NON-DEGENERATE REAL HYPERSURFACES OF A PARAQUATERNIONIC KÄHLER MANIFOLD

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Abstract. We obtain necessary and sufficient conditions for the integrability of the distributions on a non-degenerate real hypersurface of a paraquaternionic Kähler manifold. If these distributions are integrable, we show that the foliations determined by them are totally geodesic.

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Introduction. The paraquaternionic Kähler manifolds are semi-Riemannian manifolds endowed with two local almost product structures and one local almost complex structure satisfying some compatibility conditions (cf. García-Río et al [3]). The concept was defined having in mind the quaternionic Kähler manifolds which are equipped with three local almost complex structures (cf. Ishihara [4]).

In the present paper we study the geometry of non-degenerate real hypersurfaces of a paraquaternionic Kähler manifold. For the case of real hypersurfaces of a quaternionic Kähler manifold, such a study was done by Bejancu [1]. In the first section we define the distributions \( D \) and \( D^\perp \) on a non-degenerate real hypersurface \( N \) of a paraquaternionic Kähler manifold \( (M, V, g) \) and present their main properties. Then in Sect. 2, by using the second fundamental form of \( N \) we obtain necessary and sufficient conditions for the integrability of both distributions \( D \) and \( D^\perp \). Finally, in the last section, we show that in case these distributions are integrable, the foliations determined by them are totally geodesic. Also we show that the leaves of \( D \) are totally geodesic immersed in \( M \) and find a necessary and sufficient condition for leaves of \( D^\perp \) to be totally geodesic immersed in \( M \).
1. Preliminaries. Throughout the paper all manifolds are smooth and paracompact. If $M$ is a smooth manifold then we denote by $F(M)$ the algebra of smooth functions on $M$ and by $\Gamma(TM)$ the $F(M)$-module of smooth sections of the tangent bundle $TM$ of $M$. We use these notations for any other manifold or vector bundle. If not stated otherwise, we use indices: $a, b, c, \ldots \in \{1, 2, 3\}$.

Let $M$ be a manifold endowed with a paraquaternionic structure $V$, that is, $V$ is a rank-3 subbundle of $\text{End}(TM)$ which has a local basis $\{J_1, J_2, J_3\}$ on a coordinate neighbourhood $U \subset M$ satisfying (see Garcia-Rio et al [3])

\begin{align*}
\text{(a)} & \quad J_a^2 = \lambda_a I, \quad a \in \{1, 2, 3\}, \\
\text{(b)} & \quad J_1 J_2 = -J_2 J_1 = J_3, \quad \lambda_1 = \lambda_2 = -\lambda_3 = 1.
\end{align*}

A semi-Riemannian metric $g$ on $M$ is said to be adapted to the paraquaternionic structure $V$ if it satisfies

\begin{equation}
(1.2) \quad g(X, Y) + \lambda_a g(J_a X, J_a Y) = 0, \quad \forall a \in \{1, 2, 3\},
\end{equation}

for any $X, Y \in \Gamma(TM)$, and any local basis $\{J_1, J_2, J_3\}$ of $V$. As a consequence of (1.1) and (1.2) we obtain

\begin{equation}
(1.3) \quad g(J_a X, Y) + g(X, J_a Y) = 0, \quad \forall X, Y \in \Gamma(TM), \quad a \in \{1, 2, 3\}.
\end{equation}

If $\{\tilde{J}_1, \tilde{J}_2, \tilde{J}_3\}$ is a local basis of $V$ on $\tilde{U} \subset M$ and $U \cap \tilde{U} \neq \emptyset$, then we have

\begin{equation}
(1.4) \quad \tilde{J}_a = \sum_{b=1}^{3} A_{ab} J_b,
\end{equation}

where the $3 \times 3$ matrix $[A_{ab}]$ is an element of the pseudo-orthogonal group $SO(2, 1)$. The above conditions impose special dimensions for $M$ and special signatures for $g$. We take $M$ of dimension $4(m+1)$ and $g$ of neutral signature $(2(m+1), 2(m+1))$.

The triple $(M, V, g)$ is called a paraquaternionic Kähler manifold if $V$ is a parallel bundle with respect to Levi-Civita connection $\nabla$ of $g$. Then taking into account that $\nabla$ is a metric connection and by using (1.1) and (1.2) it is deduced that $(M, V, g)$ is a paraquaternionic Kähler manifold if and only if for any local basis $\{J_1, J_2, J_3\}$ of $V$ on $U \subset M$ there exist the
1-forms \( p, q, r \) on \( \mathcal{U} \) such that (cf. García-Río et al [3])

\[
\begin{align*}
(a) & \quad \left( \tilde{\nabla}_X J_1 \right) Y = q(X) J_2 Y - r(X) J_3 Y, \\
(b) & \quad \left( \tilde{\nabla}_X J_2 \right) Y = -q(X) J_1 Y + p(X) J_3 Y, \\
(c) & \quad \left( \nabla_X J_3 \right) Y = -r(X) J_1 Y + p(X) J_2 Y, \quad \forall X,Y \in \Gamma(\mathcal{T}\mathcal{U}).
\end{align*}
\]

Now, let \( N \) be a non-degenerate real hypersurface of \( (M, \mathbf{V}, g) \). This means that \( g \) induces a semi-Riemannian metric on \( N \), which we denote by the same symbol \( g \). Suppose that the unit normal vector field \( \xi \) to \( N \) on a coordinate neighborhood \( \mathcal{V} = \mathcal{U} \cap N \) is space-like. Similar results are obtained when \( \xi \) is time-like.

Then we define on \( \mathcal{V} \) the vector fields \( \xi_a = J_a \xi, a \in \{1, 2, 3\} \).

From (1.2) we deduce that

\[
\begin{align*}
(a) & \quad g(\xi_1, \xi_1) = g(\xi_2, \xi_2) = -1, \quad (b) \quad g(\xi_3, \xi_3) = 1,
\end{align*}
\]

that is, \( \xi_1 \) and \( \xi_2 \) are unit time-like vector fields, while \( \xi_3 \) is a unit space-like vector field. In a similar way we define \( \tilde{\xi}_a = \tilde{J}_a \xi \) on \( \mathcal{V} = \mathcal{U} \cap N \). Suppose \( \mathcal{V} \cap \mathcal{V} \neq \emptyset \) and by (1.4) we obtain

\[
\tilde{\xi}_a = \sum_{b=1}^{3} A_{ab} \xi_b, \quad \text{on} \quad \mathcal{V} \cap \mathcal{V}, \quad a \in \{1, 2, 3\}.
\]

Thus from (1.7) we deduce that there exists on \( N \) a globally defined distribution \( \mathcal{D}^\perp \) which is locally represented on \( \mathcal{V} \) by the orthonormal frame field \( \{\xi_1, \xi_2, \xi_3\} \). Next, we denote by \( \mathcal{D} \) the orthogonal complementary distribution to \( \mathcal{D}^\perp \) in the tangent bundle \( TN \). By using (1.1) and (1.2) we deduce that \( \mathcal{D} \) is invariant by \( J_a \), that is, we have

\[
J_a(\mathcal{D}_x) = \mathcal{D}_x, \quad \forall x \in N, \quad a \in \{1, 2, 3\}.
\]

We call \( \mathcal{D} \) the \textit{paraquaternionic distribution} on \( N \). Note that \( \mathcal{D} \) is of rank \( 4m \) and \( g \) has the signature \( (2m, 2m) \) on \( \mathcal{D} \). On the other hand, \( \mathcal{D}^\perp \) is not invariant by \( J_a \). More precisely, we have

\[
\begin{align*}
(a) & \quad J_a \xi_a = \lambda_a \xi, \quad (b) \quad J_1 \xi_2 = -J_2 \xi_1 = \xi_3, \\
(c) & \quad J_1 \xi_3 = -J_3 \xi_1 = \xi_2, \quad (d) \quad J_3 \xi_2 = -J_2 \xi_3 = \xi_1.
\end{align*}
\]
2. Integrability of distributions $\mathcal{D}$ and $\mathcal{D}^\perp$. Let $N$ be a non-degenerate real hypersurface of a paraquaternionic Kähler manifold $(M, V, g)$. Denote by $\nabla$ the Levi-Civita connection determined by $g$ on $N$. Then the equations of Gauss and Weingarten are expressed as follows:

(2.1) \[ \tilde{\nabla}_XY = \nabla_XY + h(X, Y)\xi, \]

and

(2.2) \[ \tilde{\nabla}_X\xi = -AX, \]

for any $X, Y \in \Gamma(TN)$, where $h$ is the second fundamental form of $N$ and $A$ is the shape operator of $N$. Then we have

(2.3) \[ h(X, Y) = g(AX, Y), \quad \forall X, Y \in \Gamma(TN). \]

Now, denote by $P$ the projection morphism of $TN$ to $\mathcal{D}$ with respect to the decomposition

(2.4) \[ TN = \mathcal{D} \oplus \mathcal{D}^\perp. \]

Then any vector field $Y$ on $N$ can be expressed as follows

(2.5) \[ Y = PY + \sum_{a=1}^{3} \eta_a(Y)\xi_a, \]

where $\{\eta_a\}$ are 1-forms locally defined on $N$ by

(2.6) \[ \eta_a(Y) = -\lambda_a g(Y, \xi_a), \quad a \in \{1, 2, 3\}. \]

We apply $J_1, J_2, J_3$ to (2.5) and by using (1.9) obtain

(2.7) \[
\begin{align*}
(a) \quad & J_1Y = J_1PY + \eta_1(Y)\xi + \eta_2(Y)\xi_3 + \eta_3(Y)\xi_2, \\
(b) \quad & J_2Y = J_2PY - \eta_1(Y)\xi_3 + \eta_2(Y)\xi - \eta_3(Y)\xi_1, \\
(c) \quad & J_3Y = J_3PY - \eta_1(Y)\xi_2 + \eta_2(Y)\xi_1 + \eta_3(Y)\xi.
\end{align*}
\]

**Lemma 2.1.** Let $N$ be a non-degenerate real hypersurface of a paraquaternionic Kähler manifold $(M, V, g)$. Then we have:

(2.8) \[
\begin{align*}
(a) \quad & \nabla_XJ_1Y - J_1P\nabla_XY = q(X)J_1Y - r(X)J_3Y + h(X, Y)\xi_1 \\
& + \eta_3(\nabla_XY)\xi_2 + \eta_2(\nabla_XY)\xi_3, \\
(b) \quad & \nabla_XJ_2Y - J_2P\nabla_XY = -q(X)J_1Y + p(X)J_3Y \\
& - \eta_3(\nabla_XY)\xi_1 - \eta_1(\nabla_XY)\xi_3 + h(X, Y)\xi_2, \\
(c) \quad & \nabla_XJ_3Y - J_3P\nabla_XY = -r(X)J_1Y + p(X)J_2Y \\
& + \eta_2(\nabla_XY)\xi_1 - \eta_1(\nabla_XY)\xi_2 + h(X, Y)\xi_3.
\end{align*}
\]
and

\begin{equation}
(2.9) \quad h(X, J_a Y) = \lambda_a \eta_a (\nabla_X Y), \quad \forall a \in \{1, 2, 3\},
\end{equation}

for any \(X, Y \in \Gamma(D)\).

**Proof.** Take \(X, Y \in \Gamma(D)\) in (1.5a) and by using (2.1) and (2.7a) we obtain

\begin{equation}
(2.10) \quad \{\nabla_X J_1 Y - J_1 P \nabla_X Y - h(X, Y) \xi_1 - \eta_3 (\nabla_X Y) \xi_2 - \eta_2 (\nabla_X Y) \xi_3 \\
- q(X) J_2 Y + r(X) J_3 Y\} + \{h(X, J_1 Y) - \eta_1 (\nabla_X Y)\} \xi = 0.
\end{equation}

Then (2.8a) and (2.9) (for \(a = 1\)) are obtained by taking in (2.10) the tangent and normal parts, respectively. All other equalities in (2.8) and (2.9) are deduced using similar calculations. \(\square\)

Now, we introduce two particular classes of hypersurfaces of \((M, V, g)\). We say that \(N\) is parallel \(D\)-geodesic (resp. \(D^\perp\)-geodesic) if any geodesic of \(N\) that is tangent to \(D\) (resp. \(D^\perp\)) is a geodesic of \(M\). Then, by using (2.1) we obtain the following.

**Lemma 2.2.** Let \(N\) be a non-degenerate real hypersurface of the paraquaternionic Kähler manifold \((M, V, g)\). Then we have the assertions:

(a) \(N\) is \(D\)-geodesic if and only if

\begin{equation}
(2.11) \quad h(X, Y) = 0, \quad \forall X, Y \in \Gamma(D).
\end{equation}

(b) \(N\) is \(D^\perp\)-geodesic if and only if

\begin{equation}
(2.12) \quad h(\xi_a, \xi_b) = 0, \quad \forall a, b \in \{1, 2, 3\}.
\end{equation}

Next, we prove the following.

**Lemma 2.3.** Let \(N\) be a non-degenerate real hypersurface of a paraquaternionic Kähler manifold \((M, V, g)\). Then \(N\) is \(D\)-geodesic if and only if its second fundamental form satisfies

\begin{equation}
(2.13) \quad h(X, J_a Y) = h(Y, J_a X), \quad \forall a \in \{1, 2, 3\}, \quad X, Y \in \Gamma(D).
\end{equation}
Proof. Suppose $N$ is $\mathcal{D}$-geodesic. Then from (2.11) we obtain (2.13) since $\mathcal{D}$ is invariant by $J_a$ for any $a \in \{1, 2, 3\}$. Conversely, suppose (2.13) is satisfied and by using (1.16) we deduce that
\[
h(J_3 X, Y) = h(X, J_3 Y) = h(X, J_1 J_2 Y) = h(J_1 X, J_2 Y) = h(J_2 J_1 X, Y) = -h(J_3 X, Y),
\]
for any $X, Y \in \Gamma(\mathcal{D})$. Thus $h(J_3 X, Y) = 0$ for any $X, Y \in \Gamma(\mathcal{D})$, which yields (2.11) since $J_3$ is an automorphism of the paraquaternionic distribution. □

Theorem 2.1. Let $N$ be a non-degenerate real hypersurface of a paraquaternionic Kähler manifold $(M, \mathbf{V}, g)$. Then the paraquaternionic distribution $\mathcal{D}$ is integrable if and only if $N$ is $\mathcal{D}$-geodesic.

Proof. First, by using (2.6) into (2.9) we obtain
\[
h(X, J_a Y) = -g(\nabla_X Y, \xi_a), \quad \forall a \in \{1, 2, 3\}, X, Y \in \Gamma(\mathcal{D}).
\]
Then, taking into account that $\nabla$ is torsion-free we obtain
\[
(2.14) \quad h(X, J_a Y) - h(Y, J_a X) = g([Y, X], \xi_a),
\]
for any $a \in \{1, 2, 3\}$ and $X, Y \in \Gamma(\mathcal{D})$. Thus the assertion of the theorem follows from (2.14), by using Lemma 2.3 and taking into account that $\{\xi_1, \xi_2, \xi_3\}$ is an orthonormal basis in $\Gamma(\mathcal{D}^\perp)$. □

Next, we say that $N$ is $(\mathcal{D}, \mathcal{D}^\perp)$-geodesic if its second fundamental form satisfies
\[
(2.15) \quad h(X, \xi_a) = 0, \quad \forall X \in \Gamma(\mathcal{D}), \ a \in \{1, 2, 3\}.
\]

Then we prove the following.

Theorem 2.2. Let $N$ be a non-degenerate real hypersurface of a paraquaternionic Kähler manifold $(M, \mathbf{V}, g)$. Then the distribution $\mathcal{D}^\perp$ is integrable if and only if $N$ is $(\mathcal{D}, \mathcal{D}^\perp)$-geodesic.

Proof. First, by using (1.9), (1.5c), (1.2) and (2.1) we obtain
\[
g([\xi_1, \xi_2], J_3 X) = g(\tilde{\nabla}_{\xi_1} \xi_2 - \tilde{\nabla}_{\xi_2} \xi_1, J_3 X)
\]
\[
= -g(\tilde{\nabla}_{\xi_1} J_3 \xi_2 + \tilde{\nabla}_{\xi_2} J_3 \xi_1, J_3 X)
\]
\[
= g(\nabla_{\xi_1} \xi_1, X) + g(\nabla_{\xi_2} \xi_2, X),
\]
for any $a \in \{1, 2, 3\}$ and $X, Y \in \Gamma(\mathcal{D})$. Thus $h(J_3 X, Y) = 0$ for any $X, Y \in \Gamma(\mathcal{D})$, which yields (2.11) since $J_3$ is an automorphism of the paraquaternionic distribution. □
for any $X \in \Gamma(D)$. By similar calculations we deduce that
\begin{equation}
(2.17) \quad g([\xi_2, \xi_3], J_1X) = g(\nabla_{\xi_3} \xi_3, X) - g(\nabla_{\xi_2} \xi_2, X),
\end{equation}
and
\begin{equation}
(2.18) \quad g([\xi_3, \xi_1], J_2X) = g(\nabla_{\xi_3} \xi_3, X) - g(\nabla_{\xi_1} \xi_1, X),
\end{equation}
for any $X \in \Gamma(D)$. Thus, from (2.16), (2.17) and (2.18) we infer that $D^\perp$ is integrable if and only if
\begin{align*}
g(\nabla_{\xi_1} \xi_1, X) + g(\nabla_{\xi_2} \xi_2, X) &= 0, \\
g(\nabla_{\xi_3} \xi_3, X) - g(\nabla_{\xi_2} \xi_2, X) &= 0, \quad \forall X \in \Gamma(D), \\
g(\nabla_{\xi_3} \xi_3, X) - g(\nabla_{\xi_1} \xi_1, X) &= 0,
\end{align*}
which are equivalent to
\begin{align}
(2.19) \quad g(\nabla_{\xi_a} \xi_a, X) &= 0, \quad \forall a \in \{1, 2, 3\}, \quad X \in \Gamma(D).
\end{align}
On the other hand, by using (2.1), (1.2), (1.5), (1.9a), (2.2) and (2.3) we obtain
\begin{align*}
g(\nabla_{\xi_a} \xi_a, X) &= g(\tilde{\nabla}_{\xi_a} \xi_a, X) = -\lambda_a g(J_a \tilde{\nabla}_{\xi_a} \xi_a, J_a X) \\
&= -\lambda_a g(\lambda_a \tilde{\nabla}_{\xi_a} \xi, J_a X) = g(A \xi_a, J_a X) = h(\xi_a, J_a X).
\end{align*}
Finally, taking into account that $J_a$ are automorphisms of $D$ for any $a \in \{1, 2, 3\}$ and using (2.19) and (2.20) we deduce that $D^\perp$ is integrable if and only if (2.15) is satisfied. This completes the proof of the theorem.
\[\square\]

3. Foliations on a real hypersurface of $(M, V, g)$. Let $N$ be a non-degenerate real hypersurface of a paraquaternionic Kähler manifold $(M, V, g)$. In this section we study the geometry of leaves of the foliations on $N$ provided $D$ and/or $D^\perp$ are integrable. First, we recall that a submanifold $S$ of a Riemannian manifold $Q$ is \textit{totally geodesic} if its second fundamental form vanishes identically on $S$ (cf. CHEN [2]). Then we prove the following.

**Theorem 3.1.** Let $N$ be a non-degenerate real hypersurface of a paraquaternionic Kähler manifold $(M, V, g)$, whose paraquaternionic distribution $D$ is integrable. Then any leaf of $D$ is totally geodesic immersed in $M$. 
Proof. Let $N^*$ be a leaf of $\mathcal{D}$ and $h^*$ be the second fundamental form of the immersion of $N^*$ in $N$. Then by Gauss equation we have
\begin{equation}
\nabla_X Y = \nabla_X^* Y + h^*(X, Y), \quad \forall X, Y \in \Gamma(TN^*),
\end{equation}
where $\nabla^*$ is the Levi-Civita connection on $N^*$. Then by using (2.1) and (3.1) we deduce that
\begin{equation}
\tilde{\nabla}_X Y = \nabla_X Y = \nabla_X^* Y + h^*(X, Y), \quad \forall X, Y \in \Gamma(TN^*),
\end{equation}
since $N$ is $\mathcal{D}$-geodesic (cf. Theorem 2.1). Thus by (3.2) $h^*$ is also the second fundamental form of the immersion of $N^*$ in $M$. Next, by using (1.5a), (2.7a) and (3.2) we obtain
\begin{equation}
\nabla_X^* J_1 Y + h^*(X, J_1 Y) = J_1(\nabla_X^* Y) + \eta_1(\nabla_X Y)\xi + \eta_2(\nabla_X Y)\xi_3 + \eta_3(\nabla_X Y)\xi_2 + q(X)J_2 Y - r(X)J_3 Y,
\end{equation}
for any $X, Y \in \Gamma(TN^*)$. Taking into account that $N$ is $\mathcal{D}$-geodesic, from (2.9) we infer that
\begin{equation*}
\eta_a(\nabla_X Y) = 0, \quad \forall a \in \{1, 2, 3\}, \quad X, Y \in \Gamma(TN^*). \tag{3.3}
\end{equation*}
Thus taking the vector fields from (3.3) that are normal to $N^*$ in $M$ we obtain
\begin{equation*}
h^*(X, J_1 Y) = 0, \quad \forall X, Y \in \Gamma(TN^*). \tag{3.4}
\end{equation*}
This completes the proof since $J_1$ is an automorphism of $\Gamma(TN^*)$. \hfill \Box

Now we recall that a foliation $\mathcal{F}$ on $N$ is totally geodesic if any leaf of $\mathcal{F}$ is a totally geodesic submanifold of $N$ (cf. Reinhart [5]).

Corollary 3.1. Let $N$ be a non-degenerate real hypersurface of a paraquaternionic Kähler manifold $(M, \mathcal{V}, g)$, whose paraquaternionic distribution $\mathcal{D}$ is integrable. Then the foliation determined by $\mathcal{D}$ on $N$ is totally geodesic.

Proof. Let $N^*$ be a leaf of $\mathcal{D}$. Then by Theorem 3.1 we have $h^* = 0$ on $N$. Thus by (3.1) we deduce that $N^*$ is totally geodesic immersed in $N$. Hence the foliation determined by $\mathcal{D}$ is totally geodesic. \hfill \Box

Next, we study the geometry of leaves of $\mathcal{D}^\perp$. First, we prove the following.
**Theorem 3.2.** Let $N$ be a non-degenerate real hypersurface of a paraquaternionic Kähler manifold $(M, V, g)$, whose distribution $D^\perp$ is integrable. Then the foliation determined by $D^\perp$ on $N$ is totally geodesic.

**Proof.** Let $N'$ be a leaf of $D^\perp$ and $h^\perp$ be the second fundamental form of the immersion of $N'$ in $N$. Then by using the Gauss equation for the submanifold $N'$ of $N$, (1.5) and (2.1)-(2.3) we obtain

$$
\begin{align*}
g \left( h^\perp(X, \xi_a), Y \right) &= g \left( \nabla_X \xi_a, Y \right) = g \left( J_a \nabla_X \xi, Y \right) \\
&= -g \left( \nabla_X \xi, J_a Y \right) = g \left( AX, J_a Y \right) = h(X, J_a Y),
\end{align*}
$$

for any $X \in \Gamma(TN')$, $Y \in \Gamma(D)$ and $a \in \{1, 2, 3\}$. By Theorem 2.2 $N$ is $(D, D^\perp)$-geodesic and thus (3.4) implies

$$
(3.5) \quad h^\perp(X, Z) = 0, \ \forall X, Z \in \Gamma(TN'),
$$

which completes the proof. \qed

From Corollary 3.1 and Theorem 3.2 we deduce the following corollary.

**Corollary 3.2.** Let $N$ be a non-degenerate real hypersurface of a paraquaternionic Kähler manifold $(M, V, g)$. If both distributions $D$ and $D^\perp$ are integrable then $N$ is locally a semi-Riemannian product $N^* \times N'$ where $N^*$ and $N'$ are leaves of $D$ and $D^\perp$ respectively.

Finally, we state a necessary and sufficient condition for leaves of $D^\perp$ to be totally geodesic immersed in $M$.

**Theorem 3.3.** Let $N$ be a non-degenerate real hypersurface of a paraquaternionic Kähler manifold $(M, V, g)$, whose distribution $D^\perp$ is integrable. Then any leaf of $D^\perp$ is totally geodesic immersed in $M$ if and only if $N$ is $D^\perp$-geodesic.

**Proof.** By using Gauss equation for both the immersion of $N$ in $M$ and the immersion of a leaf $N'$ of $D^\perp$ in $N$ and (3.5) we obtain

$$
(3.6) \quad g \left( \nabla_X Z, Y \right) = 0, \ \forall X, Z \in \Gamma(TN'), \ Y \in \Gamma(D).
$$

On the other hand, by using (2.2) and (2.3) we get

$$
(3.7) \quad g \left( \nabla_X Z, \xi \right) = h(X, Z), \ \forall X, Z \in \Gamma(TN').
$$
Now, we denote by \( \tilde{h} \) the second fundamental form of the immersion of \( N' \) in \( M \). Then by using Gauss equation in (3.6) and (3.7) we deduce that

\[
(3.8) \quad g(\tilde{h}(X, Z), Y) = 0, \quad \forall X, Z \in \Gamma(TN'), \ Y \in \Gamma(D),
\]

and

\[
(3.9) \quad g(\tilde{h}(X, Z), \xi) = h(X, Z), \quad \forall X, Z \in \Gamma(TN'),
\]

respectively. Thus the assertion of the theorem follows by using (3.8), (3.9) and (2.12).

\[ \square \]

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