FAMILIES OF SUBSETS AND THE COINCIDENCE OF HYPERTOPOLOGIES

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Let \((X, d)\) be a fixed metric space and \(\text{Cl}(X)\) the family of nonempty closed subsets of \(X\). In this paper we consider topologies on \(\text{Cl}(X)\), namely hypertopologies. The aim of this work is to find some families \(\mathcal{A} \subset \text{Cl}(X)\) on which two known hypertopologies coincide. The most ample families with this property are obtained.

1. Introduction. Let \((X, d)\) be a metric space. The family of nonempty subsets of \(X\) is denoted by \(\mathcal{P}(X)\), while \(\text{Cl}(X)\) denotes the family of those elements of \(\mathcal{P}(X)\) which are closed.

In this paper we will consider topologies on \(\mathcal{P}(X)\) which extend the initial topology on \(X\), i.e. if we restrict the topologies to the singleton subsets, the topology of induced subspace agrees with the initial topology on \(X\). We shall call such topologies, hypertopologies or hyperspace topologies.

Now, given two hyperspatial topologies defined on the same family \(\mathcal{A} \subset \mathcal{P}(X)\), we are studying the problem of their comparison.

During the time, this problem was solved on the most employed families, i.e. \(\text{Cl}(X)\). In the following, we will give the relation between the most known hypertopologies defined on the family \(\text{Cl}(X)\), as synthesized in [13]:
where \( \tau_H, \tau_{AW}, \tau_V, \tau_P, \tau_{bP} \) denoted the Hausdorff, Attouch-Wets, Vietoris, proximal and b-proximal topologies, respectively.

The relations described above are generally strict. Several studies have been done in order to establish the necessary and sufficient conditions on a metric space \((X, d)\) who determine the coincidence of two concrete topologies, \(\tau_1\) and \(\tau_2\) on \(\text{Cl}(X)\). By defining precisely \(\tau_1\) and \(\tau_2\), several features are obtained, specified in [6].

The results synthesized in [6] give an answer to the following question: if the family of subsets \(\text{Cl}(X)\) is fixed we search speciale classes of metric spaces who make two concrete hypertopologies to coincide.

In this paper, we will find the answer to the "dual" problem: given a fixed metric space \(X\) (for example even normed space), we are going to find some families \(A \subset \text{Cl}(X)\) on which two known hypertopologies coincide. Though this problem has never been raised previously in literature (as far as we know), there still exist some partial results, of a sufficient condition-type (see Proposition 2.1 and Proposition 2.2, Section 2).

We will obtain the most ample family (in a certain sense which will be specified in each case) with this property.

The aim of this paper is to solve this question for the pairs of topologies from the above diagram (Theorems 4.1, 5.3, 6.2 and Corollaries 7.1, 7.2).

In Section 2, we recall some notions, notations and results which we need later. Section 3 gives the stability of the Hausdorff topology with respect to boundedness and totally boundedness. In Sections 4, 5 and 6 we establish the connections between the Hausdorff and Attouch-Wets topologies, the Hausdorff and proximal topologies, the Vietoris and proximal topologies, respectively. We present some consequences and concluding remarks in Section 8.
2. Terminology and notations. Given a metric space \((X,d)\), we define the following family of subsets of \(X\):

- \(B(X) = \{ A \in \mathcal{C}l(X); \ A \text{ is a bounded set} \}\);
- \(P_k(X) = \{ A \in \mathcal{C}l(X); \ A \text{ is a totally bounded set} \}\);
- \(K(X) = \{ A \in \mathcal{C}l(X); \ A \text{ is a compact set} \}\).

We denote by

- \(S(a,\varepsilon) = \{ x \in X; d(a, x) < \varepsilon \}\)
  with \(a \in X, \varepsilon > 0\), the ball of center \(a\) and radius \(\varepsilon\);
- \(B(a,\varepsilon) = \{ x \in X; d(a, x) \leq \varepsilon \}\)
  with \(a \in X, \varepsilon > 0\), the closed ball of center \(a\) and radius \(\varepsilon\);
- \(S_\varepsilon(A) = \{ x \in X; \exists a \in A \text{ such that } d(x, a) < \varepsilon \}\),
  with \(A \subset X, \varepsilon > 0\), the \(\varepsilon\)-enlargement of \(A\).

We write \(\text{cl} A\) for the closure of the set \(A\) and \(A^c\) for the set \(X - A\).

Now we present some hypertopologies. Those topologies \(\tau\) are the supremum of two topologies, lower topology \(\tau^-\) and upper topology \(\tau^+\).

**The Hausdorff topology** ([10]) \(\tau_H\) is defined on \(A \subset \mathcal{C}l(X)\) by \(\tau_H = \tau_H^- \lor \tau_H^+\).

A base of neighborhoods for \(A_0 \in A\) with respect to the \(\tau_H^-\) topology on \(A\) is given by \(\{ U_-(A_0, \varepsilon); \varepsilon > 0 \}\), where

- \(U_-(A_0, \varepsilon) = \{ A \in A; A_0 \subset S_\varepsilon(A) \}\),

and a base of neighborhoods for \(A_0 \in A\) with respect to the \(\tau_H^+\)-topology on \(A\) is given by \(\{ U_+(A_0, \varepsilon); \varepsilon > 0 \}\), where

- \(U_+(A_0, A) = \{ A \in A; A \subset S_\varepsilon(A_0) \}\).

If \(A, B \in A\), we introduce the **Hausdorff excess of \(A\) with respect to \(B\)** by the formula

- \(e(A, B) = \sup\{ d(a, B); a \in A \}\),
  where \(d(a, B) = \inf\{ d(a, b); b \in B \}\).
Also, $\tau_H$ is given on $\mathcal{A} \subset \mathbb{C}l(X)$ by the extended real valued metric $H : \mathcal{A} \times \mathcal{A} \to [0, +\infty]$, where

$$H(A, B) = \max\{e(A, B), e(B, A)\}$$

or equivalently,

$$H(A, B) = \inf\{\varepsilon > 0; B \subset S_\varepsilon(A) \text{ and } A \subset S_\varepsilon(B)\}.$$ 

On $\mathcal{B}(X)$, $H$ is a veritable metric.

The Attouch-Wets topology ([10]) $\tau_{AW}$ on $\mathcal{A} \subset \mathbb{C}l(X)$ is supremum of the upper topology $\tau^{+}_{AW}$ and the lower topology $\tau^{-}_{AW}$.

Let be

$$U_-(A_0, x_0, \varepsilon) = \{A \in \mathcal{A}; A_0 \cap S(x_0, \frac{1}{\varepsilon}) \subset S_\varepsilon(A)\}$$

and

$$U_+(A_0, x_0, \varepsilon) = \{A \in \mathcal{A}; A \cap S(x_0, \frac{1}{\varepsilon}) \subset S_\varepsilon(A_0)\},$$

where $A_0 \in \mathcal{A}$, $\varepsilon \geq 0$ and $x_0 \in X$ is fixed arbitrarily. $A\tau^{+}_{AW}$-base of neighborhoods for $A_0 \in \mathcal{A}$ is $\{U_-(A_0, x_0, \varepsilon); \varepsilon > 0\}$ and a $\tau^{-}_{AW}$-base of neighborhoods for $A_0 \in \mathcal{A}$ is $\{U_+(A_0, x_0, \varepsilon); \varepsilon > 0\}$.

We denote by $e_\rho(A, B) = e(A \cap B(x_0, \rho), B)$ and by $H_\rho(A, B) = \max\{e_\rho(A, B), e_\rho(B, A)\}$.

A net $(A_i)_{i \in I} \subset \mathcal{A}$ is $\tau_{AW}$-convergent at $A \in \mathcal{A}$ if and only if $H_\rho(A_i, A) \to 0$ ([12], [13]).

The Vietoris topology $\tau_V$ ([10]) on $\mathcal{A} \subset \mathbb{C}l(X)$ is given by $\tau_V = \tau^{-}_V \lor \tau^{+}_V$ where a subbase for $\tau^{-}_V$ is given by all the sets

$$V^- = \{A \in \mathcal{A}; A \cap V \neq \emptyset\}$$

and a subbase for $\tau^{+}_V$ is given by all the sets

$$V^+ = \{A \in \mathcal{A}; A \subset V\}$$

with $V$ open in $X$.

We recall the following characterization of the $\tau^{-}_V$-convergence:

A net $(A_i)_{i \in I} \subset \mathcal{A}$ is $\tau^{-}_V$-convergent at $A$ iff $d(x, A) \leq \lim \inf d(x, A_0)$ for all $x \in X$. 

The proximal topology $\tau_P$ ([12], [13]) on $A \subset \text{Cl}(X)$ is $\tau_P = \tau_V^+ \lor \tau_P^+.$

The b-proximal topology $\tau_{bP}$ ([13]) on $A \subset \text{Cl}(X)$ is $\tau_{bP} = \tau_V^+ \lor \tau_{AW}^+.$

We also need the following two definitions:

**Definition 2.1.** A family $(A_i)_{i \in I}$ of subsets of a metric space $X$ is uniformly bounded if there exists a ball $S(x, \varepsilon)$ such that $A_i \subset S(x, \varepsilon)$ for all $i \in I.$

**Definition 2.2.** A family $A \in \text{Cl}(X)$ of parts of a metric space $X$ is called stable with respect to closed subsets if for any set $A \in A$ and $B \subset A$ with $B \in \text{Cl}(X)$ it follows that $B \in A.$

Now we recall two results concerning the Hausdorff, Vietoris and proximal topologies:

**Proposition 2.1.** ([7,p.43], [8, p.41]) or [11, Teor.3.3] If $(X, d)$ is a metric space, the topologies $\tau_V$ and $\tau_H(d)$ are coincident on $\mathcal{K}(X).$

**Proposition 2.2.** ([6, Lemma 5.2]) By restricting the topologies $\tau_V$ and $\tau_P$ on the family $\mathcal{K}(X)$ this topologies become coincident.

3. The stability of the Hausdorff topology with respect to boundedness and totally boundedness conditions. We first give the following lemmas:

**Lemma 3.1.** Let $(X, d)$ a metric space and $A_i, A \in \text{Cl}(X)$ for any $i \in I,$ where $I$ is a nonvoid directed set of indices.

(i) If $A$ is a bounded set and $A \in \tau_H^+ - \lim A_i,$ then $\exists i_0 \in I$ such that $\forall i \geq i_0, i \in I,$ $A_i$ is a bounded set.

(ii) If $A \in \tau_H^+ - \lim A_i$ and there exists a subnet $(B_j)_{j \in J} \subset (A_i)_{i \in I}$ such that $B_j$ are bounded sets, then $A$ is a bounded set, too.

**Proof.** i) For $\varepsilon = 1$ in the definition of the $\tau_H^+$-convergence we have $A_i \subset S_1(A),$ for all $i \geq i_1.$

ii) Let be $\varepsilon = 1$ and $A \subset S_1(A_i),$ for any $i \geq i_1.$ If $\varphi : J \longrightarrow I$ is the embedding function of $J$ into $I$ from the definition of the subnet, for
\(i_1 \in I\) there exists \(j_1 \in J\) such that \(j \geq j_1\) we have \(\varphi(j) \geq i_1\). Then \(A \subset S_1(B_j) = S_1(A_{\varphi(j)})\).

**Remark 3.1.** Let \(A \in \mathcal{C}(X)\) be a bounded set and \((A_i)_{i \in I} \subset \mathcal{C}(X)\) be a net of unbounded sets such that \(A \subset A_i\), \(\forall i \in I\). Obviously, \(\forall \varepsilon > 0\) and a fixed index \(i_0 \in I\) we have \(A \subset S_\varepsilon(A_i), \forall i \geq i_0\), which tell as that \(A \in \tau_H^{-}\)-lim \(A_i\). So, there exists some nets of unbounded sets for which there limits are bounded sets.

**Lemma 3.2.** Given a metric space \((X,d)\) let \(A, B \in \mathcal{C}(X)\) be two sets for which there exists \(\varepsilon, \delta, \eta > 0\) with the following properties:

(i) \(A\) admits a finite \(\varepsilon\)-network;
(ii) \(A \subset S_\delta(B)\);
(iii) \(B \subset S_\eta(A)\).

Then \(B\) admits a finite \(\eta + \delta + \varepsilon\)-network.

**Proof.** Let \(\{a_1, a_2, ..., a_k\} \subset A\) be a finite \(\varepsilon\)-network for \(A\). From ii), for any \(a_p\), with \(p \in \overline{1,k}\), there exists \(b_p \in B\) such that \(d(a_p, b_p) < \delta\), for every \(p \in \overline{1,k}\).

We show that \(\{b_1, ..., b_k\}\) forms a finite \(\eta + \delta + \varepsilon\)-network for \(B\):

let \(b \in B\) be fixed. From iii), there exists an \(a \in A\) with \(d(a, b) < \eta\). For this \(a \in A\) let \(p_0 \in \overline{1,k}\) be such that \(d(a, a_{p_0}) < \varepsilon\). Then

\[
d(b, b_{p_0}) \leq d(b, a) + d(a, a_{p_0}) + d(a_{p_0}, b_{p_0}) < \eta + \varepsilon + \delta.
\]

**Remark 3.2.** The previous lemma sustains that if \((X,d)\) is a metric space, \((A_i)_{i \in I} \subset \mathcal{C}(X)\) a net, and \(A \in \mathcal{C}(X)\) then:

(i) From the hypothesis, \(A = \tau_H^{-}\)-lim \(A_i\) and \(A\) totally bounded, for every \(\varepsilon > 0\) there exists \(i_\varepsilon \in I\) such that \(\forall i \geq i_\varepsilon, A_i\) has a finite \(\varepsilon\)-network;

(ii) If \(A = \tau_H^{-}\)-lim \(A_i\) and there exists a subnet \((B_j)_{j \in J}\) such that \(B_j\) are totally bounded sets, then \(A\) is totally bounded.
So, the Hausdorff convergence still keep the boundedness and the totally boundedness conditions.

**Remark 3.3.** The Attouch-Wets, Vietoris, proximal and b-proximal convergences do not conserve these properties in $\mathcal{C}(X)$; we observe this statement from a simple example:

**Example 3.1.** Let be the sequence $A_n = [-n, n] \in \mathbb{R}$ and $A = \mathbb{R}$; on $\mathbb{R}$, all the previous convergences, except $\tau_H \equiv \tau_{\ell f}$, coincide.

The set $A$ is not bounded (neither totally bounded), but we have for example, $A = \tau_{AW} - \lim A_n$ : By taking $x_0 = 0$ in the definition of Attouch-Wets topology, $e_\rho(A_n, A) = 0$ and $e_\rho(A, A_n) = \sup\{d(a, [-n, n]) ; a \in [-\rho, \rho]\} = 0$ for $n$ sufficiently large.

Obviously, $\tau_H - \lim A_n$ does not even exists!

(4. The Hausdorff and Attouch-Wets topologies. Now, we will compare also by using the diagram of first page, every two hypertopologies presented in Section 2. We will start with the Hausdorff and Attouch-Wets hypertopologies. We denoted by $H$, respectively by $AW$, $P$, $bP$ the convergences at sense of Hausdorff, Attouch-Wets, proximal and b-proximal topologies (or $H_-$, $AW_-$, $P_-$, $bP_-$, respectively $H_+$, $AW_+$, $P_+$, $bP_+$ the lower or upper corresponding convergences).

We observe that for two arbitrary subsets $A, B$ of a metric space $X$, if there exists $x_0 \in X$ and $\rho > 0$ such that $A \subset B(x_0, \rho)$, then $e_{\rho, x_0}(A, B) = e(A \cap B(x_0, \rho), B) = e(A, B)$, and we obtain:

**Proposition 4.1.** Let $(X, d)$ be a metric space, $(A_i)_{i \in I} \subset \mathcal{C}(X)$ a net with $0 \notin I$ and $A_0 \in \mathcal{C}(X)$.

(i) If $A_0 \in \mathcal{B}(X)$ and $A_0 \in AW_{-} - \lim A_i$, then $A_0 \in H_{-} - \lim A_i$.

(ii) If $(A_i)_{i \in I}$ is uniformly bounded and $A_0 \in AW_{+} - \lim A_i$, then $A_0 \in H_{+} - \lim A_i$.

(iii) If $(A_i)_{i \in I}$ is uniformly bounded and $A_0 \in bP_{-} - \lim A_i$, then $A_0 \in P_{-} - \lim A_i$.

(iv) If $(A_i)_{i \in I}$ is uniformly bounded, $A_0 \in \mathcal{B}(X)$ and $A_0 = AW - \lim A_i$, then $A_0 = H - \lim A_i$. 

Now we establish the following statement:

**Theorem 4.1.** Let \((X, d)\) be a metric space.

(i) On \(B(X)\) we have \(\tau_H \equiv \tau_{AW}^\ast\).

(ii) We denote by

\[ \mathcal{E} = \{ A \subset Cl(X); \ \tau_H \equiv \tau_{AW}^\ast \text{ on } A \}
\]

and \(A\) is nonvoid and stable at closed sets.

Then \(B(X)\) is the greatest element of \(\mathcal{E}\).

**Proof.**

i) We consider \((A_i)_{i \in I} \subset B(X)\) and \(A \in B(X)\). If \(A_i \overset{H}{\longrightarrow} A\), then \(A_i \overset{AW}{\longrightarrow} A\) (this implication have been even on \(Cl(X)\)).

Conversely, from the Proposition 4.1, i), it is sufficiently to have only \(A \in B(X)\).

ii) From i), \(B(X) \in \mathcal{E}\).

Now we show that \(B(X)\) is majorant element of the class \(\mathcal{E}\):

We suppose that we have \(A \not\subseteq B(X)\) for some \(A \in \mathcal{E}\) and let be \(A \in A \setminus B(X)\). We fixed \(x_0 \in X\). Then, for any \(n \in \mathbb{N}^*\), the closed set \(A_n = A \cap B(x_0, n)\) is contained in \(A\), so \(A_n \in A\) for any \(n \in \mathbb{N}^*\) (from the stability at closed sets of the family \(A\)).

Let be \(\rho > 0\) fixed and we choose \(n_0 = [\rho] + 1 \in \mathbb{N}^*\), where \([\rho]\) denote the greatest entire number lesser that \(\rho\). The set \(A_\rho = A \cap B(x_0, \rho)\) satisfies the inclusion \(A_\rho \subset A_n, \forall n \geq n_0\), so \(e_\rho(A, A_n) = \sup\{d(a, A_n); \ a \in A_\rho\} = 0, \forall n \geq n_0\), that is \(A \in AW \equiv \lim A_n\).

According the hypothesis, it follows that \(A \in H \equiv \lim A_n\), as \(A\) is not bounded, this is in contradiction with the Lemma 3.1, ii).

The following example shows us that the \(\tau_H\) and \(\tau_{AW}\) topologies do not coincide on \(B(X)\) (not even on \(K(X)\)):

**Example 4.1.** We choose \(A_n = \{0\} \cup \{n\}, \forall n \in \mathbb{N}^*\) and \(A = \{0\}\) compact sets in \(\mathbb{R}\). It is easy to verify that \(A_n \overset{AW}{\longrightarrow} A\), but \(A_n \overset{H}{\not\longrightarrow} A\).

Anyway, if the \(\tau_H\) and \(\tau_{AW}\) topologies coincide on a family of sets \(A \subset Cl(X)\), it doesn’t result that \(A \subset B(X)\): for example, they coincide on \(A = S(X) \cup \{X\}\).
5. The Hausdorff and proximal topologies. From Propositions 2.1 and 2.2 and on the family $K(X)$ of the compact set of a metric space $(X, d)$, the topologies $\tau_P$ and $\tau_H$ coincide. In fact, the equality of these two topologies is valid on a larger family, as follows:

**Theorem 5.1.** Let $(X, d)$ be a metric space and $P(X)$ the family of closed, totally bounded, nonempty parts of $X$. Then, on $P(X)$, $\tau_P \equiv \tau_H$.

**Proof.** We have to prove the relation $\tau_P \subseteq \tau_W$ (the other inclusion is satisfied on $Cl(X)$). We use the characterization of $\tau_P$-convergence of the nets; let be $(A_i)_{i \in I} \subset P(X)$ a net and $A \in P(X)$ such that $A \in \tau_W - \lim A_i$, that is $\lim d(a, A_i) = 0$, for any $a \in A$.

Let be $\varepsilon > 0$ and $\{a_1, \ldots, a_k\} \subset A$ a finite $\varepsilon/2$–network $A$.

We fix $a \in A$ and choose $p \in [1, k]$ such that $d(a, a_p) < \varepsilon/2$.

As $|d(a, A_i) - d(a_p, A_i)| \leq d(a, a_p) < \varepsilon/2$ it follows that $d(a, A_i) < \varepsilon/2 + d(a_p, A_i)$.

But $\lim d(a_p, A_i) = 0$. For any $p \in [1, k]$, that is: $\forall \varepsilon > 0 \exists i_{p,\varepsilon} \in I$ such that $\forall i \geq i_{p,\varepsilon}$ it results that $d(a_p, A_i) < \varepsilon/2$.

We choose $i_{p,\varepsilon} \geq i_{p,\varepsilon}$, $\forall p \in [1, k]$ with $i_{\varepsilon} \in I$. If $i \geq i_{\varepsilon}$ we have: $d(a, A_i) < \varepsilon$ and taking the supremum over all the elements $a \in A$ we find that:

$$\forall \varepsilon > 0 \exists i_{\varepsilon} \in I \text{ such that } \forall i \geq i_{\varepsilon} \implies e(A, A_i) < \varepsilon,$$

from where $A \in \tau_W - \lim A_i$. \hfill \blacksquare

**Corollary 5.1.** On family $P(X)$ of closed, nonempty and totally bounded of a metric space $(X, d)$ the equality $\tau_H = \tau_P$ is valid.

Now we will establish a converse result.

**Theorem 5.2.** If $A \subset Cl(X)$ is a family of nonempty parts of a metric space $X$, stable with respect to closed subsets, such that we have $\tau_P \equiv \tau_H$ on $A$ then $A \subset P(X)$.

**Proof.** Let suppose, by contradiction, that any net of $A$ which is $\tau_P$–convergent is also $\tau_H$–convergent, but $A$ contains at least one set which is not totally bounded. Let this be $A$:

$$\exists \varepsilon_0 > 0 \text{ such that } \forall k \in \mathbb{N} \text{, } \forall b_1, \ldots, b_k \in A \exists a \in A \text{ with } d(a, b_p) > \varepsilon_0, \forall p \in [1, k].$$
We fix \( k = 1 \) and \( a_1 \in A \); from (6.1), there exists \( a_2 \in A \) with \( d(a_1, a_2) > \varepsilon_0 \).

Applying again (6.1) for \( k = 2 \) and \( a_1, a_2 \in A \) we find \( a_3 \in A \) such that \( d(a_1, a_3) > \varepsilon_0 \) and \( d(a_2, a_3) > \varepsilon_0 \).

By an inductive approach, for \( a_1, a_2, \ldots, a_n \in A \) there exists \( a_{n+1} \in A \) such that \( d(a_{n+1}, a_i) > \varepsilon_0, \forall i \in \overline{1,n} \) and, furthermore, \( d(a_i, a_j) > \varepsilon_0 \) for any \( i, j = \overline{1,n} \) with \( i \neq j \).

We consider \( A_0 = \{a_1, a_2, \ldots, a_n, \ldots\} \), which is a closed set (having an empty set of cluster points). As \( A_0 \subset A \) and is stable with respect to closed subsets, it follows that \( A_0 \in \mathcal{A} \).

The set \( A_0 \) is not totally bounded: let be \( a_{n_1}, a_{n_2}, \ldots, a_{n_k} \in A_0 \) and we denote by \( n_0 = \max\{n_1, \ldots, n_k\} \); then the element \( a_{n_0+1} \in A_0 \) has the property that \( d(a_{n_p}, a_{n_0+1}) > \varepsilon_0, \forall p = \overline{1,k} \) (with \( \varepsilon_0 > 0 \) given by (6.1)).

Let be \( A_n = \{a_1, \ldots, a_n\} \), which is closed and totally bounded because it is finite. From the Corollary 6.1 we have that \( \mathcal{P} k(X) \subset \mathcal{A} \), so \( A_n \in \mathcal{A} \) for any \( n \in \mathbb{N}^* \). We will show that \( A_0 = \tau P - \lim A_n \).

First, \( A_0 \in \tau^{-}_V - \lim A_n \), that is \( \forall a \in A_0 \exists b_n \in A_n \text{ with } b_n \rightarrow a \):

We take \( a \in A_0 \), i.e. there exists \( n_0 \in \mathbb{N} \) with \( a = a_{n_0} \); then \( a \in A_n \), for any \( n \geq n_0 \). We consider the sequence \( b_n = a \) for \( n \geq n_0 \), and the first \( n_0 - 1 \) terms, arbitrarily chosen, with \( b_n \in A_n \). We have \( b_n \rightarrow a \).

Now \( A_0 \in \tau^{-}_H - \lim A_n \):

\[
\forall \varepsilon > 0 \exists n_\varepsilon = 1 \text{ such that } \forall n \geq 1 \text{ we have } A_n \subset A \subset S_\varepsilon(A).
\]

According to the hypothesis, it follows that the sequence \( (A_n)_{n \in \mathbb{N}} \) of totally bounded sets is \( \tau_H \)-convergent in \( \mathcal{A} \) to \( A_0 \), which is not totally bounded, in contradiction to the Remark 3.2, ii).

\[ \blacksquare \]

Without employing the Remark 3.2, the result still remains valid in the case of the topologies \( \tau^{-}_V \) and \( \tau^{-}_H \):

**Theorem 5.3.** Let \((X,d)\) be a metric space.

(i) We consider the class: \( \mathcal{E}' = \{A \subset \text{Cl}(X); A \text{ nonvoid, stable with respect to closed subset and } \tau^{-}_V = \tau^{-}_H \text{ pe } A\} \). Then \( \mathcal{P} k(X) \) is the greatest element of \( \mathcal{E}' \).

(ii) Also, \( \mathcal{P} k(X) \) is the greatest element for class
\[
\mathcal{E} = \{A \subset \text{Cl}(X); A \text{ nonvoid, stable with respect to closed subsets, } \tau_V = \tau_H \text{ on } A\}.
\]
Proof. i) From the Theorem 6.1, $P_k(X) \in \mathcal{E}'$. 

Let be $\mathcal{A} \subset \mathcal{C}(X)$ which is stable with respect to closed subsets, such that $\tau_V^\pm \equiv \tau_H^-$ on $\mathcal{A}$. We suppose that there exists $A \in \mathcal{A}$ such that $A \not\in \mathcal{P}k(X)$. By using the construction of the sequence $(A_n)$, from the proof of the Theorem 6.2, $\tau_V^-$-convergent at $A_0$, it should follow that 

$$A_0 \in \tau_H^- \lim A_n.$$ 

But for $\varepsilon_0 > 0$ given by (6.1), for any $n_0 \in \mathbb{N}^*$, we have that 

$$d(a_{n_0+1}, a_p) > \varepsilon_0, \forall p = \overline{1, n_0},$$ 

that is $a_{n_0+1} \not\in S(a_p, \varepsilon_0), \forall p = \overline{1, n_0}$. 

As $S_{\varepsilon_0}(A_{n_0}) = \bigcup_{p=1}^{n_0} S(a_p, \varepsilon_0)$ we obtain that 

$$a_{n_0+1} \not\in S_{\varepsilon_0}(A_{n_0}),$$ 

so that: 

$$\text{for } \varepsilon_0 > 0, \forall n_0 \in \mathbb{N}, \ A_0 \not\in S_{\varepsilon_0}(A_{n_0}),$$ 

which is in a contradiction with $A_0 \in \tau_H^- \lim A_n$. 

The contradiction shows that the greatest element of the class $\mathcal{E}'$ is $P_k(X)$. 

ii) It follows from the Theorem 6.2 and Corollary 6.1. 

6. The proximal and Vietoris topologies. In this section we intent to study the coincidence of the topologies $\tau_V$ and $\tau_P$. According to the Proposition 2.2, they agree on $\mathcal{K}(X)$. This is even ”the best” family on which $\tau_V \equiv \tau_P$. We will give this result also for the upper topologies, but we will restrain to the case of the linear normed space: 

**Theorem 6.1.** If $X$ is a linear normed space and it is true that $\tau_V^+ \equiv \tau_H^-$ on the set $\mathcal{A} \subset \mathcal{C}(X)$, stable with respect to the closed subsets then $\mathcal{A} \subset \mathcal{K}(X)$. 

**Proof.** Let suppose that the family $\mathcal{A} \subset \mathcal{C}(X)$, stable with respect to closed subsets, satisfies this property: any net $(A_i)_{i \in I} \subset \mathcal{A}$ being $\tau_H^-$-convergent at $A$, is also $\tau_V^+$-convergent at $A$, but $\mathcal{A} \not\subset \mathcal{K}(X)$ (we know that the $\tau_V^+$-convergence implies the $\tau_H^-$-convergence on $\mathcal{C}(X)$).
Let $A \in \mathcal{A}\setminus \mathcal{K}(X)$. Then there exists a sequence $(a_k)_{k \in \mathbb{N}} \subset A$ such that any of its subsequences $(a_{n_\ell})_{\ell}$ is divergent, that is:

$$\forall y \in X \exists \varepsilon_y > 0 \text{ such that } \forall \ell \in \mathbb{N} \exists n_{\ell,y} \geq \ell \text{ with } \|a_{n_{\ell,y}} - y\| > \varepsilon_y.$$ (7.1)

We write this condition for each term $a_k$ of the previous sequence:

$$\forall k \in \mathbb{N} \exists \varepsilon_k > 0 \text{ such that } \forall \ell \in \mathbb{N} \exists n_{\ell,k} \geq \ell \text{ with } \|a_{n_{\ell,k}} - a_k\| > \varepsilon_k.$$ (7.2)

If we take $\ell \geq k$, then $n_{\ell,k} \geq k$; we can choose the sequence of positive numbers $(\varepsilon_k)_{k \in \mathbb{N}}$ decreasing at zero: if the property is true for $\varepsilon_k > 0$, it is also valid for $\varepsilon'_k < \max \left\{ \varepsilon_k, \varepsilon_{k-1}, \frac{1}{k} \right\}$.

We assume that $a_k \neq 0$ (in the other case we exclude the terms zero from the sequence; those are in finite number: otherwise, the sequence $(a_k)_{k}$ would contain a subsequence which is convergent – at zero).

We consider $A_0 = \{a_1, a_2, \ldots, a_k, \ldots\}$. From (7.1), $A_0$ is not compact. We also observe that $A_0$ is closed, being formed by isolated points.

As $A_0 \subset A$ and $A \in \mathcal{A}$, from the stability with respect to closed subsets of the family $\mathcal{A}$ we obtain $A_0 \in \mathcal{A}$.

For $\varepsilon_k > 0$ given by (7.2), we denote by $A_k$ the set:

$$A_k = \{a_k, b_k\}, \forall k \in \mathbb{N}^*;$$

where

$$b_k = \left(1 - \frac{\varepsilon_k}{2\|a_k\|}\right) a_k.$$

$A_k$ satisfies the inclusion:

$$A_k \subseteq B \left(a_k, \frac{\varepsilon_k}{2}\right).$$

As $A_k$ is finite set, it follows that it is compact, so $A_k \in \mathcal{A}$, $\forall k \in \mathbb{N}^*$.

We will show that $A_0 \in \tau^+_H - \lim A_k$:

Let $\varepsilon > 0$ be arbitrarily fixed. Since $\varepsilon_k \searrow 0$, starting with a rank $k_\varepsilon \in \mathbb{N}$ we have $\frac{\varepsilon_k}{2} < \varepsilon$, $\forall k \geq k_\varepsilon$. Then:

$$A_k \subset B \left(a_k, \frac{\varepsilon_k}{2}\right) \subset S(a_k, \varepsilon) \subset S_\varepsilon(A_0), \forall k \geq k_\varepsilon.$$

According to the hypothesis, we obtain $A_0 \in \tau^+_V - \lim A_k.$
On the other hand, for example, for the open set

$$V = \bigcup_{k \in \mathbb{N}} S\left( a_k, \frac{\varepsilon_k}{2} \right)$$

we have $A_0 \subset V$; but $\forall k \in \mathbb{N}$, $\forall \ell \in \mathbb{N}$ with $\ell \geq k \exists n_{\ell,k} \geq \ell, k$ (given by (7.2)) such that $A_{n_{\ell,k}} \not\subset V$, namely

$$b_{n_{\ell,k}} \not\in S\left( a_k, \frac{\varepsilon_k}{2} \right),$$

where

$$b_k = \left( 1 - \frac{\varepsilon_k}{2\|a_k\|} \right) a_k \in A_k.$$ 

In order to establish these fact, let us observe that $n_{\ell,k} \geq k$ gives rise to $\varepsilon_{n_{\ell,k}} \leq \varepsilon_k$ so that

$$\|a_k - b_{n_{\ell,k}}\| \geq \|a_k - a_{n_{\ell,k}}\| - \|a_{n_{\ell,k}} - b_{n_{\ell,k}}\| > \varepsilon_k - \frac{\varepsilon_{n_{\ell,k}}}{2} \geq \varepsilon_k - \frac{\varepsilon_k}{2} = \frac{\varepsilon_k}{2}. $$

The contradiction shows that $A \subset K(X)$. 

**Theorem 6.2.** Let $X$ be a linear normed space. We consider the classes:

$$\mathcal{E} = \{ A \subset \text{Cl}(X); A \text{ nonvoid, stable with respect to closed subsets, } \tau^+_{\mathcal{V}} \equiv \tau^+_{\mathcal{H}} \text{ on } A \}$$

and

$$\mathcal{E}' = \{ A \subset \text{Cl}(X); A \text{ nonvoid, stable with respect to closed subsets, } \tau_{\mathcal{V}} \equiv \tau_p \text{ on } A \}.$$

Then $K(X)$ is the greatest element both of $\mathcal{E}$ and of $\mathcal{E}'$.

**Proof.** We apply the Proposition 2.2 and the Theorem 7.1, by using the definition of the topologies $\tau_{\mathcal{V}}$ and $\tau_p$:

$$\tau_p = \tau^-_{\mathcal{V}} \vee \tau^+_{\mathcal{H}}, \text{ and } \tau_{\mathcal{V}} = \tau^-_{\mathcal{V}} \vee \tau^+_{\mathcal{V}}.$$
Remark 6.1. By adjusting the proof above, the Theorem 7.1 (and, implicitly, and the Theorem 7.2) hold if we replace the normed space \( X \) by metric space \((X, d)\) without isolated points. This generalisation is consistent, because there are metric space without isolated points which are not normed spaces.

As every linear topological space \( \neq \{0\} \) have not isolated poits, it follows that every metrizable linear topological space which is not \( \{0\} \) haven’t isolated points.

For example, let \( s \) be the set of the sequence of real elements

\[
s = \{ x; \ x = (x_n)_{n \in \mathbb{N}^*}, \ x_n \in \mathbb{R}, \ \forall n \in \mathbb{N}^* \},
\]

with the metric

\[
d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}, \text{ where } x = (x_n)_{n \in \mathbb{N}^*}, y = (y_n)_{n \in \mathbb{N}^*}.
\]

\((s, d)\) hasn’t isolated points.

Now we suggest the small modification which make the Theorem 7.1 to be valid also for the metric space which haven’t isolated points:

For the founded sequence \((a_k) \subset A\) we have \( \varepsilon_k \searrow 0 \) given by the condition \((7.2)\). Since \( a_k \) isn’t isolated point for \( X \) it results that there exists

\[
b_k \in B \left( a_k, \frac{\varepsilon_k}{2} \right) \setminus \{a_k\}.
\]

Obviously,

\[
d(a_k, b_k) \leq \frac{\varepsilon_k}{2}.
\]

We consider

\[
A_k = \{ a_k, b_k \} \text{ compact set}
\]

and the rest of the proof holds.

7. Consequences and concluding remarks. By the previous statements, we find the following consequences:

**Corollary 7.1.** We consider a metric space \( X \) and the following classes of subsets:

\[
\mathcal{E} = \{ A \subset \text{Cl}(X); \ A \text{ nonvoid, stable with respect to closed sets, }\ A W_\_ \equiv V_\_ \text{ on } A \}\]
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and

\[ \mathcal{E}' = \{ A \in \mathcal{E}; \ AW \equiv bP \text{ on } A \}. \]

Then the greatest element of the classes \( \mathcal{E} \) and \( \mathcal{E}' \) is \( \mathcal{P}k(X) \).

**Proof.** It follows from the Theorems 4.1 and 5.3. □

**Corollary 7.2.** Let \((X, d)\) be a metric space without isolated points. We define the classes:

\[ \mathcal{E}' = \{ A \subset \text{Cl}(X); \ A \text{ nonvoid, stable with respect to closed subsets}, \ \tau_H \equiv \tau_V \text{ on } A \}. \]

Then \( \mathcal{K}(X) \) is the greatest element of the classes \( \mathcal{E}, \mathcal{E}' \) \& \( \mathcal{E}'' \).

**Proof.**
i) It is a consequence of the Theorems 5.2 and 6.2. □

For a better preview on the previous results we will give the following schemes as a synthesis, considering on a metric space \((X, d)\):

\[
\begin{align*}
H_+ & \equiv AW_+ \text{ for uniformly bounded nets,} \\
H & \equiv AW \text{ for uniformly bounded nets,} \\
& \quad \text{with bounded limit} \\
P & \equiv bP \text{ for uniformly bounded nets.}
\end{align*}
\]

The coincidences of the following topologies are considered the greatest element of the classes where they are valid:

\[
\begin{align*}
\tau_{AW}^- & \equiv \tau_H^- \ast \mathcal{B}(X) \\
\tau_H^- & \equiv \tau_V^- \ast \mathcal{P}k(X) \\
\tau_H & \equiv \tau_P \ast \mathcal{P}k(X) \\
\tau_{AW}^- & \equiv \tau_V^- \ast \mathcal{P}k(X) \\
\tau_{AW} & \equiv \tau_{bP} \ast \mathcal{P}k(X) \\
\tau_H^+ & \equiv \tau_V^+ \ast \mathcal{K}(X), \ (X, d) \text{ without isolated points} \\
\tau_P & \equiv \tau_V \ast \mathcal{K}(X), \ (X, d) \text{ without isolated points} \\
\tau_H & \equiv \tau_V \ast \mathcal{K}(X), \ (X, d) \text{ without isolated points}
\end{align*}
\]
where by $^*\mathcal{A}$ we indicate that the family $\mathcal{A} \subset \text{Cl}(X)$ has been searched among the families stable with respect to closed subsets.

REFERENCES