MIXED GENERALIZED QUASI EINSTEIN MANIFOLD
AND SOME PROPERTIES

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Abstract. In this paper we have defined mixed generalized quasi-Einstein manifold $MG(QE)_n$, which is more generalized form of Einstein manifold, quasi-Einstein manifold and generalized quasi-Einstein manifold, proved the existence theorem, gave two examples of it and studied some properties on it.

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1. Introduction. The notion of a generalized quasi-Einstein manifold was introduced by De and Ghosh [4]. According to them, a non-flat Riemannian manifold is called a generalized quasi-Einstein manifold if its Ricci-tensor $S$ of type (0,2) is non-zero and satisfies the condition

\[ S(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y). \]

Where $a, b, c$ are certain non-zero scalars and $A, B$ are two non-zero 1-forms such that for two unit vector fields $U$ and $V$ corresponding to the 1-forms $A$ and $B$ respectively

\[ g(X, U) = A(X), g(X, V) = B(X) \text{ and } g(U, V) = 0. \]

In such a case $a, b, c$ are called the associated scalars, $A, B$ respectively are called the associated main and auxiliary 1-forms and $U, V$ respectively are called the main and auxiliary generators of the manifold.

Here we generalize the notion of generalized quasi-Einstein manifold.
A non-flat Riemannian manifold \((M^n, g)(n \geq 3)\) is called mixed generalized quasi-Einstein manifold if the Ricci-tensor \(S\) of type \((0,2)\) is not identically zero and satisfies the condition
\[
S(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y) \\
+ d[A(X)B(Y) + A(Y)B(X)],
\]
where \(a, b, c, d\) are scalars of which \(b \neq 0, c \neq 0, d \neq 0\) and \(A, B\) are two non-zero 1-forms such that
\[
g(X, \rho) = A(X), \; g(X, \mu) = B(X) \text{ and } g(\rho, \mu) = 0,
\]
where \(\rho, \mu\) are unit vector fields. In such a case \(a, b, c, d\) are called associated scalars. \(A, B\) are called the associated 1-forms and \(\rho, \mu\) are called the generators of the manifold. Such \(n\)-dimensional manifold is denoted by the symbol \(MG(QE)_n\). If \(d = 0\), the manifold reduces to generalized quasi-Einstein manifold. If \(c = d = 0\), the manifold reduces to quasi-Einstein manifold. If \(b = c = d = 0\), the manifold reduces to Einstein manifold [1]. So it is our motivation to study more generalized form of Einstein manifolds, quasi-Einstein manifolds and generalized quasi Einstein manifolds which is mixed generalized quasi-Einstein manifold.

Here we have established an example of \(MG(QE)_n\) and also an example of a compact orientable \(MG(QE)_n\) without boundary.

Example of \(MG(QE)_n\) : Let \(M^n\) be a hypersurface of a Euclidean space \(E^{n+1}\) and the metric tensor \(\tilde{g}\) of \(M^n\) is induced by \(E^{n+1}\).

The Gauss equation of \(M^n\) in \(E^{n+1}\) can be written as
\[
\tilde{g}(\tilde{R}(X, Y)Z, W) = \tilde{g}(H(X, W), H(Y, Z)) - \tilde{g}(H(Y, W), H(X, Z)),
\]
where \(\tilde{R}\) is the Riemannian curvature tensor corresponding to the induced metric \(\tilde{g}\). \(H\) is the second fundamental tensor of \(M^n\) (orthonormal to \(M^n\)) and \(X, Y, Z, W\) are vector fields tangent to \(M^n\).

If \(A_\xi\) is the \((1,1)\) tensor corresponding to the normal valued second fundamental tensor \(H\), then we have [3]
\[
\tilde{g}(A_\xi(X), Y) = g(H(X, Y), \xi),
\]
where \(\xi\) is the unit normal vector field and \(X, Y\) are tangent vector fields.

Let \(H_\xi\) be the symmetric \((0,2)\) tensor associated with \(A_\xi\) in the hypersurface defined by
\[
\tilde{g}(A_\xi(X), Y) = H_\xi(X, Y).
\]
A hypersurface of a Riemannian manifold \((M^n, \tilde{g})\) is called quasi-umbilical \([3]\) if its second fundamental tensor has the form

\[(1.8) \quad H_\xi(X, Y) = \alpha \tilde{g}(X, Y) + \beta \omega(X)\omega(Y),\]

where \(\omega\) is 1-form. The vector field corresponding to the 1-form \(\omega\) is a unit vector field and \(\alpha, \beta\) are scalars. If \(\alpha = 0\) (resp. \(\beta = 0\) or \(\alpha = \beta = 0\)) holds then \(M^n\) is called cylindrical (resp. umbilical or geodesic).

We define generalized quasi umbilical hypersurface of a Riemannian manifold as follows. A hypersurface of a Riemannian manifold \((M^n, \tilde{g})\) is called generalized quasi umbilical if its second fundamental tensor has the form

\[(1.9) \quad H_\xi(X, Y) = \alpha \tilde{g}(X, Y) + \beta \omega(X)\omega(Y) + \gamma \delta(X)\delta(Y),\]

where \(\alpha, \beta, \gamma\) are scalars. The vector fields corresponding to 1-forms \(\omega\) and \(\delta\) are unit vector fields. If \(\alpha = \beta = \gamma = 0\), \(M^n\) is called geodesic. If \(\alpha = \gamma = 0\) or \(\alpha = \beta = 0\), \(M^n\) is called cylindrical. Also \(M^n\) is called umbilical when \(\beta = \gamma = 0\).

Now from (1.6), (1.7) and (1.8), we get

\[(1.10) \quad H(X, Y) = \alpha g(X, Y)\xi + \beta \omega(X)\omega(Y)\xi.\]

Since \(\xi\) is the only unit normal vector. Let us suppose that the hypersurface is generalized quasi umbilical. Then

\[(1.11) \quad H(X, Y) = \alpha g(X, Y)\xi + \beta (\omega(X)\omega(Y))\xi + \gamma \delta(X)\delta(Y)\xi.\]

Using (1.5) in (1.11) and after contraction we get

\[(1.12) \quad \overline{S}(Y, Z) = [\alpha^2(n - 1) + \alpha\beta + \alpha\gamma] g(Y, Z) + [(n - 2)\alpha\beta + \beta\gamma] \omega(Y)\omega(Z) + [(n - 2)\gamma\alpha + \beta\gamma] \delta(Y)\delta(Z) - \beta\gamma \{\omega(Y)\delta(Z) + \delta(Y)\omega(Z)\}\]

Which shows that the manifold is a mixed generalized quasi Einstein manifold \((MG(QE)_n)\).

If \(\beta = 0\) or \(\gamma = 0\), the manifold reduces to quasi-Einstein manifold. If both \(\beta = \gamma = 0\), the manifold reduces to Einstein manifold.

Example of Compact Orientable \(MG(QE)_n\) without boundary: A hypersurface of a compact orientable Riemannian manifold \((M^n, g)\) without boundary which is generalized quasi umbilical is an example of compact orientable \(MG(QE)_n\) without boundary.
In this paper we like to introduce another notion which generalizes the notion of a manifold of generalized quasi-constant curvature [5].

A Riemannian manifold is said to be a manifold of generalized quasi-constant curvature if the curvature tensor $R$ of type $(0,4)$ satisfies the condition

\[
R(X, Y, Z, W) = p[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\
+ q[g(X, W)T(Y)T(Z) - g(X, Z)T(Y)T(W)] \\
+ s[g(X, W)T(Y)T(Z)] \\
+ t[g(X, W)T(Y)T(Z)] \\
(1.13)
\]

where $p$, $q$, $s$ are scalars, $T$ and $D$ are non-zero 1-forms. $\rho$ and $\overline{\rho}$ are unit orthogonal vector fields such that

\[
g(X, \rho) = T(X), \quad g(X, \overline{\rho}) = D(X) \quad \text{and} \quad g(\rho, \overline{\rho}) = 0.
\]

A Riemannian manifold is said to be a manifold of mixed generalized quasi-constant curvature if the curvature tensor $R$ of type $(0,4)$ satisfies the condition

\[
R(X, Y, Z, W) = p[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\
+ q[g(X, W)A(Y)A(Z) - g(Y, W)A(X)A(Z)] \\
+ s[g(X, W)A(Y)A(Z)] \\
+ t[v\{A(Y)B(Z) + B(Y)A(Z)\}]g(X, W) \\
\{A(X)B(Z) + B(X)A(Z)\}g(Y, W) \\
+ t[v\{A(X)B(W) + B(X)A(W)\}]g(Y, Z) \\
\{A(Y)B(W) + B(Y)A(W)\}]g(X, Z),
\]

where $p$, $q$, $s$, $t$ are scalars. $A$, $B$ are non-zero 1-forms. $\rho$ and $\overline{\rho}$ are orthonormal unit vectors corresponding to $A$ and $B$

\[
g(X, \rho) = A(X), \quad g(X, \overline{\rho}) = B(X) \quad \text{and} \quad g(\rho, \overline{\rho}) = 0.
\]

It can be easily seen a manifold of mixed generalized quasi constant curvature defined by (1.15) is also an example of a $MG(QE)_n$. 

\[
in X, Y, Z, W
\]
In the next sections we have proved the existence theorem of mixed general-

eralized quasi Einstein manifold, studied some global properties of compact orientable \( MG(QE) \), without boundary and obtained relation between the manifold of mixed generalized quasi constant curvature and the mixed generalized quasi Einstein manifold.

2. Preliminaries.  From (1.3) and (1.4), we get

\[
S(X, X) = a|X|^2 + b|g(X, \rho)|^2 + c|g(X, \mu)|^2 + 2d|g(X, \rho)g(X, \mu)|, \ \forall X.
\]

Let \( \theta_1 \) be the angle between \( \rho \) and any vector \( X \); \( \theta_2 \) be the angle between \( \mu \) and any vector \( X \). Then

\[
\cos \theta_1 = \frac{g(X, \rho)}{\sqrt{g(\rho, \rho)}\sqrt{g(X, X)}} = \frac{g(X, \rho)}{\sqrt{g(X, X)}}, \quad \text{(as } g(\rho, \rho) = 1)\]

\[
\cos \theta_2 = \frac{g(X, \mu)}{\sqrt{g(X, X)}}.
\]

If \( b > 0, c > 0 \) and \( d > 0 \) we have from (2.1)

\[
(a + b + c + 2d)|X|^2 \geq a|X|^2 + b|g(X, \rho)|^2 + c|g(X, \mu)|^2 + 2d|g(X, \rho)g(X, \mu)| = S(X, X).
\]

Now, contracting (1.3) over \( X \) and \( Y \), we get

\[
r = na + b + c,
\]

where \( r \) is the scalar curvature of the manifold.

Putting \( X = Y = U \) in (1.3) we obtain

\[
S(U, U) = a + b.
\]

Again putting \( X = Y = V \) in (1.3) we have

\[
S(V, V) = a + c.
\]

Next, let \( Q \) be the symmetric endomorphism of the tangent space at a point corresponding to the Ricci-tensor \( S \). Then

\[
g(QX, Y) = S(X, Y) \ \forall X, Y.
\]
Let $l^2$ be the square of the length of the Ricci-tensor. Then

$$l^2 = \sum_{i=1}^{n} S(Qe_i, e_i),$$

(2.7)

where $\{e_i\}$, $i = 1, 2, 3, \ldots, n$ is an orthonormal basis of the tangent space at a point. From (1.3) we get

$$S(Qe_i, e_i) = a g(Qe_i, e_i) + b A(L e_i) A(e_i) + c B(Qe_i) B(e_i)$$

$$+ d [A(Qe_i) B(e_i) + A(e_i) B(Qe_i)$$

i.e.,

$$l^2 = (n - 2) a^2 + (a + b)^2 + (a + c)^2 + 2d^2.$$ 

(2.8)

We know in a n-dimensional ($n > 2$) Riemannian manifold, the covariant quasi conformal curvature tensor is defined as [3],

$$\tilde{C}(X, Y, Z, W) = a R(X, Y, Z, W) + b [S(Y, Z) g(X, W) - S(X, Z) g(Y, W)$$

$$+ g(Y, Z) g(QX, W) - g(X, W) g(QY, W)]$$

$$- \frac{r}{n} \left[ \frac{a}{n-1} + 2b \right] [g(Y, Z) g(X, W) - g(X, Z) g(Y, W)].$$

(2.9)

Where $a, b$ are constants.

$$g(\tilde{C}(X, Y) Z, W) = \tilde{C}(X, Y, Z, W)$$

(2.10)

$$g(R(X, Y) Z, W) = R(X, Y, Z, W).$$

(2.11)

If $a = 1, b = -\frac{1}{n-2}$ then (2.9) reduces to conformal curvature tensor and also if $r = 0$ (2.9) becomes conharmonic curvature tensor.

These results will be used in the sequel.

3. Existence theorem of a mixed generalized quasi-Einstein manifold. In this section we show under what condition there exists a mixed generalized quasi-Einstein manifold.
Theorem 3.1. If the Ricci tensor $S$ of a Riemannian manifold satisfies the relation

\begin{align*}
S(X, W)S(Y, Z) - S(Y, W)S(X, Z) &= \mu[S(Y, W)g(Z, X) + S(Z, X)g(Y, W)] \\
&\quad + \beta[g(X, W)g(Y, Z) - g(Y, W)g(Z, X)],
\end{align*}

where $\mu, \beta$ are non-zero scalars, then the manifold is a mixed generalized quasi Einstein manifold.

Proof. Let $U$ be a vector field defined by

\begin{equation}
g(X, U) = T(X), \quad \forall X \in TM.
\end{equation}

Putting $X = W = U$ in (3.1) we get

\begin{align*}
S(U, U)S(Y, Z) - S(Y, U)S(U, Z) &= \mu[S(Y, U)g(Z, U) + S(Z, U)g(Y, U)] \\
&\quad + \beta[g(U, U)g(Y, Z) - g(Y, U)g(Z, U)].
\end{align*}

Now using (3.2) and (2.4) we get

\begin{align*}
\alpha S(Y, Z) - T(QY)T(QZ) &= \mu[T(QY)T(Z) + T(QZ)T(Y)] \\
&\quad + \rho||U||^2 g(Y, Z) - T(Y)T(Z)].
\end{align*}

Where $S(U, U) = \alpha$ and $T(QY) = g(QY, U) = S(Y, U)$

\begin{align*}
S(Y, Z) &= \alpha T(QY)T(QZ) + \mu T(QY)T(Z) + T(QZ)T(Y) \\
&\quad + \rho||U||^2 g(Y, Z) - T(Y)T(Z)].
\end{align*}

Where $\alpha = \frac{1}{\rho}$. Writing $P(Y) = T(QY)$ we have

\begin{align*}
S(Y, Z) &= \alpha \rho||U||^2 g(Y, Z) + (-\alpha \rho)T(Y)T(Z) + \alpha P(Y)P(Z) \\
&\quad + \alpha \mu[T(Y)P(Z) + P(Y)T(Z)].
\end{align*}

Which shows that the manifold is a mixed generalized quasi Einstein manifold. □
4. **Sufficient condition for a compact, orientable** $MG(QE)_n (n \geq 3)$ **without boundary to be isometric to a sphere.** In this section we consider a compact, orientable $MG(QE)_n$ without boundary having constant associated scalars $a$, $b$, $c$ and $d$. Then from (2.3) and (2.6), it follows that the scalar curvature is constant and so also is the length of the Ricci-tensor.

We further suppose that $MG(QE)_n$ under consideration admits a non-isometric conformal motion generated by a vector field $X$. Since $l^2$ is constant, it follows that

\[ \mathcal{L}_X l^2 = 0, \]

where $\mathcal{L}_X$ denotes Lie differentiation with respect to $X$.

Now, it is known ([5]p.57, Theorem (4.6)) that if a compact Riemannian manifold $M$ of dimension $n > 2$ with constant scalar curvature admits an infinitesimal non-isometric conformal transformation $X$ such that $\mathcal{L}_X l^2 = 0$, then $M$ is isometric to a sphere. But a sphere is Einstein so that $b$, $c$ and $d$ vanish which is a contradiction. This leads to the following theorem.

**Theorem 4.1.** A compact orientable mixed generalized quasi Einstein manifold $MG(QE)_n (n \geq 3)$ without boundary does not admit a non-isometric conformal vector field.

5. **Killing vector field in a compact orientable** $MG(QE)_n (n \geq 3)$ **without boundary.** In this section, we consider a compact, orientable $MG(QE)_n (n \geq 3)$ without boundary with $a$, $b$, $c$, $d$ as associated scalars and $\rho$ and $\mu$ as the generators.

It is known [5] that in such a manifold $M$, the following relation holds

\[ \int_M [S(X, X)] - |\nabla X|^2 - (div X)^2]dv \leq 0, \forall X. \]

If $X$ is a killing vector field, then $div X = 0$ [5]. Hence (5.1) takes the form

\[ \int_M [S(X, X)] - |\nabla X|^2]dv = 0. \]

Let $b > 0$, $c > 0$ and $d > 0$ then by (2.2) $(a + b + c + 2d)|X|^2 \geq S(X, X)$. Therefore, $(a + b + c + 2d)|X|^2 - |\nabla X|^2 \geq S(X, X) - |\nabla X|^2$. Consequently,

\[ \int_M [(a + b + c + 2d)|X|^2 - |\nabla X|^2]dv \geq \int_M [S(X, X) - |\nabla X|^2]dv \]
and by (5.2)
\[ \int_M [(a + b + c + 2d)|X|^2 - |\nabla X|^2] dv \geq 0. \]
If \((a + b + c + 2d) < 0\), then
\[ \int_M [(a + b + c + 2d)|X|^2 - |\nabla X|^2] dv = 0. \]
Therefore, \(X = 0\). This leads to the following.

**Theorem 5.1.** If in a compact, orientable MG(QE)\(_n\) \((n \geq 3)\) without boundary the associated scalars are such that \(b > 0\), \(c > 0\), \(d > 0\) and \(a + b + c + 2d < 0\) then there exists no non-zero killing vector field in this manifold.

6. **Harmonic vector fields in a compact orientable MG(QE)\(_n\)\((n \geq 3)\) without boundary.** This section deals with harmonic vector fields in a compact orientable MG(QE)\(_n\)\((n \geq 3)\) without boundary.

Let us assume \(\theta_2 \leq \theta_1\), where \(\theta_1\) is the angle between \(\rho\) and any vector \(X\); \(\theta_2\) is the angle between \(\mu\) and any vector \(X\), then we have \(\cos \theta_2 \geq \cos \theta_1\) and \(g(X, \mu) \geq g(X, \rho)\). Therefore from (2.1), we have

\[ S(X, X) \geq (a + b + c + 2d)\{g(X, \rho)\}^2 \]
when \(a, b, c, d\) are positive. A vector field \(V\) in a Riemannian manifold \(M\) is said to be harmonic [5], if

\[ d\omega = 0 \text{ and } \delta\omega = 0, \]
where \(\omega(X) = g(X, V), \forall X\).

It is known [5] that in a compact orientable Riemannian manifold \(M\), the following relation holds for any vector field \(X\).

\[ \int_M [S(X, X) - \frac{1}{2}|d\omega|^2 + |\nabla X|^2 - (\delta\omega)^2] dV = 0, \]
where \(dV\) denotes the volume element of \(M\). Now let us consider a compact, orientable MG(QE)\(_n\)\((n \geq 3)\) without boundary. If in such a manifold, \(X\) is a harmonic vector field, then by (6.2), (6.3) reduces to

\[ \int_M [S(X, X) + |\nabla X|^2] dV = 0. \]
Hence if each of the associated scalars \( a, b, c, d \) of \( MG(QE)_n \) is greater than zero, then using (6.1), it follows from (6.4) that

\[
(6.5) \quad \int_M [(a + b + c + 2d)\{g(X, \rho)\}^2 + |\nabla X|^2]dV \leq 0.
\]

Since \( a+b+c+2d>0 \) from (6.5) we get

\[
(6.6) \quad g(X, \rho) = 0 \text{ and } \nabla X = 0, \; \forall X.
\]

From (6.6) it follows that \( X \) is orthogonal to \( \rho \) and from the second part of it follows that the vector field \( X \) is parallel.

Similarly, if \( \theta_1 \leq \theta_2 \) then we have \( g(X, \mu) = 0 \) and \( \nabla X = 0 \) \( \forall X \). Hence we have:

**Theorem 6.1.** If in a compact, orientable \( MG(QE)_n \) \( (n \geq 3) \) without boundary, each of the associated scalars \( a, b, c, d \) is greater than zero, then any harmonic vector field \( X \) in the \( MG(QE)_n \) is parallel and orthogonal to one of the generators of the manifold which makes greatest angle with the vector \( X \).

7. Relation between the manifold of mixed generalised quasi constant curvature and \( MG(QE)_n \). In this section we consider that \( MG(QE)_n \) is quasi conformally flat. Then from (2.9) we have

\[
R(X, Y, Z, W) = \frac{r}{na}\left[\frac{a'}{n - 1} + 2b'\right] \left[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\right]
\]

\[
- \frac{b'}{a'} [S(Y, Z)g(X, W) - S(X, Z)g(Y, W)]
\]

\[
+ g(Y, Z)S(X, W) - g(X, Z)S(Y, W)
\]

(7.1)

Using (1.3) in (7.1) we get

\[
R(X, Y, Z, W) = \left[\frac{r}{na}\left\{\frac{a'}{n - 1} + 2b'\right\}\right]
\]

\[
- \frac{2ab'}{\alpha} \left[ g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \right]
\]

\[
- \frac{bb'}{\alpha} \left[ g(X, W)A(Y)A(Z) - g(Y, W)A(X)A(Z) \right]
\]

(7.2)
\[ +g(Y, Z)A(X)A(W) - g(X, Z)A(Y)A(W) \]
\[-\frac{b \cdot c}{a}[g(X, W)B(Y)B(Z) - g(Y, W)B(X)B(Z)] \]
\[+g(Y, Z)B(X)B(W) - g(X, Z)B(Y)B(W) \]
\[-\frac{b \cdot d}{a} \{\{A(Y)B(Z) + B(Y)A(Z)\}g(X, W) \]
\[-\{A(X)B(Z) + B(X)A(Z)\}g(Y, W) \]
\[+\{A(X)B(W) + B(X)A(W)\}g(Y, Z) \]
\[-\{A(Y)B(W) + B(Y)A(W)\}g(X, Z) \]

Which shows that the manifold is a mixed generalized quasi-constant curvature. Thus we can state the following theorem

**Theorem 7.1.** A quasi conformally flat mixed generalized quasi-Einstein manifold is a manifold of mixed generalized quasi constant curvature.

From theorem 7.1 we can also have the following corollaries

**Corollary 7.1.** A conformally flat mixed generalized quasi-Einstein manifold is a manifold of mixed generalized quasi constant curvature.

**Corollary 7.2.** A conharmonically flat mixed generalized quasi-Einstein manifold is a manifold of mixed generalized quasi constant curvature.

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**REFERENCES**


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