ON THE BENDING OF PLATES IN THE THEORY OF ELASTIC MATERIALS WITH VOIDS

BY

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Abstract. The paper studies the field equations governing the bending of plates in the theory of elastic materials with voids. First, we present a representation of Galerkin type for the solution of the equilibrium equations of isotropic and homogeneous materials. Then, we use this solution to determine the fundamental solutions. Finally, an integral representation of Somigliana type is established.

1. Introduction. The mechanics of porous materials is of fundamental interest to the field of soil mechanics, foundation engineering as well as powder technology. However, it was only recently that the continuum mechanics approach to describe the behaviour of such porous materials yielded certain well-founded results. The research in the last decades has revealed a basic concept of the macroscopic modelling of porous solids: the volume fraction field.

An important contribution to the theory of porous elastic media has been made by NUNZIATO and COWIN [1]. They have presented a nonlinear theory for the behaviour of porous solids in which the skeletal or matrix material is elastic and the interstices are void of material. The bulk density of the body is written as the product of two fields, the matrix material density field and the volume fraction field. This representation introduces an additional degree of kinematical freedom. In [2] NUNZIATO and COWIN have derived the linear theory of elastic materials with voids.

In the present paper, based on the theory of Cowin and Nunziato, we study the bending of elastic thin plates of Mindlin type made from a material with voids. In [3] MINDLIN has introduced a model that takes into
account transverse shear deformation in the flexural motion of a plate. The transverse shear effect represent an important feature in the behaviour of several material structures (such as composite plates and plates in which the thickness to length ratio increases to 1/20 or higher) and will be taken into account in our work. For a detailed analysis of the theory of plates we refer to NAGHDI [4]. The problem of the bending of elastic plates made from a material with microstructure was considered in various papers (see e.g. ERINGEN [5], [6]). TABER [7] and THEODORAKOPOULOS and BESKOS [8] have studied the dynamical flexure of fluid–saturated porous plates in the framework of the classical plate theory and using the Biot’s theory of poroelasticity. In [9] MANOLACHI has investigated the behaviour of Kirchhoff plates with the help of complex functions. The bending of thermoelastic plates made from a material with voids has been studied by BÎRSAN [10] in the context of the dynamic theory.

In the present paper we consider a linear bending theory of porous elastic plates. The theory allows for the effect of transverse shear deformation as in the Mindlin–Timoshenko model of plates (see [11, p.13]), but we do not introduce any correction factor. In the first part we present the governing equations for the state of bending of an isotropic and homogeneous thin plate and formulate the boundary–value problem. Then we use a Galerkin representation to determine the fundamental solutions of the equilibrium equations. Finally, an integral representation of Somigliana type is established.

2. Equilibrium theory for the bending of plates. We consider a bounded regular region $B$ of the three–dimensional Euclidean space, referred to a system of rectangular axes $Ox_i$. We let $\overline{B}$ denote the closure of $B$, call $\partial B$ the boundary of $B$ and designate by $n$ the outward unit normal of $\partial B$. The Latin subscripts are understood to range over the integers 1,2,3, whereas Greek subscripts are confined to the range 1,2.

We assume that the body occupying $B$ is a linearly elastic material with voids. Let $u$ be the displacement vector field. The linear strain measure is given by

\[(1)\quad e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}).\]

We use subscripts preceded by a comma for partial differentiation with respect to the corresponding coordinate. Let $t$ be the stress tensor and
the equilibrated stress vector. We denote by $\varphi$ the change in volume fraction.

The equations of equilibrium are [2]

$$t_{ji,j} + f^*_i = 0,$$

$$h_{i,i} + g + \ell^* = 0,$$

where $f^*$ is the body force, $g$ is the intrinsic equilibrated body force and $\ell^*$ is the extrinsic equilibrated body force per unit of initial volume.

The constitutive equations for isotropic and homogeneous bodies in the linear theory of elastic materials with voids are

$$t_{ij} = \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + b\varphi\delta_{ij},$$
$$h_i = \alpha\varphi_i,$$
$$g = -be_{rr} - \xi\varphi,$$

where $\lambda, \mu, b, \alpha$ and $\xi$ are constitutive constants. To the basic equations (1)–(4) we must adjoin boundary conditions.

In what follows we assume that the region $B$ refers to the interior of a right cylinder of length $2h$ with open cross-section $\Sigma$ and the smooth lateral boundary $\Pi$. Let $L$ be the boundary of $\Sigma$. The rectangular Cartesian coordinate frame is supposed to be chosen in such a way that the plane $x_1Ox_2$ is middle plane.

We say that $S = (u_i, \varphi)$ is an admissible state on $B$ if $u_i$ and $\varphi$ are of class $C^2$ on $B$ and of class $C^1$ on $\overline{B}$. Following [4–6], an admissible state is a state of bending on $B$ provided

$$u_\alpha(x_1, x_2, x_3) = -u_\alpha(x_1, x_2, -x_3), \quad u_3(x_1, x_2, x_3) = u_3(x_1, x_2, -x_3),$$
$$\varphi(x_1, x_2, x_3) = -\varphi(x_1, x_2, -x_3), \quad (x_1, x_2, x_3) \in B.$$

In view of (1) and (4), we find that

$$t_{\alpha\beta}(x_1, x_2, x_3) = -t_{\alpha\beta}(x_1, x_2, -x_3),$$
$$t_{33}(x_1, x_2, x_3) = -t_{33}(x_1, x_2, -x_3),$$
$$t_{\alpha3}(x_1, x_2, x_3) = t_{\alpha3}(x_1, x_2, -x_3),$$
$$g(x_1, x_2, x_3) = -g(x_1, x_2, -x_3),$$
$$h_\alpha(x_1, x_2, x_3) = -h_\alpha(x_1, x_2, -x_3),$$
$$h_3(x_1, x_2, x_3) = h_3(x_1, x_2, -x_3).$$
We say that the system of body loads \((f_i^*, \ell^*)\) is compatible with a state of bending if
\[
\begin{align*}
    f_α^*(x_1, x_2, x_3) &= -f_α^*(x_1, x_2, -x_3), \\
    f_3^*(x_1, x_2, x_3) &= f_3^*(x_1, x_2, -x_3), \\
    \ell^*(x_1, x_2, x_3) &= -\ell^*(x_1, x_2, -x_3).
\end{align*}
\]

In what follows we assume that the body loads \(f_i^*\) and \(\ell^*\) are continuous on \(B\) and satisfy the conditions (7).

We derive a theory of thin plates of uniform thickness. We denote
\[
N_α = \frac{1}{2h} \int_{-h}^{h} t_{α3} dx_3, \quad \Phi = \frac{1}{2h} \int_{-h}^{h} h_3 dx_3.
\]
We assume that the functions \(t_{ij}\) and \(h_i\) are prescribed on the surfaces \(x_3 = ±h\). On the basis of (2) and (8), we obtain the equation
\[
N_{α,α} + \frac{1}{h_0} f_3 = 0,
\]
where
\[
f_3 = 2t_{33}(x_1, x_2, h) + \int_{-h}^{h} f_3^* dx_3,
\]
and \(h_0 = 2h\) is the thickness of the plate.

If we multiply by \(x_3\) the equations (2) and (3) and integrate from \(x_3 = -h\) to \(x_3 = h\), then we get
\[
M_{βα,β} - h_0 N_α + f_α = 0, \quad H_{α,α} + G - h_0 Φ + \ell = 0,
\]
where
\[
M_{αβ} = \int_{-h}^{h} x_3 t_{αβ} dx_3, \quad H_α = \int_{-h}^{h} x_3 h_α dx_3, \quad G = \int_{-h}^{h} x_3 g dx_3.
\]
\[
I = \frac{2}{3} h^3, \quad f_α = h_0 t_{3α}(x_1, x_2, h) + \int_{-h}^{h} x_3 f_α^* dx_3, \quad \ell = h_0 h_3(x_1, x_2, h) + \int_{-h}^{h} x_3 \ell^* dx_3.
\]
The functions \(f_α, f_3\) and \(\ell\) are prescribed.
Following [3, 5] we restrict our attention to the state of bending characterized by

\[ u_\alpha = x_3 v_\alpha(x_1, x_2), \quad u_3 = w(x_1, x_2), \]
\[ \varphi = x_3 \psi(x_1, x_2) \] on \( B \).

In the context of the theory of micropolar elasticity the representation (13) has been introduced by ERINGEN [5].

In view of (13) we have

\[ e_{\alpha\beta} = x_3 \varepsilon_{\alpha\beta}, \quad 2e_3 = \gamma_\alpha, \quad e_{33} = 0, \]

where

\[ \varepsilon_{\alpha\beta} = \frac{1}{2}(v_{\alpha,\beta} + v_{\beta,\alpha}), \quad \gamma_\alpha = v_\alpha + w_\alpha. \]

The quantities \( \gamma_\alpha \) represent the angles of rotation of the cross-sections \( x_\alpha = \text{constant} \) about the middle surface (see e.g. [11, p.13]).

It follows from (4), (13) and (14) that

\[ t_{\alpha\beta} = x_3(\lambda \varepsilon_{\rho\rho}\delta_{\alpha\beta} + 2\mu \varepsilon_{\alpha\beta} + b\psi\delta_{\alpha\beta}), \]
\[ t_{3\alpha} = \mu \gamma_\alpha, \quad t_{33} = x_3(\lambda \varepsilon_{\rho\rho} + b\psi), \]
\[ h_\beta = x_3 \alpha \psi_{\cdot\beta}, \quad h_3 = \alpha \psi, \quad g = -x_3(b \varepsilon_{\rho\rho} + \xi \psi). \]

By virtue of (8), (12) and (16) we find that

\[ M_{\alpha\beta} = I(\lambda \varepsilon_{\rho\rho}\delta_{\alpha\beta} + 2\mu \varepsilon_{\alpha\beta} + b\psi\delta_{\alpha\beta}), \]
\[ N_\alpha = \mu \gamma_\alpha, \quad H_\beta = \alpha I \psi_{\cdot\beta}, \]
\[ G = -I(b \varepsilon_{\rho\rho} + \xi \psi), \quad \Phi = \alpha \psi. \]

The basic equations for the bending of elastic porous plates are the equilibrium equations (9), (11), the constitutive equations (17) and the geometrical relations (15).

To the field equations we must adjoin boundary conditions. Let \( L_i \) \((i = 1, 2, ..., 6)\) be subsets of \( \partial \Sigma \) such that \( L_1 \cup L_2 = L_3 \cup L_4 = L_5 \cup L_6 = L \) and \( L_1 \cap L_2 = L_3 \cap L_4 = L_5 \cap L_6 = \emptyset \).

We define

\[ M_\alpha = M_{\beta\alpha} n_\beta, \quad N = N_{\beta} n_\beta, \quad H = H_{\beta} n_\beta \] on \( \partial \Sigma \).
and consider the following boundary conditions
\begin{align}
  v_\alpha &= \tilde{v}_\alpha \quad \text{on } L_1, \quad M_\alpha = \tilde{M}_\alpha \quad \text{on } L_2 \\
  w &= \tilde{w} \quad \text{on } L_3, \quad N = \tilde{N} \quad \text{on } L_4 \\
  \psi &= \tilde{\psi} \quad \text{on } L_5, \quad H = \tilde{H} \quad \text{on } L_6. 
\end{align}

We assume that the prescribed functions \( \tilde{v}_\alpha, \tilde{w} \) and \( \tilde{\psi} \) are continuous on their domains of definition, whereas the given functions \( \tilde{M}_\alpha, \tilde{N} \) and \( \tilde{H} \) are piecewise regular.

In view of (17) and (15) we can express the equilibrium equations (9), (11) in terms of \( v_\alpha, w \) and \( \psi \). Thus, we obtain
\begin{align}
  \mu \Delta v_\alpha + (\lambda + \mu) v_{\beta,\beta\alpha} - \mu \chi^2 (v_\alpha + w_\alpha) + b \psi_\alpha + \frac{1}{I} f_\alpha &= 0, \\
  \mu \Delta w + \mu v_{\beta,\beta} + \frac{1}{h_0} f_3 &= 0, \\
  \alpha \Delta \psi - b v_{\beta,\beta} - (\xi + \alpha \chi^2) \psi + \frac{1}{I} \ell &= 0,
\end{align}
where \( \Delta \) is the Laplace operator and \( \chi^2 = \frac{h_0}{I} \). The mixed boundary-value problem consists in the determination of the functions \( v_\alpha, w \) and \( \psi \) which satisfy the equations (20) and the boundary conditions (19).

The internal energy of the plate \( e \) (see [5]) can be written in the form
\[ 2e = I \left[ \lambda \varepsilon_{\alpha\alpha} \varepsilon_{\beta\beta} + 2 \mu \varepsilon_{\alpha\beta} \varepsilon_{\alpha\beta} + 2b \varepsilon_{\alpha\alpha} \psi + \mu \chi^2 \gamma_{\alpha\gamma} \gamma_\alpha + (\xi + \alpha \chi^2) \psi^2 + \alpha \psi_{\beta,\beta} \psi_{\beta} \right]. \]

We assume that \( e \) is a positive definite quadratic form in the variables \( \varepsilon_{\alpha\beta}, \gamma_\alpha, \psi \) and \( \psi_{\beta,\beta} \). This fact implies the following restrictions on the constitutive coefficients
\begin{align}
  \mu > 0, \quad \lambda + \mu > 0, \quad \alpha > 0, \quad \xi + \alpha \chi^2 - \frac{b^2}{\lambda + \mu} > 0.
\end{align}

The following uniqueness theorem can be proved in the classical manner (see [4], [12]).

**Theorem 1.** Assume that \( \Sigma \) is a bounded regular domain and that \( e \) is positive definite. If \( L_1 \) and \( L_3 \) are non-empty sets, then the mixed boundary-value problem (20), (19) has at most one solution.

In order to establish a reciprocity relation, we consider two solutions \( v'_\alpha, w', \psi' \) and \( v''_\alpha, w'', \psi'' \) of the equations (20) corresponding to two different
systems of body loads \( f'_a, f'_3, \ell' \) and \( f''_a, f''_3, \ell'' \), respectively. With the help of (15), (17) and (18) we define \( M'_\alpha, N', H' \) and \( M''_\alpha, N'', H'' \) corresponding to the two solutions.

Following the classical procedure (see [12]) we can prove the reciprocity relation
\[
\int_\Sigma (f'_a v''_a + f'_3 w'' + \ell' \psi'') \, da + \int_L (M'_\alpha v''_a + h_0 N' w'' + H' \psi'') \, ds = \int_\Sigma (f''_a v'_a + f''_3 w' + \ell'' \psi') \, da + \int_L (M''_\alpha v'_a + h_0 N'' w' + H'' \psi') \, ds.
\]

We introduce the notations
\[
\zeta = \xi + \alpha \chi^2, \quad \sigma^2 = \frac{1}{\alpha} \left( \zeta - \frac{b^2}{\lambda + 2\mu} \right).
\]

We see from the inequalities (21) that \( \sigma^2 \) is positive.

In the remaining of this section we present a Galerkin representation for the solution of the equilibrium equations. With the help of the associated matrices method (see [13]) we are led to the following result.

**Theorem 2.** Let
\[
v_\gamma = \alpha (\lambda + 2\mu) \Delta \Delta (\Delta - \sigma^2) V_\alpha - \
- [b^2 \Delta + (\alpha \Delta - \zeta)((\lambda + \mu) \Delta + \mu \chi^2)] \frac{\partial^2 V_\beta}{\partial x_\gamma \partial x_\beta} + \n+ \mu \chi^2 (\alpha \Delta - \zeta) \frac{\partial W}{\partial x_\gamma} - b \frac{\partial \Psi}{\partial x_\gamma},
\]
\[
w = -\mu(\Delta - \chi^2)(\alpha \Delta - \zeta) \frac{\partial V'_\gamma}{\partial x_\gamma} \n+ [b^2 \Delta + (\alpha \Delta - \zeta)((\lambda + 2\mu) \Delta - \mu \chi^2)] W + b \Psi, \n\psi = \mu b \Delta (\Delta - \chi^2) \frac{\partial V'_\gamma}{\partial x_\gamma} + \mu b \chi^2 \Delta W + (\lambda + 2\mu) \Delta \Psi,
\]
where \( V_\gamma, W \) and \( \Psi \) are functions of class \( C^8, C^6 \) and \( C^4 \) on \( \Sigma \), respectively, that satisfy the equations
\[
\Delta \Delta (\Delta - \sigma^2)(\Delta - \chi^2) V_\gamma = - \frac{1}{\alpha \mu (\lambda + 2\mu) I} f_\gamma, 
\]
\[
\Delta \Delta (\Delta - \sigma^2) W = - \frac{1}{\alpha \mu (\lambda + 2\mu) h_0} f_3, 
\]
\[
\Delta (\Delta - \sigma^2) \Psi = - \frac{1}{\alpha (\lambda + 2\mu) I} \ell.
\]
Then the functions \( v, w \) and \( \psi \) satisfy the equilibrium equations (20).

**Proof.** By virtue of (23) and (24) we have

\[
\alpha \Delta \psi - b v_{\gamma, \gamma} - (\xi + \alpha \chi^2)\psi = \\
= \mu \alpha b \Delta \Delta (\Delta - \chi^2) V_{\gamma, \gamma} + \mu \alpha b \chi^2 \Delta \Delta W + \\
+ \alpha (\lambda + 2\mu) \Delta \Delta \psi - \alpha b (\lambda + 2\mu) \Delta (\Delta - \sigma^2) V_{\gamma, \gamma} + \\
+ b [b^2 \Delta + (\alpha \Delta - \zeta)(\lambda + 2\mu) \Delta + \mu \chi^2)] \Delta V_{\beta, \beta} - \\
- \mu b \chi^2 (\alpha \Delta - \zeta) \Delta W + b^2 \Delta \psi - \mu b \zeta (\Delta - \chi^2) V_{\gamma, \gamma} - \\
- \mu b \chi^2 \zeta \Delta W - (\lambda + 2\mu) \Delta \Delta \Psi = \\
= \alpha (\lambda + 2\mu) \Delta (\Delta - \sigma^2) \Psi.
\]

In view of (25)_3, the above relation yields

\[
\alpha \Delta \psi - b v_{\gamma, \gamma} - (\xi + \alpha \chi^2)\psi = -\frac{1}{I} \ell.
\]

Thus we conclude that the equilibrium equation (20)_3 is satisfied.

Similarly, from (23), (24) and (25)_2 we obtain

\[
\Delta w + v_{\gamma, \gamma} = -\mu \Delta (\Delta - \chi^2) (\alpha \Delta - \zeta) V_{\gamma, \gamma} + \\
+ [b^2 \Delta + (\alpha \Delta - \zeta)(\lambda + 2\mu) \Delta - \mu \chi^2)] \Delta W + \\
+ b \Delta \psi + \alpha (\lambda + 2\mu) \Delta \Delta (\Delta - \sigma^2) V_{\gamma, \gamma} - \\
-[b^2 \Delta + (\alpha \Delta - \zeta)(\lambda + \mu) \Delta + \mu \chi^2)] \Delta V_{\beta, \beta} + \mu \chi^2 (\alpha \Delta - \zeta) \Delta W - b \Delta \Psi = \\
= \alpha (\lambda + 2\mu) \Delta (\Delta - \sigma^2) W = -\frac{1}{\mu h_0} f_3,
\]

and we see that equation (20)_2 is satisfied.

In an analogous manner we can prove that \( v, w, \psi \) given by (24) satisfy the equilibrium equation (20)_1. This completes the proof.

3. **Fundamental solutions.** In this section we apply Theorem 2 to obtain fundamental solutions of the equilibrium equations.

Consider a fixed point \( y(y_1, y_2) \) in the \( x_1 O x_2 \) plane and denote by \( r \) the distance from \( y \) to an arbitrary point \( x(x_1, x_2) \). Assume that we have a system of concentrated body loads acting in the point \( y \) of \( \Sigma \).

Let us examine the effect of a concentrated body force of the form

\[
f_1 = \delta(x - y), \ f_2 = f_3 = \ell = 0,
\]

where \( \delta \) is the Dirac measure.
Equations (25) are satisfied if \( V_2 = W = \Psi = 0 \) and \( V_1 \) is a solution of the equation

\[
\Delta \Delta (\Delta - \sigma^2)(\Delta - \chi^2) U = -a\delta(x - y).
\]

Here we have used the notation

\[
a = \frac{1}{\alpha \mu (\lambda + 2\mu)}.
\]

Let us consider the equations

\[
\Delta (\Delta - \sigma^2) G_1 = p, \quad \Delta (\Delta - \chi^2) G_2 = p,
\]

\[
\Delta \Delta (\Delta - \sigma^2) G_3 = p, 
\]

\[
\Delta \Delta (\Delta - \sigma^2)(\Delta - \chi^2) G_4 = p,
\]

where \( p \) is given. We note that the solutions of the above equations can be represented in the form

\[
G_1 = -\sigma^{-2}(g_2 - g_3), \quad G_2 = -\chi^{-2}(g_2 - g_4), \\
G_3 = -\sigma^{-4}(\sigma^2 g_1 + g_2 - g_3), \\
G_4 = \chi^{-2}[(\sigma^2 - \chi^2)^{-1}(G_1 - G_2) - G_3],
\]

where \( g_k \) are functions which satisfy the equations

\[
\Delta \Delta g_1 = p, \quad \Delta g_2 = p, \quad (\Delta - \sigma^2) g_3 = p, \quad (\Delta - \chi^2) g_4 = p.
\]

If we choose \( p = -a\delta(x - y) \) then the equations (32) have the solutions (33)

\[
g_1 = -\frac{a}{8\pi} \sigma^2 \ln r, \quad g_2 = -\frac{a}{2\pi} \ln r, \\
g_3 = -\frac{a}{2\pi} K_0(\sigma r), \quad g_4 = -\frac{a}{2\pi} K_0(\chi r),
\]

where \( K_0 \) is the modified Bessel function of zeroth order.

In view of (30), (31) and (33), the solution of (27) is

\[
U = -\frac{a}{2\pi} \left[ \frac{1}{4} \sigma^{-2} \chi^{-2} r^2 \ln r + \sigma^{-2} \chi^{-2} (\sigma^{-2} + \chi^{-2}) \ln r - \right. \\
-\left(\sigma^2 - \chi^2\right)^{-1}(\sigma^{-4} K_0(\sigma r) - \chi^{-4} K_0(\chi r)) \right].
\]
We denote by \( v^{(1)}_\gamma, w^{(1)}, \psi^{(1)} \) the functions obtained from (24) by putting \( V_2 = W = \Psi = 0 \) and \( V_1 = U \), where \( U \) is given by (34).

In an analogous manner we investigate the effect of the following body loads

\[
(35) \quad f_2 = \delta(x - y), \quad f_1 = f_3 = \ell = 0.
\]

In view of (25), in this case we can take \( V_1 = W = \Psi = 0 \), and \( V_2 \) satisfies the equation (27). Let \( v^{(2)}_\gamma, w^{(2)}, \psi^{(2)} \) denote the functions obtained from (24) for \( V_1 = W = \Psi = 0 \) and \( V_2 = U \). We find the following expressions for \( v^{(\beta)}_\gamma, w^{(\beta)} \) and \( \psi^{(\beta)} (\beta = 1, 2) \)

\[
(36) \quad v^{(\beta)}_\gamma = \alpha(\lambda + 2\mu)\Delta\Delta(\Delta - \sigma^2)U\delta_{\beta\gamma} - [b^2\Delta + (\alpha\Delta - \zeta)((\lambda + \mu)\Delta + \mu\chi^2)]\frac{\partial^2 U}{\partial x_\beta \partial x_\gamma},
\]

\[
 w^{(\beta)} = -\mu(\Delta - \chi^2)(\alpha\Delta - \zeta)\frac{\partial U}{\partial x_\beta},
\]

\[
 \psi^{(\beta)} = \mu b\Delta(\Delta - \chi^2)\frac{\partial U}{\partial x_\beta},
\]

where \( U \) is given in (34). By using relations of the type

\[
\frac{dK_0(\sigma r)}{dr} = -\sigma K_1(\sigma r),
\]

\[
\frac{d^2 K_0(\sigma r)}{dr^2} = \sigma^2[K_0(\sigma r) + \sigma^{-1}r^{-1}K_1(\sigma r)],
\]

where \( K_1 \) is the modified Bessel function of the first kind, then from (34)
and (36) we obtain

\[ v_\gamma^{(\beta)} = \frac{-a}{2\pi} \{ [C + 2C \ln r + (A + B)r^{-2} - B\chi^2 K_0(\chi r) - \\
- r^{-1}(A\sigma K_1(\sigma r) + B\chi K_1(\chi r))] \delta_{\beta\gamma} + \\
+ [2Cr^{-2} - 2(A + B)r^{-4} + r^{-2}(A\sigma^2 K_0(\sigma r) + \\
+ B\chi^2 K_0(\chi r)) + 2r^{-3}(A\sigma K_1(\sigma r) + \\
+ B\chi K_1(\chi r))(x_\beta - y_\beta)(x_\gamma - y_\gamma)] \}, \]

\[ w^{(\beta)} = \frac{a}{2\pi} [C + 2C \ln r + Ar^{-2} - A\sigma r^{-1} K_0(\sigma r)](x_\beta - y_\beta), \]

\[ \psi^{(\beta)} = \frac{\mu ab}{2\pi \sigma^2} [r^{-2} - \sigma r^{-1} K_1(\sigma r)](x_\beta - y_\beta). \]

In (37) we have introduced the coefficients \( A, B, C \) defined by

\[ A = \frac{\mu b^2}{(\lambda + 2\mu)\sigma^4}, \quad B = \frac{\alpha(\lambda + 2\mu)}{\chi^2}, \quad C = \frac{\mu \zeta}{4\sigma^2}. \]

In the case when the system of concentrated body loads is

\[ f_3 = \delta(x - y), \quad f_\gamma = \ell = 0, \]

we denote by \( v_3^{(3)}, w^{(3)}, \psi^{(3)} \) the corresponding displacements and volume fraction.

From equations (25) we find \( V_\gamma = \Psi = 0 \) and

\[ \Delta \Delta(\Delta - \sigma^2)W = -a\chi^{-2}\delta(x - y). \]

In view of (29), (31)_3 and (33), the solution of equation (40) is

\[ W = \frac{a}{2\pi \sigma^4 \chi^2} \left[ \ln r + \frac{1}{4} r^2 \ln r + K_0(\sigma r) \right]. \]
By substituting $V_\gamma = 0$, $\Psi = 0$ and (41) in the relations (24), we get

\[
v^{(3)}_\gamma = -\frac{a}{2\pi} [C + 2C \ln r + Ar^{-2} - A\sigma r^{-1} K_1(\sigma r)](x_\gamma - y_\gamma),
\]

\[
w^{(3)} = \frac{a}{2\pi} [- (B + \mu\alpha \sigma^{-2}) + (A - B) \ln r + Cr^2 \ln r + AK_0(\sigma r)],
\]

\[
\psi^{(3)} = \frac{\mu ab}{2\pi \sigma^2} [1 + \ln r + K_0(\sigma r)].
\]

Let us consider now the case of a concentrated extrinsic equilibrated body force. We assume that

\[
\ell = \delta(x - y), \quad f_\alpha = f_3 = 0.
\]

By virtue of (28)\textsubscript{1} and (31)\textsubscript{1}, in this case equations (25) have the solutions

\[
V_\alpha = 0, \quad W = 0, \quad \Psi = \frac{\mu a}{2\pi \sigma^2} [1 + \ln r + K_0(\sigma r)].
\]

If $v^{(4)}_\gamma$, $w^{(4)}$, $\psi^{(4)}$ denote the displacements and change in volume fraction corresponding to (43), then from (24) and (44) we obtain

\[
v^{(4)}_\gamma = -\frac{\mu ab}{2\pi \sigma^2} [r^{-2} - \sigma r^{-1} K_1(\sigma r)](x_\gamma - y_\gamma),
\]

\[
w^{(4)} = \frac{\mu ab}{2\pi \sigma^2} [1 + \ln r + K_0(\sigma r)],
\]

\[
\psi^{(4)} = \frac{1}{2\pi \alpha I} K_0(\sigma r).
\]

Relations (37), (42) and (45) give the fundamental solutions of the equilibrium equations (20). The matrix of fundamental solutions is

\[
\Gamma(x, y) = \|\gamma_{rs}\|,
\]

where $\gamma_{rs}$ $(r, s = 1, 2, 3, 4)$ are given by

\[
\gamma_{rs} = v^{(s)}_\alpha, \quad \gamma_{3s} = w^{(s)}, \quad \gamma_{4s} = \psi^{(s)}, \quad (s = 1, 2, 3, 4).
\]

From (37), (42) and (45) we see that

\[
\Gamma(x, y) = \Gamma^*(y, x),
\]
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where $\Gamma^*$ is the transposed of matrix $\Gamma$.

4. Integral representation of solution. The system (20) can be written in a matricial form. Following [14], a vector $u = (u_1, ..., u_m)$ will be considered as a column matrix so that the product of a matrix $A = \|a_{ij}\|_{m \times m}$ and the vector $u$ is a $m$ dimensional vector. The vector $u$ multiplied by the matrix $A$ will denote the matrix product between the row matrix $\|u_1, u_2, ..., u_m\|$ and the matrix $A$. We denote

$$v = (v_1, v_2, w, \psi), \ f = -(f_1, f_2, f_3, \ell).$$

The system (20) can be written in the form

\begin{equation}
D \left( \frac{\partial}{\partial x} \right) v = f,
\end{equation}

where

$$D \left( \frac{\partial}{\partial x} \right) = \left\| D_{ij} \left( \frac{\partial}{\partial x} \right) \right\|_{4 \times 4},$$

$$D_{\alpha\beta} \left( \frac{\partial}{\partial x} \right) = I \left[ \mu \delta_{\alpha\beta}(\Delta - \chi^2) + (\lambda + \mu) \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \right],$$

$$D_{\alpha 3} \left( \frac{\partial}{\partial x} \right) = -D_{3\alpha} \left( \frac{\partial}{\partial x} \right) = -\mu h_0 \frac{\partial}{\partial x_\alpha},$$

$$D_{\alpha 4} \left( \frac{\partial}{\partial x} \right) = -D_{4\alpha} \left( \frac{\partial}{\partial x} \right) = b I \frac{\partial}{\partial x_\alpha}, \quad D_{34} = D_{43} = 0,$$

$$D_{33} \left( \frac{\partial}{\partial x} \right) = \mu h_0 \Delta, \quad D_{44} \left( \frac{\partial}{\partial x} \right) = I(\alpha \Delta - \zeta).$$

We denote

$$t = (M_1, M_2, h_0 N, H)$$

and introduce the matricial operator

$$T \left( \frac{\partial}{\partial x}, n_x \right) = \left\| T_{ij} \left( \frac{\partial}{\partial x}, n_x \right) \right\|_{4 \times 4},$$
where
\[ T_{\alpha\beta} \left( \frac{\partial}{\partial x}, n_x \right) = I \left[ \mu \delta_{\alpha\beta} \frac{\partial}{\partial n} + \left( \lambda n_\alpha \frac{\partial}{\partial x_\beta} + \mu n_\beta \frac{\partial}{\partial x_\alpha} \right) \right], \]
\[ T_{\alpha3} \left( \frac{\partial}{\partial x}, n_x \right) = b I n_\alpha, \quad T_{3\alpha} \left( \frac{\partial}{\partial x}, n_x \right) = \mu h_0 n_\alpha, \]
\[ T_{33} \left( \frac{\partial}{\partial x}, n_x \right) = \mu h_0 \frac{\partial}{\partial n}, \quad T_{44} \left( \frac{\partial}{\partial x}, n_x \right) = \alpha I \frac{\partial}{\partial n}, \]
\[ T_{\alpha3} = T_{34} = T_{4\alpha} = T_{43} = 0. \]

The relation (18) can be written in the form
\[ t = T \left( \frac{\partial}{\partial x}, n_x \right) v. \]

In view of (47) and (48), the reciprocity relation (22) leads to
\[ \int_{\Sigma} \left[ v D \left( \frac{\partial}{\partial x} \right) u - u D \left( \frac{\partial}{\partial x} \right) v \right] da = \int_{L} \left[ v T \left( \frac{\partial}{\partial x}, n_x \right) u - u T \left( \frac{\partial}{\partial x}, n_x \right) v \right] ds. \] (49)

Let \( \Gamma^{(k)} (k = 1,2,3,4) \) denote the columns of the matrix \( \Gamma \). If we apply the relation (49) to a regular vector \[ v = (v_1, v_2, w, \psi) \] and to the vector field \( u(x,y) = \Gamma^{(s)}(x,y), (s = 1,2,3,4) \), then we obtain
\[ v(y) = \int_{L} \left\{ \Gamma^{*}(x,y) T \left( \frac{\partial}{\partial x}, n_x \right) v(x) - \left[ T \left( \frac{\partial}{\partial x}, n_x \right) \Gamma(x,y) \right]^{*} v(x) \right\} ds_x - \int_{\Sigma} \Gamma^{*}(x,y) D \left( \frac{\partial}{\partial x} \right) v(x) da_x. \]

Finally, if we introduce the notation
\[ \Lambda(x,y) = \left( T \left( \frac{\partial}{\partial x}, n_y \right) \Gamma(x,y) \right)^{*} \]
and take into account (46), then we arrive to the following representation of Somigliana type

\[ v(x) = \int_{L} \left[ \Gamma(x, y) T \left( \frac{\partial}{\partial y}, n_y \right) v(y) - \Lambda(x, y) v(y) \right] ds_y - \int_{\Sigma} \Gamma(x, y) D \left( \frac{\partial}{\partial y} \right) v(y) da_y. \]  

(50)

The relation (50) can be used to study the boundary–value problems of the bending of porous plates by means of the potential theory [14].

REFERENCES


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