ON SOME PROPERTIES OF THE GOULD TYPE INTEGRAL WITH RESPECT TO A MULTISUBMEASURE

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Abstract. This paper continues our work [5] concerning a Gould type integral of a bounded, real valued function with respect to a multisubmeasure taking values in $\mathcal{P}_{bf}(X)$, $X$ being a Banach space and $\mathcal{P}_{bf}(X)$ the family of all non-empty, bounded, closed subsets of $X$. We prove that the integral preserves the regularity as well as the $\sigma$-continuity of the multisubmeasure. Also, some results concerning sequences of $\mu$-integrable functions are obtained.

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1. Terminology and notations. Let $T$ be an abstract, nonvoid set, $A$ an algebra of subsets of $T$, $X$ a real Banach space, $\mathcal{P}_f(X)$ the family of all nonempty, closed subsets of $X$, $\mathcal{P}_{bf}(X)$ the family of all nonvoid, closed, bounded subsets of $X$ and $\mathcal{P}_{kc}(X)$ the family of all nonempty, compact, convex subsets of $X$.

Let also $f : T \to R$ be a real valued, bounded function.

By $\star$ we mean the Minkowski addition on $\mathcal{P}_f(X)$, that is:

$$M \star N = M + N,$$

for every $M, N \in \mathcal{P}_f(X)$.

Let $h$ be the Hausdorff pseudometric on $\mathcal{P}_f(X)$. It is well-known that $h$ becomes a metric on $\mathcal{P}_{bf}(X)$, and $(\mathcal{P}_{bf}(X), h)$, $(\mathcal{P}_{kc}(X), h)$ are complete metric spaces. We know that $h(M, N) = \max\{e(M, N), e(N, M)\}$, where $e(M, N) = \sup_{x \in M} d(x, N)$, $d(x, N)$ being the distance from $x$ to $N$ with respect
to the metric induced by the norm of $X$. We denote $|M| = h(M, \{0\})$, for every $M \in \mathcal{P}_f(X)$, where $0$ is the origin of $X$.

**Definition 1.1.** I) A partition of $T$ is a finite family $P = \{A_i\}_{i=1}^n \subset \mathcal{A}$ such that $A_i \cap A_j = \emptyset$, $i \neq j$ and $\bigcup_{i=1}^n A_i = T$.

II) Let $P = \{A_i\}_{i=1}^n$ and $P' = \{B_j\}_{j=1}^m$ be two partitions of $T$. $P'$ is said to be finer than $P$ (denoted by $P \leq P'$) if for every $j = 1, m$, there exists $i_j = 1, n$ so that $B_j \subseteq A_{i_j}$.

III) The common refinement of two partitions $P = \{A_i\}_{i=1}^n$ and $P' = \{B_j\}_{j=1}^m$ is the partition $P \wedge P' = \{A_i \cap B_j\}_{i=1}^n_{j=1}^m$.

Obviously, $P \wedge P' \geq P$ and $P \wedge P' \geq P'$.

**Definition 1.2.** Let $\mu : \mathcal{A} \mapsto \mathcal{P}_f(X)$ be a multivalued set function.

I) $\mu$ is said to be exhaustive (with respect to $h$) if $\lim_{n \to \infty} |\mu(A_n)| = 0$, for every disjoint sequence $(A_n)_n \subset \mathcal{A}$.

II) $\mu$ is said to be $o$-continuous (with respect to $h$) if $\lim_{n \to \infty} |\mu(A_n)| = 0$ for every sequence $(A_n)_n \subset \mathcal{A}$, with $A_n \searrow \emptyset$ (that is, $A_n \supseteq A_{n+1}$, for every $n \in \mathbb{N}^*$ and $\bigcap_{n=1}^\infty A_n = \emptyset$).

III) $\mu$ is said to be increasing convergent (with respect to $h$) if $\lim_{n \to \infty} h(\mu(A_n), \mu(A)) = 0$, for every sequence $(A_n)_n \subset \mathcal{A}$, with $A_n \nearrow A \in \mathcal{A}$ (that is, $A_n \subseteq A_{n+1}$, for every $n \in \mathbb{N}^*$ and $\bigcup_{n=1}^\infty A_n = A$).

IV) $\mu$ is said to be $h - \sigma$-subadditive if $|\mu(A)| \leq \sum_{n=1}^\infty |\mu(A_n)|$, for every (disjoint) sequence $(A_n)_n \subset \mathcal{A}$, with $A = \bigcup_{n=1}^\infty A_n \in \mathcal{A}$.

V) $\mu$ is said to be absolutely continuous with respect to another multivalued set function $\nu : \mathcal{A} \mapsto \mathcal{P}_f(X)$, denoted by $\mu \ll \nu$, if $\nu(A) = \{0\}$ implies $\mu(A) = \{0\}$, for every $A \in \mathcal{A}$.

If $\mu : \mathcal{A} \mapsto \mathcal{P}_f(X)$ is a multivalued set function, then we recall from [3] the following:

**Definition 1.3.** I) $\mu$ is said to be a multimeasure if:
a) \( \mu(\emptyset) = \{0\} \) and
b) \( \mu(A \cup B) = \mu(A) + \mu(B) \), for every \( A, B \in \mathcal{A} \), with \( A \cap B = \emptyset \), that is, \( \mu \) is finite additive.

II) \( \mu \) is said to be a multisubmeasure if:

a) \( \mu(\emptyset) = \{0\} \),

b) \( \mu(A \cup B) \subseteq \mu(A) + \mu(B) \), for every \( A, B \in \mathcal{A} \), with \( A \cap B = \emptyset \) and

c) \( \mu(A) \subseteq \mu(B) \), for every \( A, B \in \mathcal{A} \), with \( A \subseteq B \) (that is, \( \mu \) is monotone increasing on \( \mathcal{A} \)).

It is easy to observe that the last condition b) is equivalent to the condition b′) \( \mu(A \cup B) \subseteq \mu(A) + \mu(B) \), for every \( A, B \in \mathcal{A} \).

All over this paper we assume that \( \mu : \mathcal{A} \to P_{bf}(X) \) is a multisubmeasure. Let us also consider the following set functions associated to \( \mu \):

\[ \overline{\mu} \] defined by

\[ \overline{\mu}(A) = \sup \left\{ \sum_{i=1}^{n} |\mu(A_i)| \right\}, \text{ for every } A \in \mathcal{A}, \]

where supremum is extended over all finite partitions \( \{A_i\}_{i=1}^{n} \) of \( A \). \( \overline{\mu} \) is said to be the variation of \( \mu \);

\( \tilde{\mu} \) defined by

\[ \tilde{\mu}(A) = |\mu(A)|, \text{ for every } A \in \mathcal{A} \]

and \( \tilde{\mu} \) defined by

\[ \tilde{\mu}(A) = \inf \{\overline{\mu}(B); A \subseteq B, B \in \mathcal{A}\}, \text{ for every } A \subseteq T. \]

We have observed in [3] that \( \overline{\mu} \) is a finite additive set function on \( \mathcal{A} \) and \( \tilde{\mu} \) is a submeasure in Drewnowski’s sense [2] on \( \mathcal{A} \). We also note that \( \tilde{\mu}(A) = \overline{\mu}(A) \), for every \( A \in \mathcal{A} \).

**Definition 1.4.** We say that a property \( (P) \) holds \( \mu \)-almost everywhere if the property \( (P) \) is valid on \( T \setminus A \), with \( \tilde{\mu}(A) = 0 \).

**Definition 1.5.** We say that a sequence of functions \( (f_n)_{n}, \) where \( f_n : T \to R \) for every \( n \in N \), is convergent in submeasure to \( f \) (denoted by \( f_n \xrightarrow{\mu} f \)) if for every \( \delta > 0 \), \( \lim_{n \to \infty} \tilde{\mu}(B_n(\delta)) = 0 \), where

\[ B_n(\delta) = \{t \in T; |f_n(t) - f(t)| \geq \delta\}. \]
Definition 1.6. A multisubmeasure $\mu : \mathcal{A} \rightarrow \mathcal{P}_f(X)$ is said to be of finite variation if $\mu(T) < \infty$.

We also recall the following (see [1], [3] and [7]):

If, particularly, $T$ is a locally compact Hausdorff space, let $\mathcal{B}$ (respectively, $\mathcal{B}'$) be the borelian $\delta$-ring (respectively, $\sigma$-ring) generated by the compact subsets of $T$ and $\mathcal{B}_0$ (respectively $\mathcal{B}'_0$) be the Baire $\delta$-ring (respectively, $\sigma$-ring) generated by the compact subsets of $T$ which are $G_\delta$ (that is, countable intersection of open sets).

If $T$ is a compact space, then $\mathcal{B}'$ becomes a $\sigma$-algebra, and if $T$ is a metrizable compact space, then $T$ is $G_\delta$, so, $\mathcal{B}'_0$ becomes a $\sigma$-algebra, too.

In the assumption for $T$ to be a locally compact space, we have introduced in [3] the following notions:

Definition 1.7. i) A set $A \in \mathcal{A}$ is said to be $R'_l$-regular (with respect to $\mu$) if for every $\varepsilon > 0$, there exists a compact set $K \in \mathcal{A}, K \subset A$ such that $|\mu(B)| < \varepsilon$, for every $B \in \mathcal{A}, B \subset A \setminus K$.

ii) A set $A \in \mathcal{A}$ is said to be $R'_r$-regular (with respect to $\mu$) if for every $\varepsilon > 0$, there exists an open set $D \in \mathcal{A}, A \subset D$ such that $|\mu(B)| < \varepsilon$, for every $B \in \mathcal{A}, B \subset D \setminus A$.

iii) A set $A \in \mathcal{A}$ is said to be $R_l$-regular (with respect to $\mu$) if for every $\varepsilon > 0$, there exists a compact set $K \in \mathcal{A}, K \subset A$ such that $h(\mu(B), \mu(A)) < \varepsilon$, for every $B \in \mathcal{A}, K \subset B \subset A$.

iv) A set $A \in \mathcal{A}$ is said to be $R_r$-regular (with respect to $\mu$) if for every $\varepsilon > 0$, there exists an open set $D \in \mathcal{A}, A \subset D$ such that $h(\mu(B), \mu(A)) < \varepsilon$, for every $B \in \mathcal{A}, A \subset B \subset D$.

Definition 1.8. $\mu$ is said to be $R'_l$-regular ($R'_r$-regular, $R_l$-regular, $R_r$-regular, respectively) on $\mathcal{A}$ if every set $A \in \mathcal{A}$ is $R'_l$-regular ($R'_r$-regular, $R_l$-regular, $R_r$-regular, respectively).

We remind from [3] and [4] the following:

Theorem 1.9. i) $R'_l$-regularity is equivalent to $R_l$-regularity on $\mathcal{A}$ (for multisubmeasures of finite variation);

ii) $R'_l$-regularity is equivalent to o-continuity on $\mathcal{B}'_0$;
iii) $R'_l$-regularity implies $o$-continuity on $\mathcal{B}'$;

iv) $R'_l$-regularity is equivalent to $R'_r$-regularity on $\mathcal{B}$.

2. The Gould type integral with respect to a multisubmeasure. In the sequel, without any special assumptions, $T$ will be an abstract nonvoid set, $\mu : \mathcal{A} \rightarrow \mathcal{P}_{bf}(X)$ a multisubmeasure of finite variation and $f : T \rightarrow R$ a real valued, bounded function.

Definition 2.1. I) $f$ is said to be $\tilde{\mu}$-totally-measurable on $(T, \mathcal{A}, \mu)$ if for every $\varepsilon > 0$ there exists a partition $P_\varepsilon = \{A_i\}_{i=0,n}$ of $T$ such that:

i) $\tilde{\mu}(A_0) < \varepsilon$ and

ii) $\sup_{t,s \in A_i} |f(t) - f(s)| = \text{osc}(f, A_i) < \varepsilon$, for every $i = 1, n$.

II) $f$ is said to be $\tilde{\mu}$-totally-measurable on $B \in \mathcal{A}$ if its restriction $f|_B$ of $f$ to $B$ is $\tilde{\mu}$-totally measurable on $(B, \mathcal{A}_B, \mu_B)$, where $\mu_B = \mu|_{A_B}$ and $A_B = \{A \cap B; A \in \mathcal{A}\}$.

Let $\sigma(P) = \sum_{i=1}^n f(t_i)\mu(A_i)$, for every partition $P = \{A_i\}_{i=1,n}$ of $T$ and every $t_i \in A_i, i = 1, n$.

Definition 2.2. I) $f$ is said to be $\mu$-integrable on $T$ if the net $(\sigma(P))_{P \in (P, \leq)}$ is convergent in $(\mathcal{P}_{bf}(X), h)$ (where $P$ is the set of all partitions of $T$ and $\leq$ the order relation on $P$ given in the definition 1.1 II), for every choice of the points $t_i \in A_i$; its limit is called the integral of $f$ on $T$ with respect to the multisubmeasure $\mu$, denoted by $\int_T f \, d\mu$.

Hence, $f$ is $\mu$-integrable on $T$ if there exists a set $I \in \mathcal{P}_{bf}(X)$ such that for every $\varepsilon > 0$ there exists a partition $P_\varepsilon$ of $T$ so that for every other partition $P = \{A_i\}_{i=1,n}$ with $P \geq P_\varepsilon$ and every choice of points $t_i \in A_i, i = 1, n, we have

(5) $h(\sigma(P), I) < \varepsilon$.

II) If $B \in \mathcal{A}$, $f$ is said to be $\mu$-integrable on $B$ if the restriction $f|_B$ of $f$ to $B$ is $\mu$-integrable on $(B, \mathcal{A}_B, \mu_B)$.

Obviously, if there exists, the integral is unique.
Let us note that, all over this paper, if we deal with a $\mathcal{P}_{kc}(X)$-valued multisubmeasure, then the Minkowski addition "$\cdot$" changes in fact into "$+$".

In the sequel, we shall use the following results that we have established in [5]:

**Theorem 2.3.** If $f$ is $\mu$-integrable on $T$ and $\alpha \in R$, then $\alpha f$ is $\mu$-integrable on $T$ and

$$\int_T \alpha f d\mu = \alpha \int_T f d\mu. \quad (6)$$

**Theorem 2.4.** Suppose that $\mu : A \mapsto \mathcal{P}_{kc}(X)$ and $f, g : T \mapsto R$ are two bounded, $\mu$-integrable functions on $T$ so that $f(t) \cdot g(t) \geq 0$, for every $t \in T$. Then $f + g$ is $\mu$-integrable on $T$ and

$$\int_T (f + g) d\mu = \int_T f d\mu + \int_T g d\mu. \quad (7)$$

**Theorem 2.5.** Let $f, g : T \mapsto R$ be two bounded, $\mu$-integrable functions. Then:

i)

$$h \left( \int_T f d\mu, \int_T g d\mu \right) \leq \sup_{t \in T} |f(t) - g(t)| \cdot \overline{\Pi}(T) \quad (8)$$

ii)

$$\left| \int_T f d\mu \right| \leq \sup_{t \in T} |f(t)| \cdot \overline{\Pi}(T). \quad (9)$$

In the sequel, we remind from [5] an important class of functions which are $\mu$-integrable, where $\mu : A \mapsto \mathcal{P}_{kc}(R)$ is the multisubmeasure induced by a submeasure $\nu : A \mapsto R_+$ of finite variation, defined by:

$$\mu(A) = [0, \nu(A)], \text{ for every } A \in \mathcal{A}. \quad \text{for every } A \in \mathcal{A}.$$

**Theorem 2.6.** Let $\mu : A \mapsto \mathcal{P}_{kc}(R)$ be the multisubmeasure induced by a submeasure $\nu : A \mapsto R_+$ of finite variation and $f : T \mapsto R_+$ be a
bounded, \( \tilde{\mu} \)-totally-measurable function on \( T \). Then \( f \) is \( \mu \)-integrable on \( T \) and, moreover,

\[
\int_T f \, d\mu = [0, \int_T f \, d\nu]
\]

(\( \int_T f \, d\nu \) represents the Gould integral [6] of \( f \) with respect to the variation \( \nu \) of \( \nu \), which is a finite additive set function on \( A \)).

**Theorem 2.7.** Let \( B, C \in A \), with \( B \cap C = \emptyset \). If \( f \) is \( \mu \)-integrable on \( B \) and \( C \), then \( f \) is \( \mu \)-integrable on \( B \cup C \), and, moreover,

\[
\int_{B \cup C} f \, d\mu = \int_B f \, d\mu + \int_C f \, d\mu.
\]

**Theorem 2.8.** Let \( B, C \in A \), with \( B \subseteq C \). Then

\[
\int_B f \, d\mu \subseteq \int_C f \, d\mu
\]
supposing the existence of both integrals.

**Theorem 2.9.** If \( \mu : A \to \mathcal{P}_{kc}(X) \), \( f : T \to R \) is \( \mu \)-integrable on \( T \) and \( B \in A \) is an arbitrarily set, then \( f \) is \( \mu \)-integrable on \( B \).

**Corollary 2.10.** If \( \mu : A \to \mathcal{P}_{kc}(X) \) and \( f \) is \( \mu \)-integrable on \( T \), then:

i) \( M : A \to \mathcal{P}_{kc}(X) \), defined by \( M(A) = \int_A f \, d\mu \), for every \( A \in A \), is a monotone multimeasure;

ii) \( M \ll \mu \).

We establish in the following some properties of the integral.

**Theorem 2.11.** Let \( \mu : A \to \mathcal{P}_{kc}(X) \). Then:

i) If \( \mu \) is \( o \)-continuous (increasing convergent, respectively) on \( A \), then \( M \) is \( o \)-continuous (increasing convergent, respectively) on \( A \);

ii) If \( \mu \) is \( h \)-\( \sigma \)-subadditive, then \( M \) is a \( h \)-multimeasure, that is, \( M(\emptyset) = 0 \) and \( \lim_{n \to \infty} h(M(A), \sum_{k=1}^{\infty} M(A_k)) = 0 \), for every disjoint sequence of sets \((A_n)_{n \in N^*} \subset A \), with \( A = \bigcup_{n=1}^{\infty} A_n \in A \).
Proof. i) Let us suppose that $\mu$ is o-continuous on $\mathcal{A}$. Then (see [3]), $\overline{\mu}$ is o-continuous on $\mathcal{A}$. Let $(A_n)_n \subset \mathcal{A}$, $A_n \not\subseteq \emptyset$. From the o-continuity of $\overline{\mu}$ on $\mathcal{A}$, we get that for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $\overline{\mu}(A_n) < \frac{\varepsilon}{\alpha}$, for every $n \geq n_0$, where $\alpha = \sup_{t \in T}|f(t)|$. The inequality $|M(A_n)| \leq \alpha \cdot \overline{\mu}(A_n)$, for every $n \in N^*$ yields $|M(A_n)| < \varepsilon$, for every $n \geq n_0$, that is, $M$ is o-continuous on $\mathcal{A}$.

We suppose now that $\mu$ is increasing convergent on $\mathcal{A}$. Let $(A_n)_n \subset \mathcal{A}$ be such that $A_n \not\supseteq A \in \mathcal{A}$.

We shall prove that for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ so that $h(M(A_n), M(A)) < \varepsilon$, for every $n \geq n_0$. From theorem 2.5 ii) we get that:

$$h(M(A_n), M(A)) = h\left(\int_{A_n} f \, d\mu, \int_{A_n} f \, d\mu\right)$$

$$= h\left(\int_{A_n} f \, d\mu, \int_{A_n} f \, d\mu + \int_{A \setminus A_n} f \, d\mu\right)$$

$$\leq |\int_{A \setminus A_n} f \, d\mu| \leq \alpha \cdot \overline{\mu}(A \setminus A_n) = \alpha \cdot (\overline{\mu}(A) - \overline{\mu}(A_n)).$$

We prove in the sequel that there exists $n_0(\varepsilon) \in \mathbb{N}$ so that $\overline{\mu}(A) - \overline{\mu}(A_n) < \frac{\varepsilon}{\alpha}$, for every $n \geq n_0$.

Indeed, if $\{B_i\}_{i=1}^{1, m} \subset \mathcal{A}$ is an arbitrary partition of $A$, then

$$\sum_{i=1}^{m} |\mu(B_i)| \leq \sum_{i=1}^{m} |\mu(B_i \cap A_n)| + \sum_{i=1}^{m} h(\mu(B_i), \mu(B_i \cap A_n)),$$

for every $n \in N^*$.

Since $\{B_i \cap A_n\}_{i=1, m}$ is a partition of $A_n$, for every $n \in N^*$, we have

$$\sum_{i=1}^{m} |\mu(B_i)| \leq \overline{\mu}(A_n) + \sum_{i=1}^{m} h(\mu(B_i), \mu(B_i \cap A_n)).$$

But $\mu$ is increasing convergent and $B_i \cap A_n \not\subseteq B_i \cap A = B_i$, for every $i = 1, m$; therefore, for every $i = 1, m$, there exists $n_0(\varepsilon) \in \mathbb{N}$ so that for every $n \geq n_0$, $h(\mu(B_i), \mu(B_i \cap A_n)) < \frac{\varepsilon}{2^i \alpha}$. Consequently,

$$\sum_{i=1}^{m} |\mu(B_i)| \leq \overline{\mu}(A_n) + \sum_{i=1}^{m} \frac{\varepsilon}{2^i \alpha} < \overline{\mu}(A_n) + \frac{\varepsilon}{\alpha},$$

for every $n \geq n_0 = \max\{n_0^i\}_{i=1, m}$.

Taking the supremum on the left side we get that $\overline{\mu}(A) \leq \overline{\mu}(A_n) + \frac{\varepsilon}{\alpha}$ and, finally, $h(M(A_n), M(A)) < \alpha \cdot \frac{\varepsilon}{\alpha} = \varepsilon$, for every $n \geq n_0$, as claimed.
ii) Let \((A_n)_n \subset \mathcal{A}\) be a disjoint sequence of sets, with \(\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}\). Because \(\mu\) is \(h-\sigma\)-subadditive, then, equivalently, it is \(o\)-continuous (see [3]), hence \(M\) is \(o\)-continuous, too. Since \(B_n = \bigcup_{k=n+1}^{\infty} A_n \setminus \emptyset\) and \((B_n)_{n \in \mathbb{N}^*} \subset \mathcal{A}\), then for every \(\varepsilon > 0\), there exists \(n_0(\varepsilon) \in \mathbb{N}\) such that \(|M(B_n)| < \varepsilon\), for every \(n \geq n_0\).

Because \(M\) is finite additive we get that \(M(A) = \sum_{k=1}^{n} M(A_k) + M(B_n)\), which implies

\[
h(M(A), \sum_{k=1}^{n} M(A_k)) = h(\sum_{k=1}^{n} M(A_k) + M(B_n), \sum_{k=1}^{n} M(A_k)) = |M(B_n)| < \varepsilon, \text{ for every } n \geq n_0.
\]

This means that \(M\) is indeed a \(h\)-multimeasure, since, obviously, \(M(\emptyset) = \{0\}\). □

**Theorem 2.12.** Let \(T\) be a locally compact Hausdorff space and \(\mu : \mathcal{A} \rightarrow \mathcal{P}_{kc}(X)\). Then:

i) If \(\mu\) is \(R_{l}^{r}\)-regular on \(\mathcal{A}\), then \(M\) is \(R_{l}^{r}\)-regular on \(\mathcal{A}\). Also, if \(\inf_{t \in T} f(t) = m > 0\) and \(M\) is \(R_{l}^{r}\)-regular on \(\mathcal{A}\), then \(\mu\) is \(R_{l}^{r}\)-regular on \(\mathcal{A}\);

ii) If \(\inf_{t \in T} f(t) = m > 0\), then \(\mu\) is \(R_{l}\)-regular on \(\mathcal{A}\) if and only if \(M\) is \(R_{l}\)-regular on \(\mathcal{A}\);

iii) If \(T\) is a compact space and \(\inf_{t \in T} f(t) = m > 0\), then \(\mu\) is \(R_{r}^{r}\)-regular on \(\mathcal{B}\) if and only if \(M\) is \(R_{r}^{r}\)-regular on \(\mathcal{B}\);

iv) If \(T\) is a compact space, \(\inf_{t \in T} f(t) = m > 0\) and \(M\) is \(R_{r}\)-regular on \(\mathcal{B}\), then \(\mu\) is \(R_{r}\)-regular on \(\mathcal{B}\).

**Proof.** i) If \(\mu\) is \(R_{l}^{r}\)-regular on \(\mathcal{A}\) then, equivalently, \(\overline{\mu}\) is \(R_{l}^{r}\)-regular on \(\mathcal{A}\) (see [3]). One can easily get then that \(M\) is \(R_{l}^{r}\)-regular on \(\mathcal{A}\), using the inequality \(|M(A)| \leq \alpha \cdot \overline{\mu}(A)\), which holds for every \(A \in \mathcal{A}\).

In what concerns the converse, we suppose that, additionally, \(\inf_{t \in T} f(t) = m > 0\). Let \(A \in \mathcal{A}\). Because \(M\) is \(R_{l}^{r}\)-regular in \(\mathcal{A}\), then for every \(\varepsilon > 0\), there exists a compact set \(K \subset \mathcal{A}\) so that \(K \subset A\) and \(|M(B)| < \frac{\varepsilon}{2}\), for every \(B \in \mathcal{A}\), with \(B \subset A \setminus K\).
But $f$ is $\mu$-integrable on $T$, so it is $\mu$-integrable on an arbitrary, fixed set $B \in \mathcal{A}$, with $B \subset A \setminus K$. Consequently, there is a partition $P_\varepsilon = \{B_i\}_{i=1}^{p} \subset \mathcal{P}_B$ such that

$$h(\int_B f \, d\mu, \sum_{i=1}^{p} f(t_i)\mu(B_i)) < \frac{\varepsilon}{2},$$

for every $t_i \in B_i$, $i = 1, p$.

Therefore, $|\sum_{i=1}^{p} f(t_i)\mu(B_i)| \leq h(\int_B f \, d\mu, \sum_{i=1}^{p} f(t_i)\mu(B_i)) + |M(B)| < \varepsilon$.

But, on the other hand, $m \mu(B) \subseteq m \sum_{i=1}^{p} \mu(B_i) \subseteq \sum_{i=1}^{p} f(t_i)\mu(B_i)$, so,

$$|m \mu(B)| \leq |\sum_{i=1}^{p} f(t_i)\mu(B_i)|,$$

which implies that $|m \mu(B)| < \varepsilon$, that is, $|\mu(B)| < \frac{\varepsilon}{m}$. This means $\mu$ is $R'_r$-regular in $A$.

So, if $\inf_{t \in T} f(t) = m > 0$, then $\mu$ is $R'_r$-regular on $A$ if and only if $M$ is $R'_r$-regular on $A$.

ii) We use the fact that $M$ is also a multisubmeasure on $A$ and $R'_r$-regularity is equivalent to $R'_r$-regularity for multisubmeasures of finite variation.

iii) Since $T$ is a compact space, then $A = \mathcal{B}$ is an algebra. Because on $\mathcal{B}$ $R'_r$-regularity is equivalent to $R'_r$-regularity (see [3]), we have the conclusion.

iv) Let $A \in \mathcal{B}$ and $\varepsilon > 0$.

Because $M$ is $R_r$-regular in $A$, then there exists an open set $D \in \mathcal{B}$, $A \subset D$, so that $h(M(A), M(B)) < \varepsilon$, for every $B \in \mathcal{B}$, with $A \subset B \subset D$.

But $h(M(A), M(B)) = h(M(A), M(A) + M(B\setminus A)) = |M(B\setminus A)|$ since $M : \mathcal{B} \to \mathcal{P}_{kc}(X)$. Therefore, $|M(B\setminus A)| < \varepsilon$, for every $B \in \mathcal{B}$, with $A \subset B \subset D$.

Now, let $B \in \mathcal{B}$, with $B \subset D\setminus A$. Then $B = B\setminus A = (A \cup B)\setminus A$ and, since $A \subset A \cup B \subset D$, it follows that $|M(B)| < \varepsilon$, that is, $M$ is $R'_r$-regular on $B$. From iii) we get that $\mu$ is $R'_r$-regular, hence it is $R_r$-regular on $B$ (see [3]), as claimed.

We point out now other properties of the integral.

**Theorem 2.13.** Let $\mu : A \to \mathcal{P}_{bf}(X)$ and $f, g : T \to R$ be two bounded functions on $T$, so that:

i) $f$ is $\mu$-integrable on $T$ and
ii) \( f = g \) \( \mu \)-a.p.t.

Then \( g \) is \( \mu \)-integrable on \( T \) and \( \int_T f \, d\mu = \int_T g \, d\mu \).

**Proof.** Let \( \varepsilon > 0 \) be arbitrarily. Since \( f \) is \( \mu \)-integrable on \( T \), there exists a partition \( P_\varepsilon = \{A_i\}_{i=1}^n \in \mathcal{P}_T \) so that

\[
h(\sigma(P), \int_T f \, d\mu) < \frac{\varepsilon}{2}, \text{ for every } P \in \mathcal{P}_T, \text{ with } P \geq P_\varepsilon.
\]

Let \( E \subset T \) be such that \( f = g \) on \( T \setminus E \) and \( \tilde{\mu}(E) = 0 \). From the definition of \( \tilde{\mu} \), there exists a set \( A \in \mathcal{A} \) so that \( E \subseteq A \) and \( \tilde{\mu}(A) < \frac{\varepsilon}{4M} \), where

\[
M = \max \{\sup_{t \in T} |f(t)|, \sup_{t \in T} |g(t)|\}.
\]

We consider the partitions \( P_0 = \{A, T \setminus A\} \in \mathcal{P}_T \) and \( P_0 \land P_\varepsilon = \{A \cap A_i, A_i \setminus A\}_{i=1}^{1,m} \in \mathcal{P}_T \). Let also \( P = \{B_j\}_{j=1}^{1,m} \in \mathcal{P}_T \), with \( P \geq P_0 \land P_\varepsilon \) and \( t_j \in B_j, j = 1, m \) be arbitrarily.

Because \( P \geq P_0 \land P_\varepsilon \), then for every \( j = 1, m \), there exists \( i_j = 1, n \) with \( B_j \subset A \cap A_{i_j} \) or \( B_j \subset A_{i_j} \setminus A \). Without any loss of generality, we suppose that \( B_j \subset A \cap A_{i_j} \), for every \( j = 1, p \) and \( B_j \subset A_{i_j} \setminus A \), for every \( j = p+1, m \) (we may have only one of these situations).

We shall prove that \( h(\int_T f \, d\mu, \sum_{j=1}^{m} g(t_j) \mu(B_j)) < \varepsilon \) (consequently, \( g \) is \( \mu \)-integrable on \( T \) and \( \int_T f \, d\mu = \int_T g \, d\mu \)).

Indeed,

\[
h(\int_T f \, d\mu, \sum_{j=1}^{m} g(t_j) \mu(B_j)) \leq h(\int_T f \, d\mu, \sum_{j=1}^{m} f(t_j) \mu(B_j))
\]

\[
+ h(\sum_{j=1}^{m} f(t_j) \mu(B_j), \sum_{j=1}^{m} g(t_j) \mu(B_j))
\]

\[
< \frac{\varepsilon}{2} + \sum_{j=1}^{m} |f(t_j) - g(t_j)| \cdot |\mu(B_j)|.
\]

Because \( B_j \subset A \cap A_{i_j} \) it follows that \( |\mu(B_j)| \leq |\mu(A \cap A_{i_j})| \), for every \( j = 1, p \).
Similarly, \( |\mu(B_j)| \leq |\mu(A_{ij} \setminus A)| \), for every \( j = p + 1, m \); therefore,
\[
h(\int_T f \, d\mu, \sum_{j=1}^m g(t_j)\mu(B_j)) < \frac{\varepsilon}{2} + \sum_{j=1}^p |f(t_j) - g(t_j)| \cdot |\mu(A \cap A_{ij})|
\]
\[
+ \sum_{j=p+1}^m |f(t_j) - g(t_j)| \cdot |\mu(A_{ij} \setminus A)|.
\]

But \( f = g \) on \( T \setminus A \), so \( f(t_j) = g(t_j) \), for every \( t_j \in A_{ij} \setminus A \), with \( j = p + 1, m \). This implies that
\[
h(\int_T f \, d\mu, \sum_{j=1}^m g(t_j)\mu(B_j)) < \frac{\varepsilon}{2} + \sum_{j=1}^p |f(t_j) - g(t_j)| \cdot |\mu(A \cap A_{ij})|
\]
\[
\leq \frac{\varepsilon}{2} + 2M \cdot \sum_{j=1}^p |\mu(A \cap A_{ij})| \leq \frac{\varepsilon}{2} + 2M \cdot m(A) < \varepsilon.
\]

The proof is thus finished. \( \square \)

**Remark 2.14.** i) If \( \mu : A \to \mathcal{P}_{bf}(X) \), then \( \mu(A) \subseteq \int_A d\mu \), supposing the existence of the integral. If \( \mu : A \to \mathcal{P}_{kc}(X) \) and \( m = \inf_{t \in T} f(t) > 0 \), then
\( m \cdot \mu(A) \subseteq \int_A f d\mu \).

Indeed, for every \( \varepsilon > 0 \), there exists \( P_\varepsilon = \{A_i\}_{i=1}^n \in \mathcal{P}_A \) so that
\[
h(\sigma(P), \int_A d\mu) < \varepsilon, \text{ for every } P \in \mathcal{P}_A, \text{ with } P \geq P_\varepsilon.
\]

Particularly, \( h(\sum_{i=1}^n \mu(A_i), \int_A d\mu) < \varepsilon \).

Therefore,
\[
e(\mu(A), \int_A d\mu) \leq e(\mu(A), \sum_{i=1}^n \mu(A_i)) + e(\sum_{i=1}^n \mu(A_i), \int_A d\mu)
\]
\[
= e(\sum_{i=1}^n \mu(A_i), \int_A d\mu) < \varepsilon,
\]

for every \( \varepsilon > 0 \), hence \( \mu(A) \subseteq \int_A d\mu \).
In what concerns the second statement, we have
\[
e(m \cdot \mu(A), \int_A f d\mu) \\
\leq e(m \cdot \mu(A), \sum_{i=1}^n f(t_i)\mu(A_i)) + e(\sum_{i=1}^n f(t_i)\mu(A_i), \int_A f d\mu) \\
< \varepsilon + e(m \cdot \mu(A), m \sum_{i=1}^n \mu(A_i)) + e(\sum_{i=1}^n \mu(A_i), \sum_{i=1}^n f(t_i)\mu(A_i)) = \varepsilon,
\]
for every \(\varepsilon > 0\),
which completes the proof.

ii) On the other hand, if we deal with the Gould integral [6] with respect to a finite additive set function \(m : A \rightarrow X\), \(X\) being a Banach space, we have \(\int_A dm = m(A)\).

iii) Although \(\mu(A) \subseteq \int_A d\mu := M(A)\), we observe that \(\mu(A) = M(A)\).

Indeed, because \(\mu(A) \subseteq \int_A d\mu\), then \(|\mu(A)| \leq |\int_A d\mu|\), hence \(\mu(A) \leq M(A)\).

In order to prove \( \geq \), let \(\{B_i\}_{i=1}^n \in \mathcal{P}_A\) be an arbitrary partition. We shall demonstrate that \(\sum_{i=1}^n |M(B_i)| \leq \mu(A)\).

Since for every \(i = 1, \ldots, n\) and every \(\varepsilon > 0\) there exists \(P_{\varepsilon} = \{B_j^i\}_{j=1}^{q_i} \in \mathcal{P}_{B_i}\) so that for every \(P \geq P_{\varepsilon}\), with \(P \in \mathcal{P}_{B_i}\), we have \(h(\sigma(P), \int_{B_i} d\mu) < \frac{\varepsilon}{2\varepsilon}\), we get that, particularly, \(h(\sum_{j=1}^{q_i} \mu(B_j^i), \int_{B_i} d\mu) < \frac{\varepsilon}{2}\), for every \(i = 1, \ldots, n\).

Therefore, \(\sum_{i=1}^n |\int_{B_i} d\mu| \leq \sum_{i=1}^n h(\sum_{j=1}^{q_i} \mu(B_j^i), \int_{B_i} d\mu) + \sum_{i=1}^n |\sum_{j=1}^{q_i} \mu(B_j^i)| < \sum_{i=1}^n \frac{\varepsilon}{2\varepsilon} + \sum_{i=1}^n \sum_{j=1}^{q_i} |\mu(B_j^i)| < \varepsilon + \sum_{i=1}^n \mu(B_i) = \varepsilon + \mu(A)\), for every \(\varepsilon > 0\), which implies that \(\sum_{i=1}^n |M(B_i)| \leq \mu(A)\), hence \(\mu(A) \geq M(A)\), as claimed.

Of course, the existence of all the integrals that appear here can be assured if we suppose that, for instance, \(\mu : A \mapsto \mathcal{P}_{kc}(X)\) and \(f\) is \(\mu\)-integrable on \(T\).

3. Sequences of \(\mu\)-integrable functions. It is easy to prove the following:
Theorem 3.1. Let $f, g : T \mapsto R$ be two bounded and $\tilde{\mu}$-totally-measurable functions. Then:

i) $f + g$ is $\tilde{\mu}$-totally-measurable;

ii) $\lambda f$ is $\tilde{\mu}$-totally-measurable, for every $\lambda \in R$;

iii) $f^2$ and $fg$ are $\tilde{\mu}$-totally-measurable.

Theorem 3.2. We suppose that $(f_n)_n : T \mapsto R$ is a sequence of $\tilde{\mu}$-totally-measurable, bounded functions so that $f_n \xrightarrow{\mu} f$, where $f : T \mapsto R$ is a bounded function. Then $f$ is $\tilde{\mu}$-totally-measurable.

Proof. Let $\varepsilon > 0$. For every $n \in N$, $f_n$ is $\tilde{\mu}$-totally-measurable, hence, for every $n \in N$, there exists a partition of $T$, $P^n_\varepsilon = \{A^n_i\}_{i=0,m_n}$, so that

$\tilde{\mu}(A^n_0) < \frac{\varepsilon}{2^2}$ and $\sup_{t,s \in A^n_i} |f_n(t) - f_n(s)| < \frac{\varepsilon}{2^2}$, for every $i = 1, m_n$.

Since $f_n \xrightarrow{\mu} f$, we get that for every $\delta > 0$, $\lim \tilde{\mu}(B_n(\delta)) = 0$, where $B_n(\delta) = \{t \in T, |f_n(t) - f(t)| < \delta\}$. Then for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \in N$ so that $\tilde{\mu}(B_{n_0}(\frac{\varepsilon}{3})) < \frac{\varepsilon}{2}$. From the definition of $\tilde{\mu}$, there is a set $C_{n_0} \in A$, so that $B_{n_0}(\frac{\varepsilon}{3}) \subseteq C_{n_0}$ and $\tilde{\mu}(C_{n_0}) = \tilde{\mu}(C_{n_0}) < \frac{\varepsilon}{2}$.

Let then be the partition

$P_\varepsilon = \{C_{n_0} \cup A_0^{n_0}, A_1^{n_0} \cap cC_{n_0}, A_2^{n_0} \cap cC_{n_0}, \ldots, A_{m_{n_0}}^{n_0} \cap cC_{n_0}\}$

of $T$.

Since $\tilde{\mu}(C_{n_0} \cup A_0^{n_0}) = \tilde{\mu}(C_{n_0} \cup A_0^{n_0}) \leq \tilde{\mu}(C_{n_0}) + \tilde{\mu}(A_0^{n_0}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} \leq \varepsilon$, it only remains to prove that $\sup_{t,s \in C_{n_0} \cap A_1^{n_0}} |f(t) - f(s)| < \varepsilon$, for every $i = 1, m_{n_0}$.

Indeed,

$\sup_{t,s \in C_{n_0} \cap A_1^{n_0}} |f(t) - f(s)| \leq \sup_{t \in C_{n_0}} |f(t) - f_{n_0}(t)| + \sup_{t,s \in A_{n_0}^{n_0}} |f_{n_0}(t) - f_{n_0}(s)|$

$+ \sup_{s \in C_{n_0}} |f_{n_0}(s) - f(s)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3 \cdot 2^2} + \frac{\varepsilon}{3} \leq \varepsilon$,

for every $i = 1, m_{n_0}$, as claimed. \qed

Theorem 3.3. Let $\mu : A \mapsto \mathcal{P}_{kc}(X)$ and $(f_n)_n : T \mapsto \mathbb{R}$ be a uniformly bounded sequence of $\mu$-integrable functions so that $f_n \xrightarrow{\mu} f$, where $f : T \mapsto \mathbb{R}$.
$R$ is a bounded function. Then $f$ is $\mu$-integrable on $A$ and

\[ \lim_{n \to \infty} \int_A f_n \, d\mu = \int_A f \, d\mu \text{ (with respect to } h), \text{ for every } A \in \mathcal{A}. \]

**Proof.** Let $M' = \bar{\mu}(T), M_1 = \sup_{t \in T} |f(t)|, M_2 = \sup_{t \in T, n \in \mathbb{N}} |f_n(t)| \text{ and } M = \max(M_1, M_2).

Since $f_n \xrightarrow{\mu} f$, we get that for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ so that $\bar{\mu}(B_n(\frac{\varepsilon}{4M})) < \frac{\varepsilon}{4M}$, for every $n \geq n_0$. Particularly, $\bar{\mu}(B_{n_0}(\frac{\varepsilon}{4M})) < \frac{\varepsilon}{4M}$. Consequently, from the definition of $\bar{\mu}$, there is a set $C_{n_0} \in \mathcal{A}$ such that $B_{n_0}(\frac{\varepsilon}{4M}) \subseteq C_{n_0}$ and $\bar{\mu}(C_{n_0}) = \bar{\mu}(C_{n_0}) < \frac{\varepsilon}{4M}$.

Let $A \in \mathcal{A}$ be arbitrarily, but fixed. We prove first that $f$ is $\mu$-integrable on $C_{n_0}$ (it will also be $\mu$-integrable on $C_{n_0} \cap A$, see theorem 2.9). Indeed, for every $\varepsilon > 0$, there exists the partition $P_\varepsilon = \{C_{n_0}\} \in \mathcal{P}_{C_{n_0}}$ so that, for every other partition $P' = \{D_l\}_{l=1}^p \in \mathcal{P}_{C_{n_0}}$ with $P' \succeq P_\varepsilon$ and for every $t_l \in D_l$, $l = \overline{1, p}$ and every $c \in C_{n_0}$, we have:

\[
\begin{align*}
&h(\sigma(P'), f(c)\mu(C_{n_0})) \\
= &\quad h(\sum_{l=1}^p f(t_l)\mu(D_l), f(c)\mu(C_{n_0})) \leq | \sum_{l=1}^p f(t_l)\mu(D_l) | + |f(c)\mu(C_{n_0})| \\
\leq &\quad M_1 \sum_{l=1}^p |\mu(D_l)| + M_1 |\mu(C_{n_0})| \leq 2M_1 \bar{\mu}(C_{n_0}) < 2M_1 \frac{\varepsilon}{4M} \leq \varepsilon/2.
\end{align*}
\]

Similarly, for every partition $P'' \in \mathcal{P}_{C_{n_0}}$, with $P'' \succeq P_\varepsilon$, we get that $h(\sigma(P''), f(c)\mu(C_{n_0})) < \frac{\varepsilon}{2}$, hence $h(\sigma(P'), \sigma(P'')) < \varepsilon$.

This means $(\sigma(P))_{P \in \mathcal{P}_{C_{n_0}}}$ is a Cauchy, hence a convergent net in the complete metric space $\mathcal{P}_{kc}(X)$.

Therefore, $f$ is $\mu$-integrable on $C_{n_0}$.

We prove now that $f$ is $\mu$-integrable on $A$. Since $f$ is $\mu$-integrable on $A \cap C_{n_0}$, then, according to theorem 2.7, it is sufficient to prove that $f$ is $\mu$-integrable on $A \backslash C_{n_0}$.

Because $f_n$ is $\mu$-integrable on $T$, for every $n \in \mathbb{N}$, we get that, particularly, $f_n$ is $\mu$-integrable on $A \backslash C_{n_0}$; there exists then a partition $P'_{\varepsilon_0} = \{A_i\}_{i=1}^{\infty \sigma_0} \in \mathcal{P}_{A \backslash C_{n_0}}$ so that for every $P \in \mathcal{P}_{A \backslash C_{n_0}}$, with $P \succeq P'_{\varepsilon_0}$, $h(\sigma(P), \sigma(P'_{\varepsilon_0})) < \frac{\varepsilon}{2}$. 
Let $P = \{D_j\}_{j=1}^{l} \in \mathcal{P}_{A \setminus C_{n_0}}$ with $P \geq P^{n_0}_{\varepsilon}$ be arbitrarily, but fixed.

For every $t_j \in D_j$, $j = 1, \ldots, l$ and every $c_i \in A_i$, $i = 1, \ldots, m_{n_0}$, we have:

\[
\begin{align*}
    h(\sum_{j=1}^{l} f(t_j) \mu(D_j), \sum_{i=1}^{m_{n_0}} f(c_i) \mu(A_i)) &
    \leq h(\sum_{j=1}^{l} f(t_j) \mu(D_j), \sum_{j=1}^{l} f^{n_0}(t_j) \mu(D_j)) \\
    &+ h(\sum_{j=1}^{l} f^{n_0}(t_j) \mu(D_j), \sum_{i=1}^{m_{n_0}} f^{n_0}(c_i) \mu(A_i)) \\
    &+ h(\sum_{i=1}^{m_{n_0}} f^{n_0}(c_i) \mu(A_i), \sum_{i=1}^{m_{n_0}} f(c_i) \mu(A_i)) \\
    &\leq \sum_{j=1}^{l} |\mu(D_j)| \sup_{t_j \in D_j \subset A \setminus C_{n_0}} |f(t_j) - f^{n_0}(t_j)| \\
    &+ \varepsilon + \sum_{i=1}^{m_{n_0}} \mu(A_i) \sup_{c_i \in A_i \subset A \setminus C_{n_0}} |f(c_i) - f^{n_0}(c_i)| \\
    &\leq M' \varepsilon + \varepsilon + M' \varepsilon < \varepsilon,
\end{align*}
\]

hence $f$ is indeed $\mu$-integrable on $A \setminus C_{n_0}$.

In the sequel, we shall prove the equality (12). Since $f_n$ is $\mu$-integrable on $T$ for every $n \in \mathbb{N}$, from theorem 2.9 we get that $f_n$ is $\mu$-integrable on $A$, for every $n \in \mathbb{N}$. Hence there exists $\int_{A} f_n d\mu$ for every $n \in \mathbb{N}$ and $\int_{A} f d\mu$.

Let us use, the same as before, the sets $B_n(\frac{\varepsilon}{6M'})$, with $n \geq n_0$. From the definition of $\tilde{\mu}$ we get that for every $n \geq n_0$ there is a set $C_n \in \mathcal{A}$ such that $B_n(\frac{\varepsilon}{6M'}) \subseteq C_n$ and $\tilde{\mu}(C_n) = \pi(C_n) < \frac{\varepsilon}{4M}$. Consequently, for every $n \geq n_0$, we have:

\[
\begin{align*}
    h(\int_{A} f_n d\mu, \int_{A} f d\mu) &= h(\int_{A \setminus C_n} f_n d\mu, \int_{A \setminus C_n} f d\mu + \int_{A \cap C_n} f_n d\mu, \int_{A \setminus C_n} f d\mu + \int_{A \cap C_n} f d\mu) \\
    &\leq h(\int_{A \setminus C_n} f_n d\mu, \int_{A \setminus C_n} f d\mu) + h(\int_{A \cap C_n} f_n d\mu, \int_{A \cap C_n} f d\mu)
\end{align*}
\]
\[ \sup_{t \in A \setminus C_n} |f_n(t) - f(t)| \leq \mu(A \setminus C_n) + \sup_{t \in A \cap C_n} |f_n(t) - f(t)| \cdot \mu(A \cap C_n) \]
\[ < \frac{\varepsilon}{6M'} + 2M \cdot \mu(C_n) < \frac{\varepsilon}{2} + 2M \cdot \frac{\varepsilon}{4M} = \varepsilon. \]
This completes the proof. \( \square \)

**Remark 3.4.** Particularly, if \( \mu : A \to \mathcal{P}_{\kappa\mu}(R) \) is the multisubmeasure induced by a submeasure of finite variation \( \nu : A \to \mathbb{R}^+ \) and if \((f_n) : T \to R_+\) is a uniformly bounded sequence of \( \tilde{\mu}\)-totally-measurable functions such that \( f_n \xrightarrow{\mu} f \), where \( f : T \to R \) is a bounded function then

\[ \lim_{n \to \infty} \int_A f_n d\mu = \int_A f d\mu \quad \text{(with respect to } h) \], for every \( A \in \mathcal{A} \).

Indeed, from theorem 2.6 and theorem 2.9 we get that for every \( n \in N^* \), \( f_n \) is \( \mu \)-integrable on \( A \), for every \( A \in \mathcal{A} \).

By applying theorem 3.3, the proof finishes.

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