ON BOOLEAN, NEAT, PURE AND STONE IDEALS

BY

LADISLAV BERAN∗

Abstract. We prove that the class of Boolean ideals is rhombically hereditary in the class of modular lattices with 0. Similar results are obtained also for pure ideals, Stone ideals and for neat ideals.

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1. Introduction. Let $\mathcal{K}$ denote a class of lattices and let $\mathcal{J}$ be a class of ideals. The class $\mathcal{J}$ will be called rhombically hereditary in $\mathcal{K}$ if the implication

$$(P \lor Q \in \mathcal{J} \And P \land Q \in \mathcal{J}) \Rightarrow (P \in \mathcal{J} \And Q \in \mathcal{J})$$

is true for any ideals $P, Q$ of $L$ where $L$ denotes any lattice of $\mathcal{K}$.

The first known result on the rhombic heredity was established in [8] and can be reformulated as follows: The class of principal ideals is rhombically hereditary in the class $\mathcal{D}$ of distributive lattices. This statement was generalized to an analogous assertion where the class $\mathcal{D}$ is replaced by the class $\mathcal{M}$ of modular lattices [1, Thm 1].

All unexplained terminology is standard as in [2] and [9].

2. Boolean, neat, pure and Stone ideals. An element $x$ of a lattice $L$ with 0 and 1 is said to be Boolean [3] if there exists at least one complement $y$ of $x$, i.e., an element $y \in L$ such that $x \land y = 0$ and $x \lor y = 1$.

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The set of all Boolean elements in \( L \) will be denoted by \( B(L) \). M. Cherciu [6] denotes this set by \( C(L) \) which corresponds to a French terminology (\( \text{élément chrysippiens} \)) (see also [10] and [11]).

Analogously, an ideal \( I \) of a lattice \( L \) with 0 will be here called a Boolean ideal of \( L \), if it is a Boolean element of the ideal lattice \( \text{Id}L \) of \( L \), i.e., if there exists an ideal \( J \) of \( L \) such that \( I \cap J = \{0\} \) and \( I \lor J = L \).

**Theorem 1.** Let \( L \) be a modular lattice with 0 and 1. If \( a, b \in L \) are such that \( a \lor b \) and \( a \land b \) are Boolean, then \( a \) and \( b \) are also Boolean.

**Proof.** Put \( s := a \lor b \) and \( p := a \land b \), let \( s^* \) denote a complement of \( s \) and \( p' \) denote a complement of \( p \).

We will first show that \( s' := (s^* \lor p) \land p' \) is a complement of \( s \). Note that

\[
\text{(1)} \quad s' \leq p'.
\]

By modularity,

\[
(s \land p') \lor s' = (s \land p') \lor [(s^* \lor p) \land p'] = [(s \land p') \lor s^* \lor p] \land p' = \{[s \lor (p' \lor p)] \lor s^*\} \land p' = p'.
\]

One observes that

\[
s \lor s' \geq (s \land p') \lor s' = p'
\]

and so \( s \lor s' \geq p \lor p' = 1 \). By modularity,

\[
s \land s' = s \land (s^* \lor p) \land p' = [(s \land s^*) \lor p] \land p' = 0.
\]

Let \( a' := s' \lor (p' \land b) \). Using modularity, we get

\[
a \lor a' \geq p \lor a' = p \lor s' \lor (p' \land b) = s' \lor [(p \lor p')] \land b = s' \lor b.
\]

Hence \( a \lor a' \geq s' \lor a \lor b = 1 \). Now, by modularity and (1), \( a' = s' \lor (p' \land b) = p' \land (b \lor s') \). Therefore, using modularity once again we see that

\[
a \land a' \leq s \land a' = s \land p' \land (b \lor s') = p' \land [b \lor (s \land s')] = p' \land b.
\]

Consequently, \( a \land a' \leq p' \land b \land a = 0 \). Thus \( a' \) is a complement of \( a \). \( \square \)
Corollary 2. The class of Boolean ideals is rhombically hereditary in the class of modular lattices with 0.

Proof. The ideal lattice $\text{Id} L$ of a modular lattice $L$ with 0 is modular [9, p. 41] and bounded. □

An ideal $I$ of a lattice $L$ with 0 is said to be pure [4] if, given any $a \in L$, there exists an ideal $I$ of $L$ such that $P \wedge I = \{0\}$ and $P \vee I = P \vee (a)$.

Corollary 3. The class of pure ideals is rhombically hereditary in the class of modular lattices with 0 and 1.

Proof. Combine Theorem 1 with [4, Theorem 3.1]. □

An ideal $I$ of a lattice $L$ with 0 and 1 is said to be Stone (see, e.g. [6] or [7]) if for any $i \in I$ there exists an element $j \in B(L)$ such that $i \leq j$ and $j \in I$.

Remark 4. If $I$ is a principal ideal of $L$, say $I = (i_0)$, then it is a Stone ideal if and only if $i_0 \in B(L)$.

Corollary 5. The class of principal Stone ideals is rhombically hereditary in the class of modular lattices with 0 and 1.

Proof. Suppose that $I, J \in \text{Id} L$ are such that $I \vee J$ and $I \wedge J$ are principal Stone ideals. Then $I \vee J = (s)$ and $I \wedge J = (p)$ for suitable $s, p \in B(L)$. Using the result mentioned above [1, Thm 1], we can see that $I = (a)$ and $J = (b)$. The elements $a$ and $b$ belong to $B(L)$ by Theorem 1. Hence $I$ and $J$ are Stone ideals by Remark 4. □

Theorem 6. The class of Stone ideals is rhombically hereditary in the class of distributive lattices with 0 and 1.

Proof. Let $A$ and $B$ be ideals of a bounded distributive lattice $L$ and suppose that $A \vee B$ and $A \wedge B$ are Stone ideals. Choose $a$ in $A$ and $b \in B$. Now, referring to the fact that $A \vee B$ is a Stone ideal, we see that there exists $s \in B(L)$ such that $a \vee b \leq s \in A \vee B$.

Since $s \in A \vee B$ and $L$ is distributive, there exist [9, p. 81] elements $a_0 \in A$ and $b_0 \in B$ such that $s = a_0 \vee b_0$.

Let $a_1 := a \vee a_0$, $b_1 := b \vee b_0$. Then

\[ a_1 \vee b_1 = s. \]
By assumption, \( A \land B \) is a Stone ideal and \( a_1 \land b_1 \in A \land B \). Hence there exists \( p \in B(L) \) such that \( a_1 \land b_1 \leq p \).

Put \( a_2 := a_1 \lor p_1, b_2 := b_1 \lor p_1 \) where \( p_1 := p \land s \). By distributivity, \( p_1 \in B(L) \). Evidently,

\[
(3) \quad a_1 \land b_1 \leq p \land s = p_1.
\]

One has also

\[
a_2 \lor b_2 = a_1 \lor b_1 \lor p_1 = s \lor p_1 = s.
\]

From (3) we deduce

\[
(4) \quad a_2 \land b_2 = (a_1 \lor p_1) \land (b_1 \lor p_1) = (a_1 \land b_1) \lor p_1 = p_1.
\]

Since \( p_1 \) and \( s \) belong to \( B(L) \), we can conclude from Theorem 1 that \( a_2, b_2 \in B(L) \). Then in view of \( a \leq a_2 \) and \( b \leq b_2 \) it is immediate that \( A \) and \( B \) are Stone ideals. \( \square \)

**Remark 7.** It is an open question whether the class of Stone ideals is rhombically hereditary in the class of modular lattices with 0 and 1.

An element \( i \) of a lattice \( L \) with 0 is called *neat* in \( L \) [5] if for any \( s \) covering \( i \) there exists a relative complement \( i^+ \) of \( i \) in \( [0, s] \), i.e.,

\[
i < s \Rightarrow \exists i^+ \in L \, i \lor i^+ = s \quad \& \quad i \land i^+ = 0.
\]

Neat ideals can be defined in a similar manner: An ideal \( I \) of a lattice \( L \) with 0 is said to be neat if for any ideal \( S \in \text{Id} L \) covering \( I \) in \( \text{Id} L \) there exists an ideal \( I^+ \in \text{Id} L \) such that \( I \lor I^+ = S \) and \( I \land I^+ = \{0\} \).

**Lemma 8.** If \( L \) is a modular lattice and \( x \prec y \) in \( L \), then either \( x \land z = y \land z \) or \( x \land z \prec y \land z \) for any \( z \in L \). In the latter case, \( y/x \nless y \land z/x \land z \), i.e., \( x \lor (y \land z) = y \) and \( x \land (y \land z) = x \land z \).

The proof is omitted since it is straightforward. \( \square \)

**Theorem 9.** Let \( a, b \) be arbitrary elements of a modular lattice \( L \) with 0. If both \( a \lor b \) and \( a \land b \) are neat in \( L \), then \( a \) and \( b \) are also neat elements in \( L \).

**Proof.** Suppose that

\[
(5) \quad a \prec s
\]
and distinguish between two cases.

**Case I:** $a \land b \neq b \land s$. From Lemma 8 and (5) we can deduce easily that in this case

$$a \land b \prec b \land s.$$  

Since $a \land b$ is neat, there exists $c \in L$ such that

$$ (a \land b) \lor c = b \land s \quad \& \quad (a \land b) \land c = 0. $$

To finish our argument in the first case we will show that $c$ is a relative complement of $a$ in $[0, s]$.

Now $c \leq s \land b \leq s$ and $a \leq s$ imply that $a \lor c \leq s$. By (7), $a \lor c \geq (a \land b) \lor c = s \land b$ so that $a \lor c \geq (s \land b) \lor a$. From Lemma 8 and (6) we get $s/a \land s \land b/a \land b$. Thus $a \lor (s \land b) = s$ and, therefore, $a \lor c = s$.

On the other hand, by (7) we have $a \land c \leq (a \land b) \land c = 0$. Hence $c$ is a relative complement of $a$ in $[0, s]$.

**Case II:** $a \land b = b \land s$. Then $s \not\leq a \lor b$.

Indeed, suppose $a \prec s \leq a \lor b$. It follows that $a \lor b \leq s \land b \leq a \lor b$, i.e., $b \lor s = b \lor a$ and, at the same time, $b \land s = b \land a$ by assumption. This, together with modularity, yields $a = s$, a contradiction.

Since $a \prec s$, the dual of Lemma 8 implies that either $a \lor b = s \lor b$ or $a \lor b \prec s \lor b$. However, we have seen that $a \lor b = s \lor b$ is impossible. Hence $a \lor b \prec s \lor b$.

Taking into account that $a \lor b$ is neat, we infer that there exists $d \in L$ such that

$$ (a \lor b) \lor d = s \lor b \quad \& \quad (a \lor b) \land d = 0. $$

By modularity and (8),

$$ a \lor [s \land (b \lor d)] = s \land (a \lor b \lor d) = s \land (s \lor b) = s. $$

Therefore,

$$ s/a \land s \land (b \lor d)/a \land (b \lor d). $$

By modularity,

$$ a \land (b \lor d) \prec s \land (b \lor d). $$
Next, observe that the modularity of $L$ and (8) yield
\[ b \lor [a \land (b \lor d)] = (b \lor a) \land (b \lor d) = b \lor [(b \lor a) \land d] = b \]
and so $a \land (b \lor d) \leq b$. It follows that $a \land (b \lor d) \leq a \land b$ and, consequently,
\[ a \land (b \lor d) = a \land b. \tag{11} \]

Combining (10) with (11), we get $a \land b \prec s \land (b \lor d)$. Since $a \land b$ is neat, there exists an element $e \in L$ such that
\[ (a \land b) \lor e = s \land (b \lor d) \quad \& \quad (a \land b) \land e = 0. \tag{12} \]

Our aim is to show that $e$ is a relative complement of $a$ in $[0, s]$. To this end, observe first that by (12) we have
\[ a \lor e \geq (a \land b) \lor e = s \land (b \lor d). \]

Hence, from (9), $a \lor e \geq a \lor [s \land (b \lor d)] = s$. Also $a \prec s$ and, by (12), $e \leq s \land (b \lor d) \leq s$. This implies that $a \lor e \leq s$. Thus $a \lor e = s$. Moreover, by (12), (5) and (11),
\[ a \land e \leq a \land [s \land (b \lor d)] = a \land (b \lor d) = a \land b. \]

Therefore, $a \land e \leq (a \land b) \land e$. From (12) we get $a \land e = 0$. \hfill \Box

**Corollary 10.** The class of neat ideals is rhombically hereditary in the class of modular lattices with 0.

**Proof.** Let $A, B \in \text{Id} L$ be such that $A \lor B$ and $A \land B$ are neat ideals of a modular lattice $L$. Since $\text{Id} L$ is a modular lattice with the unit equal to $L$ and with the zero equal to $\{0\}$, it is sufficient to apply Theorem 9. \hfill \Box

**REFERENCES**


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