ULTIMATE BOUNDEDNESS AND PERIODICITY RESULTS FOR CERTAIN THIRD ORDER NONLINEAR MATRIX DIFFERENTIAL EQUATIONS

BY

M.O. OMEIKE

Abstract. This paper extends some known results on the boundedness of solutions and the existence of periodic solutions of certain vector equations to matrix equations.

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1. Introduction. Let \( \mathcal{M} \) denote the space of all real \( n \times n \) matrices, \( \mathbb{R}^n \) the real \( n \)-dimensional Euclidean space and \( \mathbb{R} \) the real line \( -\infty < t < \infty \). We shall be concerned here with certain properties of solutions of differential equations of the form

\[
\ddot{X} + AX + B\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X})
\]

where \( X : \mathbb{R} \to \mathcal{M} \) is the unknown, \( A, B \in \mathcal{M} \) are constants, \( H : \mathcal{M} \to \mathcal{M} \) and \( P : \mathbb{R} \times \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M} \). The specific properties we shall be interested in are the ultimate boundedness of all solutions and the existence of periodic solutions when \( P \) is periodic in \( t \).

In [8], TEJUMOLA establishes conditions under which all solutions of the matrix differential equation,

\[
\ddot{X} + A\dot{X} + H(X) = P(t, X, \dot{X}),
\]

are stable, bounded and periodic (depending on the choice of \( P \)). These results are extended to the equation (1.1).
For the special case in which (1.1) is an $n$-vector equation (so that $X : \mathbb{R} \to \mathbb{R}^n$, $H : \mathbb{R}^n \to \mathbb{R}^n$ and $P : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$) a number of boundedness, stability and existence of periodic solutions results have been established by Ezeilo and Tejumola [4], Afuwape [1], Meng [5] and others for a number of various vector third order differential equations. The conditions obtained in each of these previous investigations are generalizations of the well-known Routh-Hurwitz conditions

$$a > 0, \quad c > 0, \quad ab - c > 0$$

for the stability of the trivial solution of the linear differential equation

$$\ddot{x} + ax + b\dot{x} + cx = 0$$

with constant coefficients. Our present investigations are akin to those of Tejumola [8], Meng [5], Afuwape [1] and we shall provide extensions of their results to matrix differential equations of the form (1.1).

2. Notations and definitions. Some standard matrix notation will be used. For any $X \in \mathcal{M}$, $X^T$ and $x_{ij}$, $i, j = 1, 2, \ldots, n$ denote the transpose and the elements of $X$ respectively while $(x_{ij}) (y_{ij})$ will sometimes denote the product matrix $XY$ of the matrices $X, Y \in \mathcal{M}$. $X_i = (x_{i1}, x_{i2}, \ldots, x_{in})$ and $X^j = (x_{1j}, x_{2j}, \ldots, x_{nj})$ stand for the $i$th row and $j$th column of $X$ respectively and $X = (X_1, X_2, \ldots, X_n)$ is the $n^2$ column vector consisting of the $n$ rows of $X$.

We shall denote by $JH(X)$ the $n^2 \times n^2$ generalised Jacobian matrix associated with the function $H : \mathcal{M} \to \mathcal{M}$ and evaluated at $X$; that is, $JH(X)$ is the matrix associated with the Jacobian determinant $\frac{\partial(H_1, H_2, \ldots, H_n)}{\partial(X_1, X_2, \ldots, X_n)}$. Corresponding to the constant matrix $A \in \mathcal{M}$ we define an $n^2 \times n^2$ matrix $\tilde{A}$ consisting of $n^2$ diagonal $n \times n$ matrix $(a_{ij}I_n) (I_n$ being the unit $n \times n$ matrix) and such that $(a_{ij}I_n)$ belongs to the $i$th $- n$ row and $j$th $- n$ column (that is, counting $n$ at a time) of $\tilde{A}$. In the special case $n = 2$, $\tilde{A}$ is the $4 \times 4$ matrix

$$\begin{pmatrix}
a_{11}I_2 & a_{12}I_2 \\
a_{21}I_2 & a_{22}I_2
\end{pmatrix}.$$

Next we introduce an inner product $\langle \cdot, \cdot \rangle$ and a norm $\| \cdot \|$ on $\mathcal{M}$ as follows. For arbitrary $X, Y \in \mathcal{M}$, $\langle X, Y \rangle = \text{trace } XY^T$. It is easy to check
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that \( \langle X, Y \rangle = \langle Y, X \rangle \) and that \( \| X - Y \|^2 = \langle X - Y, X - Y \rangle \) defines a norm of \( M \). Indeed, \( \| X \| = | X |_{n^2} \) where \( | \cdot |_{n^2} \) denotes the usual Euclidean norm in \( \mathbb{R}^{n^2} \) and \( X \in \mathbb{R}^{n^2} \) is as defined above.

Lastly the symbol \( \delta \), with or without subscripts, denote finite positive constants whose magnitudes depend only on \( A, B, H \) and \( P \). Any \( \delta \), with a subscript, retains a fixed identity throughout while the unnumbered ones are not necessarily the same each time they occur.

3. Statement of results. It will be assumed throughout the sequel that \( H \in C'(M) \) and that \( P \in C(\mathbb{R} \times M \times M \times M) \). Further, \( H \) and \( P \) satisfy conditions for the existence of solutions of (1.1) for any set of preassigned initial conditions.

Theorem 1. Let \( H(0) = 0 \) and suppose that

(i) the Jacobian matrix \( JH(X) \) of \( H(X) \) is symmetric and furthermore that the eigenvalues \( \lambda_i(JH(X)) \) of \( JH(X), (i = 1, 2, \ldots, n) \) satisfy for \( X \in M \),

\[
(3.1) \quad 0 < \delta_h \leq \lambda_i(JH(X)) \leq \Delta_h
\]

where \( \delta_h, \Delta_h \) are finite constants;

(ii) the matrices \( \tilde{A}, \tilde{B}, JH(X) \) are associative and commute pairwise. The eigenvalues \( \lambda_i(A) \) of \( A \) and \( \lambda_i(B) \) of \( B \) (\( i = 1, 2, \ldots, n^2 \)) satisfy

\[
(3.2) \quad 0 < \delta_a \leq \lambda_i(A) \leq \Delta_a
\]

\[
(3.3) \quad 0 < \delta_b < \lambda_i(B) \leq \Delta_b
\]

where \( \delta_a, \delta_b, \Delta_a, \Delta_b \) are finite constants. Furthermore,

\[
(3.4) \quad \Delta_h \leq k\delta_a\delta_b
\]

where
\[
\alpha > 0, 0 < \beta < 1 \text{ are some constants,}
\]

(iii) \( P \) satisfies

\[
\|P(t, X, Y, Z)\| \leq \delta_0 + \delta_1(\|X\| + \|Y\| + \|Z\|)
\]

for all arbitrary \( X, Y, Z \in \mathcal{M} \), where \( \delta_0 \geq 0, \delta_1 \geq 0 \) are constants and \( \delta_1 \) is sufficiently small.

Then every solution \( X(t) \) of (1.1) satisfies

\[
\|X(t)\| \leq \Delta_1, \quad \|\dot{X}(t)\| \leq \Delta_1, \quad \|\ddot{X}(t)\| \leq \Delta_1
\]

for all \( t \) sufficiently large, where \( \Delta_1 \) is a constant the magnitude of which depends only on \( \delta_0, \delta_1, A, B, H \) and \( P \).

This result provides an extension of a result of Afuwape [1] and Meng [5] for an \( n \)-vector.

**Theorem 2.** Suppose, further to the conditions of Theorem 1, that \( P \) satisfies \( P(t, X, Y, Z) = P(t + \omega, X, Y, Z) \) uniformly for all \( X, Y, Z \in \mathcal{M} \). Then (1.1) admits of at least one periodic solution with period \( \omega \).

4. **Some preliminaries.** The following results will be basic to the proofs of Theorems 1 and 2.

**Lemma 1 ([8]).** Let \( H(0) = 0 \) and assume that the matrices \( \tilde{A} \) and \( JH(X) \) are symmetric and commute for all \( X \in \mathcal{M} \). Then

\[
\langle H(X), AX \rangle = \int_0^1 X^T \tilde{A} JH(\sigma X) X d\sigma
\]

**Lemma 2 ([1]).** If \( D \) is a real symmetric \( \ell \times \ell \) matrix, then for any \( X \in \mathbb{R}^\ell \) we have

\[
\delta_d \|X\|^2 \leq \langle DX, X \rangle \leq \Delta_d \|X\|^2,
\]

where \( \delta_d, \Delta_d \) are the least and greatest eigenvalues of \( D \), respectively.

**Lemma 3 ([1]).** Let \( Q, D \) be any two real \( \ell \times \ell \) commuting symmetric matrices. Then
(i) the eigenvalues $\lambda_i(QD)$ $(i = 1, 2, \cdots, \ell)$ of the product matrix $QD$ are all real and satisfy
\[
\max_{1 \leq j, k \leq \ell} \lambda_j(Q)\lambda_k(D) \geq \lambda_i(QD) \geq \min_{1 \leq j, k \leq \ell} \lambda_j(Q)\lambda_k(D);
\]

(ii) the eigenvalues $\lambda_i(Q + D)$ $(i = 1, 2, \cdots, \ell)$ of the sum of matrices $Q$ and $D$ are real and satisfy
\[
\left\{ \max_{1 \leq j \leq \ell} \lambda_j(Q) + \max_{1 \leq k \leq \ell} \lambda_k(D) \right\} \geq \lambda_i(Q + D) \geq \left\{ \min_{1 \leq j \leq \ell} \lambda_j(Q) + \min_{1 \leq k \leq \ell} \lambda_k(D) \right\}.
\]

Proof of Theorem 1. Let us for convenience, replace Eq.(1.1) by the equivalent system form
\[
\begin{align*}
\dot{X} &= Y \\
\dot{Y} &= Z \\
\dot{Z} &= -AZ - BY - H(X) + P(t, X, Y, Z).
\end{align*}
(4.3)
\]

Our main tool in the proof is the scalar Lyapunov function
\[ V : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R} \]
adapted from [5] and defined for any function $X, Y, Z \in \mathcal{M}$ by
\[
2V = \left\{ \langle \beta(1 - \beta)BX, BX \rangle + \langle 2\alpha A^{-1}BY, Y \rangle + \langle \beta BY, Y \rangle \right. \\
\left. + \langle \alpha A^{-1}Z, Z \rangle + \langle \alpha(Z + AY), Y + A^{-1}Z \rangle \\
\langle Z + AY + (1 - \beta)BX, Z + AY + (1 - \beta)BX \rangle \right\}
(4.4)
\]
where $\alpha > 0, \ 0 < \beta < 1$ are some constants. For each term of this function it is clear that
\[
\beta(1 - \beta)\delta_b\|X\|^2 \leq \langle \beta(1 - \beta)BX, BX \rangle
\]
\[
= \beta(1 - \beta)\sum_{i=1}^{n} |BX_i|^2
\leq \beta(1 - \beta)\Delta_b\|X\|^2,
(4.4a)
\]
\[
2\alpha \Delta_a^{-1}\delta_b\|Y\|^2 \leq \langle 2\alpha A^{-1}BY, Y \rangle = 2\alpha \sum_{i=1}^{n} |A^{-1}BY_i|^2
\leq 2\alpha \delta_a^{-1}\Delta_b\|Y\|^2.
(4.4b)
\]
In a similar manner,

\[(4.4c)\quad \beta \delta_b \|Y\| \leq \langle \beta BY, Y \rangle = \beta \sum_{i=1}^{n} |BY|^2 \leq \beta \Delta_b \|Y\|^2,\]

\[(4.4d)\quad \alpha \Delta_a^{-1} \|Z\|^2 \leq \langle \alpha A^{-1} Z, Z \rangle \leq \alpha \delta_a^{-1} \|Z\|^2,\]

\[(4.4e)\quad 0 \leq \langle \alpha (Z + AY), Y + A^{-1} Z \rangle \leq \nu (\|Y\|^2 + \|Z\|^2),\]

and

\[(4.4f)\quad 0 \leq \langle Z + AY + (1 - \beta)BX, Z + AY + (1 - \beta)BX \rangle \leq \sum_{i=1}^{n} |Z_i + AY_i + (1 - \beta)BX_i|^2 \leq \mu (\|Z\|^2 + \|Y\|^2 + \|X\|^2),\]

for some positive constants \(\nu, \mu\). The estimates above are valid since

\[\sum_{i=1}^{n} |X_i|^2 = \sum_{i=1}^{n} |X|^2 \leq |X|^2 \quad \text{for any } X \in \mathcal{M}.\]

Combining these estimates (4.4a – 4.4f) in (4.4) we obtain that

\[(4.5)\quad \delta_2 (\|X\|^2 + \|Y\|^2 + \|Z\|^2) \leq 2V \leq \delta_3 (\|X\|^2 + \|Y\|^2 + \|Z\|^2),\]

\[\delta_2 = \min\{\beta (1 - \beta) \delta_b; 2\alpha \Delta_a^{-1} \delta_b + \beta \delta_b; \alpha \Delta_a^{-1}\}\]

and

\[\delta_3 = \max\{\beta (1 - \beta) \Delta_b + \mu; 2\alpha \Delta_a^{-1} \Delta_b + \beta \Delta_b + \nu + \mu; \alpha \delta_a^{-1} + \nu + \mu\}.\]

From (4.5), we have that \(V(X, Y, Z) \to \infty\) as \(\|X\|^2 + \|Y\|^2 + \|Z\|^2 \to \infty\). To prove our result, it suffices to prove that there exists a constant \(\Delta_1 > 0\) such that

\[(4.6)\quad \|X\|^2 + \|Y\|^2 + \|Z\|^2 \leq \Delta_1, \quad \text{for } t \geq T(X_0, Y_0, Z_0),\]

for any solution \((X, Y, Z)\) for (4.3), \((X_0 = X(0), Y_0 = Y(0), Z_0 = Z(0))\).
Let \((X, Y, Z)\) be any solution of (4.3). Then the total derivative of \(V\) with respect to \(t\) along this solution path is

\[(4.7) \quad \dot{V} = -U_1 - U_2 - U_3 + U_4\]

where

\[
U_1 = \left\langle \frac{1 - \beta}{2} BX, H(X) \right\rangle + \langle \beta ABY, Y \rangle + \left\langle \frac{\alpha}{2} Z, Z \right\rangle
\]

\[
U_2 = \left\langle \frac{1 - \beta}{2} BY, H(X) \right\rangle + \langle \alpha BY, Y \rangle + \langle (A + \alpha I)Y, H(X) \rangle
\]

\[
U_3 = \left\langle \frac{1 - \beta}{4} BX, H(X) \right\rangle + \left\langle \frac{\alpha}{2} Z, Z \right\rangle + \langle (I + 2\alpha A^{-1})Z, H(X) \rangle
\]

\[
U_4 = \langle (1 - \beta)BX + (A + \alpha I)Y + (I + 2\alpha I)Y + (I + 2\alpha A^{-1})Z, P(t, X, Y, Z) \rangle.
\]

To arrive at (4.6), we first prove the following:

**Lemma 4.** Subject to a conveniently chosen value for \(k\) in (3.5), we have for all \(X, Y, Z\)

\[U_j \geq 0, \quad (j = 2, 3).\]

**Proof of Lemma 4.** For strictly positive constants \(k_1, k_2\) conveniently chosen later, we have

\[(4.8a) \quad \langle (\alpha I + A)Y, H(X) \rangle
\]

\[= \left\| k_1 (\alpha I + A)^{1/2} Y + 2^{-1}k_1^{-1}(\alpha I + A)^{1/2} H(X) \right\|^2
\]

\[-\langle k_1^{-1} (\alpha I + A)Y, Y \rangle
\]

\[-4^{-1}k_1^{-2}((\alpha I + A)H(X), H(X))\]

and

\[(4.8b) \quad \langle (I + 2\alpha A^{-1})Z, H(X) \rangle
\]

\[= \left\| k_2 (I + 2\alpha A^{-1})^{1/2} Z + 2^{-1}k_2^{-1}(I + 2\alpha A^{-1})^{1/2} H(X) \right\|^2
\]

\[-\langle k_2^{-1} (I + 2\alpha A^{-1})Z, Z \rangle
\]

\[-\langle 4^{-1}k_2^{-2}(I + 2\alpha A^{-1})H(X), H(X) \rangle,\]
thus,

\[
U_2 = \|k_1(\alpha I + A)^{1/2}Y + 2^{-1}k_1^{-1}(\alpha I + A)^{-1/2}H(X)\|^2 \\
+ (4^{-1}(1 - \beta)BX - 4^{-1}k_1^{-1}(\alpha I + A)H(X), H(X)) \\
+ \langle (\alpha B - k_1^2(\alpha I + A)Y, Y \rangle
\]

and

\[
U_3 = \|k_2(I + 2\alpha A^{-1})^{1/2}Z + 2^{-1}k_2^{-1}(I + 2\alpha A^{-1})^{-1/2}H(X)\|^2 \\
+ \langle (1 - \beta)4^{-1}BX - 4^{-1}k_2^{-2}(I + 2\alpha A^{-1})H(X), H(X) \rangle \\
+ \langle \left[ \frac{\alpha}{2} I - k_2^2(I + 2\alpha A^{-1}) \right] Z, Z \rangle
\]

By Lemmas 1, 2 and 3, we obtain

\[
U_2 \geq \left\{ \int_0^1 \sigma \int_0^1 X^T \left[ \frac{1 - \beta}{4} \tilde{B} \\
- \frac{1}{4k_1^2} \left( \alpha \tilde{I} + \tilde{A} \right) JH(\sigma X) \right] JH(\tau \sigma X) X d\tau d\sigma \\
+ Y^T \left[ \alpha \tilde{B} - k_1^2(\alpha \tilde{I} + \tilde{A}) \right] Y \right\}.
\]

and

\[
U_3 \geq \left\{ \int_0^1 \sigma \int_0^1 X^T \left[ \frac{1 - \beta}{4} \tilde{B} \\
- \frac{1}{4k_2^2} \left( \alpha \tilde{I} + 2\alpha \tilde{A}^{-1} \right) JH(\sigma X) \right] JH(\tau \sigma X) X d\tau d\sigma \\
+ Z^T \left[ \alpha \tilde{I} - k_2^2(\tilde{I} + 2\alpha \tilde{A}^{-1}) \right] Z \right\}.
\]

Furthermore, by using Lemmas 2 and 3, we obtain

\[
U_2 \geq \left\{ \delta_h \left[ \frac{1 - \beta}{4} \delta_b - \frac{1}{4k_1^2}(\alpha + \Delta_a) \right] \|X\|^2 \\
+ \left[ \alpha \delta_b - k_1^2(\alpha + \Delta_a) \right] \|Y\|^2 \right\},
\]

and

\[
U_3 \geq \left\{ \delta_h \left[ \frac{1 - \beta}{4} \delta_b - \frac{1}{4k_2^2}(1 + 2\alpha \delta_a^{-1}) \right] \|X\|^2 \\
+ \left[ \frac{\alpha}{2} - k_2^2(1 + 2\alpha \delta_a^{-1}) \right] \|Z\|^2 \right\}.
\]
Thus, we obtain, for all $X, Y$ in $M$,

\[(4.10a) \quad U_2 \geq 0\]

if $k^2_1 \leq \frac{\alpha \delta_b}{\alpha + \Delta_a}$ with

\[(4.11a) \quad \Delta_h \leq \frac{k^2_1 (1 - \beta) \delta_b}{(\alpha + \Delta_a)} \leq \frac{\alpha (1 - \beta) \delta_b^2}{(\alpha + \Delta_a)^2}\]

and for all $X, Z$ in $M$,

\[(4.10b) \quad U_3 \geq 0\]

if $k^2_2 \leq \frac{\alpha \delta_a}{2(\alpha + 2\alpha)}$ with

\[(4.11b) \quad \Delta_h \leq \frac{k^2_2 (1 - \beta) \delta_a \delta_b}{2(\alpha + \delta_a)} \leq \frac{\alpha (1 - \beta) \delta_a^2 \delta_b}{2(2\alpha + \delta_a)^2}.\]

Combining all the inequalities in (4.10) and (4.11), we have for all $X, Y, Z$ in $M$, $U_2 \geq 0$ and $U_3 \geq 0$, if

\[\Delta_h \leq k \delta_a \delta_b\]

with

\[(4.12) \quad k = \min \left\{ \frac{\alpha (1 - \beta) \delta_b}{\delta_a (\alpha + \Delta_a)^2}; \frac{\alpha (1 - \beta) \delta_a}{2(\alpha + 2\alpha)^2} \right\} < 1.\]

This completes the proof of Lemma $4$. \hfill \Box

Finally, we are left with estimates for $U_1$ and $U_4$. From (4.7), we clearly have

\[(4.13) \quad U_1 = \frac{1 - \beta}{2} \int_0^1 X^T B J H (\sigma X) X \, d\sigma + \beta Y^T \hat{A} \hat{B} Y + \frac{\alpha}{2} Z^T Z\]

\[\geq \frac{1 - \beta}{2} \delta_b \delta_h \|X\|^2 + \beta \delta_a \delta_b \|Y\|^2 + \frac{\alpha}{2} \|Z\|^2\]

\[\geq \delta_4 (\|X\|^2 + \|Y\|^2 + \|Z\|^2)\]
where \( \delta_4 = \min \left\{ \frac{\delta_b (1 - \beta)}{\beta \delta_b; \frac{\alpha}{2}} \right\} \). Since \( P(t, X, Y, Z) \) satisfies (3.6), by Schwarz’s inequality, we obtain

$$
\begin{align*}
|U_4| & \leq \left\{ (1 - \beta) \Delta_b \|X\| + (\alpha + \Delta_a) \|Y\| \\
& \quad + (1 + 2\alpha \delta^{-1}_a) \|Z\| \right\} \|P(t, X, Y, Z)\| \\
& \leq \delta_5 (\|X\| + \|Y\| + \|Z\|)[\delta_0 + \delta_1 (\|X\| + \|Y\| + \|Z\|)] \\
& \leq 3\delta \delta_5 (\|X\|^2 + \|Y\|^2 + \|Z\|^2) \\
& \quad + 3^{1/2} \delta_0 \delta_5 (\|X\|^2 + \|Y\|^2 + \|Z\|^2)^{1/2},
\end{align*}
$$

(4.14)

where \( \delta_5 = \max\{ (1 - \beta) \Delta_b; (\alpha + \Delta_a); (1 + 2\alpha \delta^{-1}_a) \} \).

Combining inequalities (4.10), (4.13) and (4.14) in (4.7), we obtain

$$
\dot{V} \leq -2\delta_6 (\|X\|^2 + \|Y\|^2 + \|Z\|^2) + \delta_7 (\|X\|^2 + \|Y\|^2 + \|Z\|^2)^{1/2},
$$

(4.15)

where \( \delta_6 = \frac{1}{2} (\delta_4 - 3\delta_1 \delta_5) \), \( \delta_1 < 3^{-1} \delta_5^{-1} \delta_4 \), \( \delta_7 = 3^{1/2} \delta_0 \delta_5 \).

If we choose \( (\|X\|^2 + \|Y\|^2 + \|Z\|^2)^{1/2} \geq \delta_8 = \delta_7 \delta_5^{-1} \), inequality (4.15) implies that

$$
\dot{V} \leq -\delta_0 (\|X\|^2 + \|Y\|^2 + \|Z\|^2).
$$

(4.16)

Then there exists \( \delta_9 \) such that

$$
\dot{V} \leq -1 \text{ if } \|X\|^2 + \|Y\|^2 + \|Z\|^2 \geq \delta_9^2.
$$

The remainder of the proof of Theorem 1 may now be obtained by use of the estimates (4.5) and (4.16) and an obvious adaptation of the Yoshizawa type reasoning employed in [5].

5. Proof of Theorem 2. The proof of this theorem follows as in the proof of Theorem 3([5]).

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Department of Mathematical Sciences,
University of Agriculture,
Abeokuta,
NIGERIA
moomeike@yahoo.com