suppose that \( F \) is continuous, then
\[
Fu = F(\lim_n gx_n) = \lim_n Fgx_n.
\]
As \( F(X) \subseteq g(X) \), \( z \in Fu \) and \( z = gv \) for some \( v \in X \). We shall show that \( Fv = gv \). Now
\[
\delta(gv, Fv) = \delta(z, Fv) = \lim_n \delta(Fx_n, Fv) \leq \lim_n \phi(d(gx_n, gv), D^r(gx_n, Fx_n), D^r(gv, Fv), D^r(gv, Fv), 0) \leq \psi(d^r(gv, Fv))
\]
which is possible only when \( \{gv\} = Fv \) since \( \psi \in \Psi \). Then by repeating the arguments of Case I, it is shown that \( z = gv = Fv \) is a common fixed point of \( F \) and \( g \).

Condition (5.1) implies the uniqueness of \( z \). Finally suppose that \( g \) is continuous at \( z \), as the inequality (5.1) implies the inequality (3.1), the continuity of \( F \) is proved by repeating the arguments given in the proof of Theorem 3.1. This completes the proof.

**Theorem 5.2.** Let \( X \) be a compact metric space and let \( F : X \rightarrow B(X) \) and \( g : X \rightarrow X \) be two continuous mappings satisfying

\[
\delta(Fx, Fy) < \max\{d(gx, gy), D(gx, Fx), D(gy, Fy), \\
1/2[D(gx, Fy) + D(gy, Fx)]\}
\]

for all \( x, y \in X \) with right hand side being not zero. Further suppose that
(i) \( F(X) \subseteq g(X) \) and
(ii) \( \{F, g\} \) are limit coincidentally pseudo-commuting.

Then \( F \) and \( g \) have a unique common fixed point.

**Proof.** The proof is similar to Theorem 4.2 and now the desired conclusion follows by an application of Theorem 5.1.

**Theorem 5.3.** Let \( F : X \to B(X) \) and \( g : X \to X \) be two mappings satisfying (5.1) and (5.2). Suppose further that
(i) \( F(X) \subseteq g(X) \),
(ii) any one of \( F(X) \) or \( g(X) \) is complete, and
(iii) \( \{F, g\} \) are coincidentally pseudo-commuting.

Then \( F \) and \( g \) have a unique common fixed point \( z \in X \) and if \( g \) is continuous at \( z \), then \( F \) is also continuous at \( z \).

**Proof.** Let \( x \in X \) be arbitrary and define a sequence \( \{y_n\} \) in \( X \) by (3.3). Clearly \( \{y_n\} \) is well defined in view of hypothesis (i). Moreover \( \{y_n\} \subseteq F(X) \subseteq g(X) \).

Denote by \( \beta_n = \delta(X_n, X_{n+1}) \), \( n \in \mathbb{N} \), where \( X_n = Fx_{n-1}, n \in \mathbb{N} \).

Now there are two cases:

**Case I.** Suppose that \( \beta_s = \delta(X_s, X_{s+1}) = 0 \) for some \( s \in \mathbb{N} \), then we have
\[
\{y_s\} = Fx_{s-1} = \{gx_s\} = Fx_s = \{gx_{s+1}\} = \{y_{s+1}\}.
\]

Since \( F \) and \( g \) are coincidentally pseudo–commuting we get \( gFx_s \in Fgx_s \), i.e., \( D(gFx_s, Fgx_s) = 0 \).

Now using the arguments similar to Case I of Theorem 5.3, it is proved that \( z = gx_s = Fx_s \) is a common fixed point of \( F \) and \( g \).

**Case II.** Suppose that \( \beta_n = \delta(X_n, X_{n+1}) \neq 0 \) for each \( n \in \mathbb{N} \) from Case II of Theorem 4.1 it follows that \( \{y_n\} \) is a Cauchy sequence in \( F(X) \subseteq g(X) \).

Suppose that \( g(X) \) is complete then there is a point \( u \in X \) such that
\[
l \lim_{n} Fx_n = \lim_{n} gx_n = z \quad \text{and} \quad z = gu.
\]

We show that \( Fu = gu \). By (5.1) we get
\[
\delta^r(Fu, gu) = \lim_{n} \delta^r(Fu, Fx_n) \leq \lim_{n} \phi(gu, gx_n), D^r(gu, Fu), D^r(gx_n, Fx_n) \leq \phi(0, \delta^r(Fu, gu), 0, \delta^r(gx_n, Fx_n)) = \omega(0, \delta^r(Fu, gu), 0, \delta^r(Fu, gu)) \leq \psi(\delta^r(Fu, gu))
\]

which is possible only when \( Fu = gu \) since \( \psi \in \Psi \). Similarly if \( F(X) \) is complete, there is a point \( z \in F(X) \subseteq g(X) \). Such that \( z = gu \) for some
u ∈ X. The rest of the proof is similar to Theorem 5.1. This completes the proof.

**Theorem 5.4.** Let $F : X \rightarrow B(X)$ and $g : X \rightarrow X$ be two mappings satisfying (5.1) and (5.2). Further suppose that
(i) $F(X) \subseteq g(X)$,
(ii) $g$ is compact, and
(iii) $\{F, g\}$ are coincidentally pseudo-commuting.

Then $F$ and $g$ have a unique common fixed point.

**6. Examples.** In this section we give some examples concerning the application of the abstract theory developed in the previous section.

**Example 6.1.** Let $X = [0, 1]$ be equipped with the Euclidean metric $d$. Consider the two mappings $F : [0, 1] \rightarrow 2^{[0, 1]} \setminus \{\emptyset\}$ and $g : [0, 1] \rightarrow [0, 1]$ defined by $Fx = [x^3/3, x/2]$ and $gx = x/2$. Obviously $F([0, 1]) = [0, 1/2] \subseteq [0, 1/2] = g([0, 1/2])$ which is closed and hence complete. The set of all coincidence points of $F$ and $g$ is $C = \{0\}$. Also $F$ and $g$ are continuous and coincidentally commuting on $[0, 1]$, since $(Fg)(0) = (gF)(0)$. Finally $F$ and $g$ satisfy condition (3.1) with $r = 1$ and

$$\phi(t_1, t_2, t_3, t_4, t_5) = (1/4) \max\{t_1, t_2, t_3, 1/2(t_4 + t_5)\}.$$ 

Thus all the conditions of Theorem 3.1 are satisfied and hence $F$ and $g$ have a unique common fixed point, viz., 0.

**Example 6.2.** Let $X = \mathbb{R}^+$ be equipped with the usual Euclidean metric $d$. Define $F : \mathbb{R}^+ \rightarrow 2^{\mathbb{R}^+} \setminus \{\emptyset\}$ and $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by $Fx = [x/8 + 1, (x/2) + 1]$ and $gx = (x/2) + 1$ for $x \in \mathbb{R}^+$.

Clearly $F$ and $g$ satisfy condition (3.1).

With $r = 1$ and $\phi(t_1, t_2, t_3, t_4, t_5) = (1/4) \max\{t_1, t_2, t_3, 1/2(t_4 + t_5)\}$.

Again $F$ and $g$ satisfy hypotheses (i) and (iii) of Theorem 3.1 but $F$ and $g$ have no any common fixed point in $\mathbb{R}^+$, because $F$ and $g$ are not coincidentally commuting on $\mathbb{R}^+$. Here the set $C$ of all coincidence points of $F$ and $g$ is $C = \{0\}$, however,

$$(gF)(0) = g(F0) = g(1) = 3/2 \neq 9/8 = F(g0) = (Fg)(0).$$

Hence the hypothesis (ii) of coincidentally commutativity of $F$ and $g$ cannot be relaxed in Theorem 3.1.
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