AFFINE STRUCTURE ON COMPLEX FOLIATED MANIFOLDS

BY

GHEORGHE MUNTEANU and CRISTIAN IDA

Abstract. In this paper the affine complex manifold notion is defined and some local and global aspects on such a manifold: complex Liouville vector, semispray, complex Lagrange structure, etc., are studied. These notions are introduced here by analogy with the real case ([V1, V2, V3]), but there exist specific particularities.

Mathematics Subject Classification 2000: 53B40, 53C12, 57R30.

Key words: complex Lagrange structures, affine foliations, cohomology.

1. Affine complex manifold. Let $\mathcal{M}$ be a complex foliated manifold ([V1]) with the same dimension and codimension $n$ and $u = (z^i, \eta^j)_{i=1}^m$ complex coordinates in a local map $(U_\alpha, \varphi_\alpha)$.

Definition 1.1. $\mathcal{M}$ is said to be an affine complex manifold if the local changes $(U_\alpha, \varphi_\alpha) \rightarrow (U_\beta, \varphi_\beta)$ are given by

\begin{equation}
(z'^i = z'^i(z) \quad ; \quad \eta'^j = \frac{\partial z'^i}{\partial z^j}\eta^j + B'^i(z))
\end{equation}

where $z'^i$ and $B'^i$ are holomorphic functions on $z^i$ variables and $\det(\frac{\partial z'^i}{\partial z^j}) \neq 0$.

The leafs of this manifold, denoted by $\mathcal{V}$, are characterized by $z^i = \text{const}$.

Let $J$ be the natural complex structure of the manifold and $T'\mathcal{M}$ and $T''\mathcal{M} = \overline{T'\mathcal{M}}$ be its holomorphic and antiholomorphic subbundles, respectively. By $T_C\mathcal{M} = T'\mathcal{M} \oplus T''\mathcal{M}$ we denote the complexified tangent bundle.
From (1.1) it results the following changes for the natural local frames on $T'_uM$:

\[
\begin{align*}
\frac{\partial}{\partial z^k} &= \frac{\partial z^i}{\partial z^k} \frac{\partial}{\partial z^i} + \left( \frac{\partial^2 z^i}{\partial z^j \partial z^k} \eta^j + \frac{\partial B^i}{\partial z^k} \right) \frac{\partial}{\partial \eta^i}, \\
\frac{\partial}{\partial \eta^k} &= \frac{\partial z^i}{\partial z^k} \frac{\partial}{\partial \eta^i}.
\end{align*}
\]

By conjugation over all in (1.2) we obtain the change rules of the local frames on $T''_uM$, and then the behaviour of the $J$ complex structure

\[
\begin{align*}
J(\frac{\partial}{\partial z^k}) &= i \frac{\partial}{\partial z^k} ; \\
J(\frac{\partial}{\partial \eta^k}) &= 0 ; \\
J(\frac{\partial}{\partial \bar{z}^k}) &= -i \frac{\partial}{\partial \bar{z}^k} ; \\
J(\frac{\partial}{\partial \bar{\eta}^k}) &= -i \frac{\partial}{\partial \bar{\eta}^k}.
\end{align*}
\]

Note that inside of (1.1) we can take into account the more general affine changes

\[
\eta^i' = A^i_j(z) \eta^j + B^i(z),
\]

where $A^i_j$ are also holomorphic functions.

As well as in (1.1) we can consider the particular case $B^i = 0$, when $M$ is identified with the holomorphic tangent bundle of a complex manifold, $M = T'M$. In this last situation we will say that $T_C M$ is of holomorphic vector type.

However, the general discussion $A^i_j \neq \frac{\partial z^i}{\partial z^j}$ limits the opportunity of other significant structures on $M$ be considered. Such important structure for the geometry of $T_C M$, excluded by the general case, is

\[
(1.4) \quad S(\frac{\partial}{\partial z^k}) = \frac{\partial}{\partial \eta^k} ; \quad S(\frac{\partial}{\partial \eta^k}) = 0 ; \quad S(\frac{\partial}{\partial \bar{z}^k}) = \frac{\partial}{\partial \bar{\eta}^k} ; \quad S(\frac{\partial}{\partial \bar{\eta}^k}) = 0.
\]

**Proposition 1.1.** $S$ is a global tangent structure acting on $T_C M$. Moreover, $S$ is integrable.

**Proof.** We easily check that $S^2 = 0$ and the (1.4) definition is preserved at (1.2) changes. Since $N_s(X, Y) := [SX, SY] - S[SX, Y] - S[X, SY] + S^2 [X, Y] = 0$, for any $X, Y \in \Gamma T_C M$, it follows that $S$ is integrable. □

**Proposition 1.2.** $(S, J, J \circ S = S \circ J)$ determines a commutative and integrable semiquaternionic structure ([Mu1]) on $T_C M$.

Consider $T'V = span(\frac{\partial}{\partial \eta^i}) \subset T'uM$, the vertical distribution, which in view of (1.2) is an integrable one.

Let $\Gamma_\alpha := \eta^i \frac{\partial}{\partial \eta^i}$ be the Liouville complex field, ([Mu2]), locally considered in the map $(U_\alpha, \varphi_\alpha)$. Note that if $T_C M$ is of holomorphic vectorial
type, then $\Gamma_{\alpha}$ glue up to a global defined vector field. Let us pay more attention how other requests ensure the global existence of the Liouville complex vector field. This is one important question in the following.

By $\mathcal{A}^0_{pr}(\mathcal{M}, \mathcal{V})$ we denote the module of affine leafwise holomorphic vertical functions, namely $\{a_i(z)\eta^i + b(z)\}$, where $a_i, b \in \Omega^0_{pr}(\mathcal{M})$ are holomorphic functions on $\mathcal{M}$. For the sheaf of corresponding germs we have the following exact sequence

\[
0 \to \Omega^0_{pr}(\mathcal{M}) \xrightarrow{i} \mathcal{A}^0_{pr}(\mathcal{M}, \mathcal{V}) \xrightarrow{pr} \Omega^0_{pr}(\mathcal{M}) \otimes T^\ast \mathcal{V} \to 0
\]

explicitly given by $b \mapsto a_i \eta^i + b \xrightarrow{pr} a_i dz^i$.

We note that $\Gamma_{\alpha \beta} := \Gamma_{\beta} - \Gamma_{\alpha} = (\frac{\partial}{\partial \bar{z}^i} \eta^j + B^i - \eta^i) \frac{\partial}{\partial \eta^j}$ is a complex vector field on $U_{\alpha} \cap U_{\beta}$ with its coefficients in $\mathcal{A}^0_{pr}(\mathcal{M}, \mathcal{V})$, and therefore $\Gamma_{\alpha \beta}$ yields a cocycle, $(\delta \Gamma)_{\alpha \beta \gamma} = \Gamma_{\alpha \beta} - \Gamma_{\alpha \gamma} + \Gamma_{\beta \gamma} = 0$; hence its class in Cech cohomology, $[\Gamma_{\alpha}]$, will be with coefficients in $\mathcal{A}^0_{pr}(\mathcal{M}, \mathcal{V})$. Thus $[\Gamma_{\alpha}] \in H^1(\mathcal{M}, \mathcal{A}^0_{pr}(\mathcal{M}, \mathcal{V}))$, the first group of cohomology, and will be called the *complex linear obstruction*. Clearly, if $[\Gamma_{\alpha}] = 0$, then the Liouville complex field is globally defined on $\mathcal{M}$ and it will be simply denoted by $\Gamma$. Moreover, it results that $\Gamma_{\alpha} = \Gamma|_{U_{\alpha}} + E_{\alpha}$, where $E_{\alpha} = E^i(z) \frac{\partial}{\partial \eta^i}$ is a projectable vector field on $U_{\alpha}$, called the Euler complex field.

In [P-M] we proved a Grothendieck-Dolbeault lemma on a complex manifold endowed with a complex Finsler metric. We can consider here $\Gamma_{\alpha} = \bar{\eta}^j \frac{\partial}{\partial \eta^j}$, the conjugate Liouville complex vector and the differential operator $d = d' + d''$ on the affine complex manifold $\mathcal{M}$. All the same, a Grothendieck-Dolbeault lemma on an affine complex manifold (endowed possible with a complex Finsler metric) seems to be a difficult problem. Hence $d''$, usually, don’t determine a fine resolution and consequently to prove the classical isomorphism between $H^1(\mathcal{M}, \mathcal{A}^0_{pr}(\mathcal{M}, \mathcal{V}))$ and the 1-dimensional de Rham group is not anyway possible. Remains for a further work to furnish some examples for which we can do it.

Now with respect to $\Gamma_{\alpha}$, as in the classical way (a bit generalized), [M-A, Mu2, V3], we will introduce the affine complex semispray.

**Definition 1.2.** An affine complex semispray in $(U_{\alpha}, \varphi_{\alpha})$ is a complex vector field $G_{\alpha}$ on $T' \mathcal{M}$, for which $S \circ G_{\alpha} - \Gamma_{\alpha}$ is projectable, i.e. $S \circ G_{\alpha} = \Gamma_{\alpha} + A(z) \frac{\partial}{\partial \eta^j}$.
By this definition it is easily deduced that $G_\alpha$ have the following writing:

$$G_\alpha = (\eta^i + A^i) \frac{\partial}{\partial z^i} - 2G^i \frac{\partial}{\partial \eta^i},$$

(1.6)

where $G^i(z, \eta)$ are complex valued functions on $\mathcal{M}$, called the coefficients of the semispray.

If we consider $(U_\beta, \varphi_\beta)$ other local chart and $G_\beta$ an affine complex semispray with local coefficients given by $G^i'(z', \eta')$, using (1.2) and (1.6), directly is obtained

$$A'^i = A^k \frac{\partial z'^i}{\partial z^k} - B'^i;$$

$$G'^i = G^k \frac{\partial z'^i}{\partial z^k} - \frac{1}{2}(\eta'^k - A'^k)(\partial^2 z'^i \frac{\partial}{\partial z^j} \partial \eta^j + \partial B'^i \frac{\partial}{\partial z^k}).$$

(1.7)

Now, let us make a useful remark. The affine character of the complex semispray is related to the chosen local chart. If in $(U_\alpha, \varphi_\alpha)$, we operate in (1.1) the following translation for the coordinates: $\tilde{z}^i = z^i$ and $\tilde{\eta}^i = \eta^i - A^i$, then $G_\alpha$ looses his affine character and, in such an atlas, $G_{\alpha \beta} := G_\beta - G_\alpha$ has coefficients in the form $a_{ij}(z)\eta^j \eta^i + a_i(z)\eta^j + b(z)$, which belongs to the sheaves of 2-jets vertical affine leafwise germs, denoted by $\mathcal{J}^2A_{pr}^0(\mathcal{M}, \mathcal{V})$.

It results that

$$0 \rightarrow \Omega^0_{pr}(\mathcal{M}) \overset{i}{\rightarrow} A^0_{pr}(\mathcal{M}, \mathcal{V}) \overset{i^2}{\rightarrow} J^2A^0_{pr}(\mathcal{M}, \mathcal{V}) \overset{\pi}{\rightarrow} \Omega^0_{pr}(\mathcal{M}) \otimes (T^*\mathcal{M})^2 \rightarrow 0$$

is an exact sequence.

Obviously, in the mentioned atlas, $G_{\alpha \beta}$ defines a cocycle in the Céch cohomology, its coefficients being in $J^2A_{pr}^0(\mathcal{M}, \mathcal{V})$. Let $[G_\alpha]$ be the cohomology class defined by $G_\alpha$: $[G_\alpha] \in H^1(M, J^2A^0_{pr}(\mathcal{M}, \mathcal{V}))$ and its vanishing is a condition for the global existence of the complex semispray $G_\alpha$, called the second order obstruction.

The interest for the existence of a global complex semispray is major. It is well known, [Mu2], that in our purpose of ”linearizing” for the geometry of $\mathcal{M}$, a special meaning plays the complex nonlinear connection (in brief (c.n.c.)), which is intrinsically related to the notion of complex semispray. Therefore, subsequently, let us analyze in which consist the obstructions for globalization of a (c.n.c.).

Let $T'\mathcal{H}$ be a supplementary (transversal) distribution of $T'\mathcal{V}$ in $T'\mathcal{M}$, locally spanned in $(U_\alpha, \varphi_\alpha)$ by \{ $\frac{\partial}{\partial \tilde{x}^i} = \frac{\partial}{\partial z^i} - N^j_k \frac{\partial}{\partial \eta^j}$ \}. From (1.2) and (1.4) it
results $S(\delta \frac{\delta}{\delta z}) = \partial \partial^\eta$, and then the global existence of the tangent structure $S$ implies

$$\frac{\delta}{\delta z^k} = \frac{\partial z^i}{\partial \eta^k} \frac{\delta}{\delta z^k} \frac{\delta}{\delta z^i}. \tag{1.8}$$

Hence, at the local changes of the charts, we have the following rules of change for the coefficients $N^i_k(z, \eta)$ of the (c.n.c.)

$$\frac{\partial z^i}{\partial \bar{\eta}^k} N^j_m = N^i_k \frac{\partial z^i}{\partial \eta^k} \frac{\partial}{\partial \bar{\eta}^m} \bar{\eta}^j - \frac{\partial^2 z^i}{\partial \eta^k \partial z^m} \eta^j + \frac{\partial B^i}{\partial z^k} \frac{\partial}{\partial \bar{\eta}^m} \bar{\eta}^i. \tag{1.9}$$

**Proposition 1.3.** In the translated atlas $\{(\tilde{z}^i = z^i; \bar{\eta}^i = \eta^i - A^i)\}$, in which $G_{\alpha\beta}$ has coefficients in the shaves of 2-jets vertical affine leafwise germs, $N^i_k = \partial G^j / \partial \bar{\eta}^k$ determine the coefficients of a (c.n.c.).

**Proof.** By differentiate (1.7), in the mentioned atlas, yields

$$\frac{\partial G^m}{\partial \bar{\eta}^m} N^i_k \frac{\partial z^i}{\partial \bar{\eta}^i} = \frac{1}{2} \left( \frac{\partial^2 z^i}{\partial \eta^k \partial z^m} \eta^j + \frac{\partial B^i}{\partial z^k} \frac{\partial}{\partial \bar{\eta}^m} \bar{\eta}^i \right) - \frac{1}{2} \frac{\partial^2 z^i}{\partial \eta^k \partial z^m} \eta^j.$$

Since $\frac{\partial G^m}{\partial \bar{\eta}^m} = \frac{\partial z^i}{\partial \eta^k} N^i_k$, precisely the (1.9) rule of change results for the coefficients of the (c.n.c.). $\square$

It deduces that the global existence of $G_\alpha$, that is $[G_\alpha] = 0$, is a sufficient restriction for the global existence of a (c.n.c.).

We recall, [Mu2], that a (c.n.c.) can exist independently of a complex semispray (for instance, the Chern-Finsler (c.n.c.) does not come anyway from a semispray). From the definition of the adapted frames $\frac{\delta}{\delta z^k}$ and (1.9) we may conclude that the obstructions for the existence of a (c.n.c.) are linear but their coefficients are not always affine projectable functions.

We can continue our talk with the study of the global existence for other geometric objects, related to a (c.n.c.), such as: $d$–complex linear connection, their torsions and curvatures, etc. But these may be the topic for a forthcoming paper.

### 2. Complex Lagrange structures.

Let $L_\alpha : M \rightarrow \mathbb{R}$ be a Lagrangian function, defined on $U_\alpha \subset M$, domain of local chart.

**Definition 2.1.** We say that a family $\{M, L_\alpha\}$ is a local complex Lagrange structure on $M$, if there exists an atlas such that $g_{ij} = \partial^2 L_\alpha / \partial \eta^i \partial \bar{\eta}^j$ glue up to a global Hermitian metric on $M$. 

If \( \{M, L_\alpha\} \) defines a global Lagrange structure on \( M \), by integration of \( g_{ij} \), we obtain a Lagrangian \( L : M \to \mathbb{R} \) such \( L_\alpha = L \big|_{U_\alpha} + l_\alpha \), where \( l_\alpha \) is an affine real valued vertical form on \( M \), i.e. there exist \( \tilde{A}_i(z) \) and \( \tilde{B}(z) \in \mathbb{R}^+ \) such that

\[
L_\alpha = L \big|_{U_\alpha} + \tilde{A}_i (\eta^i + \bar{\eta}^i) + \tilde{B}.
\]

On the intersection \( U_\alpha \cap U_\beta \) we can define a cocycle, \( L \) being closed with respect to differential \( (\delta L)_{\alpha\beta\gamma} := L_{\alpha\beta} - L_{\alpha\gamma} + L_{\beta\gamma} = 0 \).

**Proposition 2.1.** Denote by \( [L_\alpha] \) the cohomology class defined by the cocycle \( L_{\alpha\beta} \).

Then \( [L_\alpha] \in H^1(M, A^0_R(M, \mathcal{V} + \bar{\mathcal{V}})) \) and \( [L_\alpha] = 0 \) yields \( L \) is globally defined.

Let us see when \( [L_\alpha] = 0 \). Certainly, generally the class \( [L_\alpha] \) is distinct to \( [G_\alpha] \).

By analogy with (1.5) we can construct an exact sequence over affine real valued functions, without requesting their holomorphy,

\[
0 \to \Phi^0_R(M) \overset{i}{\to} A^0_R(M, \mathcal{V} + \bar{\mathcal{V}}) \overset{\pi}{\to} \Phi^{(1,0)}_R(M) \to 0,
\]

explicitly given by the correspondence \( \tilde{B} \to \tilde{A}_i (\eta^i + \bar{\eta}^i) + \tilde{B} \to \tilde{A}_i d\eta^i \).

This induces the following exact sequence of cohomology groups

\[
0 \to H^1(M, \Phi^0_R(M)) \overset{\iota}{\to} H^1(M, A^0_R(M, \mathcal{V} + \bar{\mathcal{V}})) \overset{\pi}{\to} H^1(M, \Phi^{(1,0)}_R(M)) \overset{\delta}{\to} H^2(M, \Phi^{(2,0)}_R(M)) \overset{\delta}{\to} ...
\]

Let be \( [L_\alpha]_1 = \pi^* [L_\alpha] \in H^1(M, \Phi^{(1,0)}_R(M)) \). If \( [L_\alpha]_1 = 0 \) then \( [L_\alpha]_1 \in ker \pi^* = Im i^* \) and therefore there exists \( [L_\alpha]_2 \in H^1(M, \Phi^{(0)}_R(M)) \) such that \( i^* [L_\alpha]_2 = [L_\alpha]_1 \). Hence we can state

**Proposition 2.2.** \( [L_\alpha] = 0 \) if and only if \( [L_\alpha]_1 = [L_\alpha]_2 = 0 \).

These are the main obstructions to a global complex Lagrange structure on \( M \).

In a complex Lagrange space the Euler \((1,0)\)-form plays a special rôle ([Mu2]). With respect to a local Lagrange structure \( L_\alpha \), the Euler form is \( \theta_\alpha = \frac{\partial L_\alpha}{\partial \eta^i} dz^i \). The difference \( C_{\alpha\beta} := \theta_\beta - \theta_\alpha = \{ \bar{A}_j \frac{\partial \bar{z}^j}{\partial \bar{z}^{\alpha}} - \bar{A}_\alpha \} dz^i \) defines a
cocycle and its cohomology class is in $H^1(M, \Phi_R^{1,0}(M))$. Then $[L_\alpha]_1 = 0$ implies $\theta_\alpha$ is global defined and conversely. So, $\theta_\alpha$ glue up to a global complex Euler form and $L_\alpha$ differ from $L |_{U_\alpha}$ only by the term $\bar{B}$.

On the other hand, for a fixed globally defined (c.n.c.) which spans the distribution $T^\prime \mathcal{H}$, we can consider the dual adapted base $\delta \eta^i = d\eta^i + N^j_i dz^j$ and accordingly the following form $\omega_\alpha := \frac{\partial L_\alpha}{\partial \eta^i} \delta \eta^i = \frac{\partial L_\alpha}{\partial \bar{\eta}^i} \delta \bar{\eta}^i + A_i \delta \eta^i$.

Since $\delta \eta^i$ is globally defined, it results that $\theta_\alpha$ is globally defined if and only if $\omega_\alpha$ is globally defined, too.

We note that

$$d'' \omega_\alpha = \frac{\partial^2 L_\alpha}{\partial \eta^i \partial \bar{\eta}^j} dz^i \wedge \delta \bar{\eta}^j = g_{i\bar{j}} dz^i \wedge \delta \bar{\eta}^j := \Theta.$$  

Since $g_{i\bar{j}}$ the metric tensor of a local complex Lagrange structure is globally defined, then $\Theta$ is global too. Thus, in conclusion we obtain

**Proposition 2.3.** $[L_\alpha]_1 = 0$ if and only if $\Theta$ is a $d''\omega$-exact form.

Let us obtain further a characterization condition for $[L_\alpha]_2 = 0$. Let us assume that $[L_\alpha]_1 = 0$, namely $\Theta = d''\omega$ is an exact form. Then $\theta_\alpha = \omega |_{U_\alpha} + \xi_\alpha$, where $\xi_\alpha = \xi_i(z) dz^i$ is a foliated $(1,0)$-form on $M$. From (2.1) it results

$$\frac{\partial (L_\beta - L_\alpha)}{\partial \eta^i} = \xi_i - \xi_i;$$

that means $L_\beta - L_\alpha = (\xi_i - \xi_i)(\eta^i + \bar{\eta}^i) + B$, where $(\xi_i - \xi_i)(z)$ and $B(z)$ are real valued on $U_\alpha \cap U_\beta$.

After the change of the local complex structure, $\tilde{L}_\alpha := L_\alpha - \xi_i(\eta^i + \bar{\eta}^i)$, is obtained a cocycle $\tilde{L}_{\alpha\beta} = \tilde{L}_\beta - \tilde{L}_\alpha$ which coincides with $L_{\alpha\beta}$, but $[L_\alpha]_1 = 0$ implies $\tilde{L}_\alpha = L |_{U_\alpha} + \tilde{B}$.

Obviously we have $(d'' + d'') \tilde{B} = 0$, because $\tilde{B}$ depends only on $z$.

**Theorem 2.1.** The family $\{L_\alpha\}$ defines a global complex Lagrange structure on $M$ if and only if, for a fixed global horizontal distribution $T^\prime \mathcal{H}$, the 2-form $\Theta$ is exact and there exist a family $\{\tilde{L}_\alpha\}$ of local complex Lagrangians such that $\{(d'' + d'') \tilde{L}_\alpha\}$ define a 1-global form on $M$.

Finally we remark that the above discussion can be easily applied for the case of complex Lagrange geometry, when $M = T^\prime M$ is endowed with a local complex Lagrangian, intensively studied in [Mu2].
REFERENCES


Received: 30.09.2004

Transilvania Univ. of Brašov,
Faculty of Mathematics and Informatics,
gh.munteanu@inf.o.unitbv.ro,