GENERAL NATURAL KÄHLER STRUCTURES OF
CONSTANT HOLOMORPHIC SECTIONAL CURVATURE
ON TANGENT BUNDLES

BY

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Dedicated to Prof. R. Miron on the occasion of his 80-th birthday
and to the memory of
Prof. Gh. Gheorghiev on the occasion of his 100-th birthday

Abstract. We study some properties of the curvature tensor field of the natural
Kählerian structure \((G, J)\) of general type on \(TM\). Namely, we are interested in finding
the conditions under which the Kählerian manifold \((TM, G, J)\) has constant holomorphic
sectional curvature. The main result is that a certain proportionality factor is expressed
as a rational function of the two essential parameters, involved in the definition of the
integrable almost complex structure \(J\) on \(TM\), their derivatives and the values of the
constant sectional curvature of the base manifold \((M, g)\) and the constant holomorphic
sectional curvature of \((TM, G, J)\).

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1. Introduction. The tangent bundle \(TM\) of a Riemannian manifold \((M, g)\) has many nice geometric properties, and furnishes important examples arising in various geometric classifications.

It is well known (see [17], [21], [22]) that the splitting of the tangent bundle to \(TM\) into the vertical and horizontal distributions, defined by the

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Levi Civita connection of $g$ on $M$, and the corresponding Sasaki metric lead to an almost Kähler structure on $TM$. Moreover, the possibility to consider vertical, complete and horizontal lifts on $TM$ leads to interesting geometric structures, studied in the last years (see [1], [2], [7], [8], [19], [20]), and to interesting relations with some problems in Lagrangian and Hamiltonian mechanics.

The second author has studied some properties of a natural lift $G$, of diagonal type, of the Riemannian metric $g$ and a natural almost complex structure $J$ of diagonal type on $TM$ (see [12], [11], [10], and see also [14], [15]). The condition for $(TM, G, J)$ to be a Kähler Einstein manifold leads to the conditions for $(M, g)$ to have constant sectional curvature, and for $(TM, G, J)$ to have constant sectional holomorphic curvature or to be a locally symmetric space.

In the paper [9], the second author has presented a general expression of the natural almost complex structures on $TM$. In the definition of the natural almost complex structure $J$ of general type there are involved eight parameters (smooth functions of the density energy on $TM$). However, from the condition for $J$ to define an almost complex structure, four of the above parameters can be expressed as (rational) functions of the other four parameters. A Riemannian metric $G$ which is a natural lift of general type of the metric $g$ depends on other six parameters. From the conditions for $G$ to be Hermitian with respect to $J$, one gets that these six parameters can be expressed with the help of the first eight parameters involved in definition of $J$ and two proportionality factors. From the integrability condition for $J$, we get (beside the condition for the base manifold to be of constant sectional curvature) that other two parameters involved in the definition of $J$ can be expressed as functions of other two essential parameters and their first order derivatives. Thus a natural Hermitian structure $(G, J)$ of general type depends on four essential parameters (two essential parameters involved in the definition of the integrable almost complex structure $J$ and two proportionality factors). From the condition for $(G, J)$ to be almost Kählerian, we get that the second proportionality factor is the derivative of the first one. The family of natural Kählerian structures $(G, J)$ of general type on $TM$ depends on three essential coefficients (two are involved in the expression of $J$, and the third one is the first proportionality coefficient).

In the present paper we study some properties of the curvature tensor field of the natural Kählerian structure $(G, J)$ of general type on $TM$. Namely, we are interested in finding the conditions under which the Kähle-
rian manifold \((TM, G, J)\) has constant holomorphic sectional curvature. The main result is that the first proportionality factor is expressed as a rational function of the first two essential parameters, their derivatives and the values of the constant sectional curvature of the base manifold \((M, g)\) and the constant holomorphic sectional curvature of \((TM, G, J)\). Some quite long computations have been done by using the Mathematica package RICCI for doing tensor calculations.

The manifolds, tensor fields and other geometric objects we consider in this paper are assumed to be differentiable of class \(C^\infty\) (i.e. smooth). We use the computations in local coordinates in a fixed local chart but many results may be expressed in an invariant form by using the vertical and horizontal lifts. The well known summation convention is used throughout this paper, the range of the indices \(h, i, j, k, l, r\) being always \(\{1, \ldots, n\}\).

2. Preliminary results. Let \((M, g)\) be a smooth \(n\)-dimensional Riemannian manifold and denote its tangent bundle by \(\tau : TM \rightarrow M\). Recall that \(TM\) has a structure of a \(2n\)-dimensional smooth manifold, induced from the smooth manifold structure of \(M\). This structure is obtained by using local charts on \(TM\) induced from usual local charts on \(M\). If \((U, \varphi) = (U, x^1, \ldots, x^n)\) is a local chart on \(M\), then the corresponding induced local chart on \(TM\) is \((\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \ldots, x^n, y^1, \ldots, y^n)\), where the local coordinates \(x^i, y^j, i, j = 1, \ldots, n\), are defined as follows. The first \(n\) local coordinates of a tangent vector \(y \in \tau^{-1}(U)\) are the local coordinates in the local chart \((U, \varphi)\) of its base point, i.e. \(x^i = x^i \circ \tau\), by an abuse of notation. The last \(n\) local coordinates \(y^j, j = 1, \ldots, n\), of \(y \in \tau^{-1}(U)\) are the vector space coordinates of \(y\) with respect to the natural basis in \(T_{\tau(y)}M\) defined by the local chart \((U, \varphi)\). Due to this special structure of differentiable manifold for \(TM\), it is possible to introduce the concept of \(M\)-tensor field on it. The \(M\)-tensor fields are defined by their components with respect to the induced local charts on \(TM\) (hence they are defined locally), but they can be interpreted as some (partial) usual tensor fields on \(TM\). However, the essential quality of an \(M\)-tensor field on \(TM\) is that the local coordinate change rule of its components with respect to the change of induced local charts is the same as the local coordinate change rule of the components of a usual tensor field on \(M\) with respect to the change of local charts on \(M\). More precisely, an \(M\)-tensor field of type \((p, q)\) on \(TM\) is defined by sets of \(n^{p+q}\) components (functions depending on \(x^i\) and \(y^j\)), with \(p\) upper indices and \(q\) lower indices, assigned to induced
local charts \( (\tau^{-1}(U), \Phi) \) on \( TM \), such that the local coordinate change rule of these components (with respect to induced local charts on \( TM \)) is that of the local coordinate components of a tensor field of type \((p, q)\) on the base manifold \( M \) (with respect to usual local charts on \( M \)), when a change of local charts on \( M \) (and hence on \( TM \)) is performed (see [6] for further details); e.g., the components \( y^i, i = 1, \ldots, n \), corresponding to the last \( n \) local coordinates of a tangent vector \( y \), assigned to the induced local chart \( (\tau^{-1}(U), \Phi) \) define an \( M \)-tensor field of type \((1, 0)\) on \( TM \). A usual tensor field of type \((p, q)\) on \( M \) may be thought of as an \( M \)-tensor field of type \((p, q)\) on \( TM \). If the considered tensor field on \( M \) is covariant only, the corresponding \( M \)-tensor field on \( TM \) may be identified with the induced (pullback by \( \tau \)) tensor field on \( TM \). Some useful \( M \)-tensor fields on \( TM \) may be obtained as follows. Let \( u : [0, \infty) \to \mathbb{R} \) be a smooth function and let \( \|y\|^2 = g_{\tau(y)}(y, y) \) be the square of the norm of the tangent vector \( y \in \tau^{-1}(U) \). If \( \delta_j^i \) are the Kronecker symbols (in fact, they are the local coordinate components of the identity tensor field \( I \) on \( M \)), then the components \( u(\|y\|^2)\delta_j^i \) define an \( M \)-tensor field of type \((1, 1)\) on \( TM \). Similarly, if \( g_{ij}(x) \) are the local coordinate components of the metric tensor field \( g \) on \( M \) in the local chart \((U, \varphi)\), then the components \( u(\|y\|^2)g_{ij} \) define a symmetric \( M \)-tensor field of type \((0, 2)\) on \( TM \). The components \( g_{0i} = y^k g_{ki} \) define an \( M \)-tensor field of type \((0, 1)\) on \( TM \).

Denote by \( \nabla \) the Levi Civita connection of the Riemannian metric \( g \) on \( M \). Then we have the direct sum decomposition

\[
(1) \quad TT M = VTM \oplus HTM
\]

of the tangent bundle to \( TM \) into the vertical distribution \( VTM = \text{Ker} \tau_* \) and the horizontal distribution \( HTM \) defined by \( \nabla \). The set of vector fields \( \left( \frac{\partial}{\partial y^i}, \ldots, \frac{\partial}{\partial y^n} \right) \) on \( \tau^{-1}(U) \) defines a local frame field for \( VTM \) and for \( HTM \) we have the local frame field \( \left( \frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n} \right) \), where

\[
\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{0i} \frac{\partial}{\partial y^h}, \quad \Gamma^h_{0i} = y^k \Gamma^h_{ki},
\]

and \( \Gamma^h_{ki}(x) \) are the Christoffel symbols of \( g \).

The set \( \left( \frac{\partial}{\partial y^i}, \ldots, \frac{\partial}{\partial y^n}, \frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n} \right) \) defines a local frame on \( TM \), adapted to the direct sum decomposition (1). Remark that

\[
\frac{\partial}{\partial y^i} = \left( \frac{\partial}{\partial x^i} \right)^V, \quad \frac{\delta}{\delta x^i} = \left( \frac{\partial}{\partial x^i} \right)^H,
\]
where $X^V$ and $X^H$ denote the vertical and horizontal lift of the vector field $X$ on $M$ respectively. We can use the vertical and horizontal lifts in order to obtain invariant expressions for some results in this paper. However, we should prefer to work in local coordinates since the formulas are obtained easier and, in a certain sense, they are more natural.

We can easily obtain the following

**Lemma 2.1.** If $n > 1$ and $u, v$ are smooth functions on $TM$ such that

$$ug_{ij} + vg_{0i}g_{0j} = 0,$$

on the domain of any induced local chart on $TM$, then $u = 0, v = 0$.

**Remark.** In a similar way we obtain from the condition

$$u\delta^i_j + vg_{0j}y^i = 0$$

the relations $u = v = 0$.

Consider the energy density of the tangent vector $y$ with respect to the Riemannian metric $g$

$$t = \frac{1}{2}\|y\|^2 = \frac{1}{2}g_{\tau(y)}(y, y) = \frac{1}{2}g_{ik}(x)y^i y^k, \ y \in \tau^{-1}(U).$$

Obviously, we have $t \in [0, \infty)$ for all $y \in TM$.

Denote by $C = y^i \frac{\partial}{\partial y^i}$ the Liouville vector field on $TM$ and by $\tilde{C} = y^i \frac{\delta}{\delta x^i}$ the similar horizontal vector field on $TM$. Consider the real valued smooth functions $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ defined on $[0, \infty) \subset \mathbb{R}$. A natural 1-st order almost complex structure on $TM$, defined by the Riemannian metric $g$, is obtained just like the natural 1-st order lifts of $g$ to $TM$ are obtained in [5], [4].

**Theorem 2.2.** The natural tensor field $J$ of type $(1, 1)$ on $TM$, given by

$$J \frac{\delta}{\delta x^i} = a_1(t) \frac{\partial}{\partial y^i} + b_1(t)g_{0i}C + a_4(t)\frac{\delta}{\delta x^i} + b_4(t)g_{0i}\tilde{C},$$

and

$$J \frac{\partial}{\partial y^i} = a_3(t)\frac{\partial}{\partial y^i} + b_3(t)g_{0i}C - a_2(t)\frac{\delta}{\delta x^i} - b_2(t)g_{0i}\tilde{C},$$

defines an almost complex structure on $TM$, if and only if $a_4 = -a_3, b_4 = -b_3$ and the coefficients $a_1, a_2, a_3, b_1, b_2$ and $b_3$ are related by

$$a_1a_2 = 1 + a_3^2, \quad (a_1 + 2tb_1)(a_2 + 2tb_2) = 1 + (a_3 + 2tb_3)^2.$$
Remark. From the conditions (4) we have that the coefficients $a_1, a_2, a_1 + 2tb_1, a_2 + 2tb_2$ cannot vanish and have the same sign. We assume that $a_1 > 0, a_2 > 0, a_1 + 2tb_1 > 0, a_2 + 2tb_2 > 0$ for all $t \geq 0$.

Remark. The relations (4) allow us to express two of the coefficients $a_1, a_2, a_3, b_1, b_2, b_3$ as functions of the other four; e.g. we have:

\begin{align*}
(5) \quad a_2 &= \frac{1 + a_3^2}{a_1}, \quad b_2 = \frac{2a_3b_3 - a_2b_1 + 2tb_3^2}{a_1 + 2tb_1}.
\end{align*}

The integrability condition for the above almost complex structure $J$ on a manifold $M$ is characterized by the vanishing of its Nijenhuis tensor field $N_J$, defined by

\begin{align*}
N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y],
\end{align*}

for all vector fields $X$ and $Y$ on $M$.

The integrability conditions obtained in [9] are given in the theorem:

**Theorem 2.3.** Let $(M, g)$ be an $n(> 2)$-dimensional connected Riemannian manifold. The almost complex structure $J$ defined by (3) on $TM$ is integrable if and only if $(M, g)$ has constant sectional curvature $c$ and the coefficients $b_1, b_2, b_3$ are given by:

\begin{align*}
(6) \quad \begin{cases} 
  b_1 = \frac{2c^2ta_2^2 + 2cta_1a'_2 + a_1a'_1 - c + 3ca^2}{a_1 - 2ta_1 - 2cta_2 - 4ct^2a_2^2}, \\
  b_2 = \frac{2ta_1^2 - 2ta_1a'_1 + ca_1^2 + 2cta_2a'_2 + a_1a'_2}{a_1 - 2ta_1 - 2cta_2 - 4ct^2a_2^2}, \\
  b_3 = \frac{a_1a'_1^2 + 2a_2a_1a'_2 + 4cta_2a_3 - 2cta_2a'_3}{a_1 - 2ta_1 - 2cta_2 - 4ct^2a_2^2}.
\end{cases}
\end{align*}

Remark. The second relation in (5) (or in (4)) is identically fulfilled by the expressions $b_1, b_2, b_3$ in (6).

Remark. In the case where $a_3 = 0$ it follows $b_3 = 0$ too, and we have:

\begin{align*}
 a_2 &= \frac{1}{a_1}, \quad b_1 = \frac{a_1a'_1 - c}{a_1 - 2ta'_1}, \quad b_2 = \frac{c - a_1a'_1}{a_1(a_1^2 - 2ct)}
\end{align*}

(compare with the corresponding expressions from [11] and [18]).
In the paper [9], the second author studied the conditions under which a Riemannian metric $G$ of natural type on $TM$, defined by

$$G\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = c_1 g_{ij} + d_1 g_0i g_0j = G^{(1)}_{ij},$$

and

$$G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = c_2 g_{ij} + d_2 g_0i g_0j = G^{(2)}_{ij},$$

and

$$G\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = c_3 g_{ij} + d_3 g_0i g_0j = G^{(3)}_{ij},$$

where $c_1, c_2, c_3, d_1, d_2, d_3$ are six smooth functions of the density energy on $TM$, is almost Hermitian with respect to the general almost complex structure $J$, i.e.

$$G(JX, JY) = G(X, Y),$$

for all vector fields $X, Y$ on $TM$. He proved the following result

**Theorem 2.4.** The family of natural, Riemannian metrics $G$ on $TM$ such that $(TM, G, J)$ is an almost Hermitian manifold, is given by (7), provided that the coefficients $c_1, c_2, c_3, d_1, d_2, d_3$ are related to the coefficients $a_1, a_2, a_3, b_1, b_2, b_3$ by the following proportionality relations

$$\frac{c_1}{a_1} = \frac{c_2}{a_2} = \frac{c_3}{a_3} = \lambda$$

$$\frac{c_1 + 2td_1}{a_1 + 2tb_1} = \frac{c_2 + 2td_2}{a_2 + 2tb_2} = \frac{c_3 + 2td_3}{a_3 + 2tb_3} = \lambda + 2t\mu,$$

where the proportionality coefficients $\lambda > 0$ and $\lambda + 2t\mu > 0$ are functions depending on $t$.

**Remark.** In the case where $a_3 = 0$, it follows that $c_3 = d_3 = 0$ and we obtain the almost Hermitian structure considered in [18]. Remark that the functions used in [18] are slightly different of the functions used in the present paper. Moreover, if $\lambda = 1$ and $\mu = 0$, we obtain the almost Kählerian structure considered in [12].

Considering the two-form $\Omega$ defined by the almost Hermitian structure $(G, J)$ on $TM$

$$\Omega(X, Y) = G(X, JY),$$

for all vector fields $X, Y$ on $TM$, the second author obtained the following result:
Proposition 2.5. The expression of the 2-form $\Omega$ in the local adapted frame $\left( \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}, \frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n} \right)$ on $TM$, is given by

$$\Omega\left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = 0, \ \Omega\left( \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right) = 0, \ \Omega\left( \frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j} \right) = \lambda g_{ij} + \mu g_{0i}g_{0j}$$

or, equivalently

$$\Omega = (\lambda g_{ij} + \mu g_{0i}g_{0j}) \tilde{\nabla} y^i \wedge dx^j,$$

where $\tilde{\nabla} y^i = dy^i + \Gamma^i_{0h} dx^h$ is the absolute differential of $y^i$.

Next, by calculating the exterior differential of $\Omega$, he obtained:

Theorem 2.6. The almost Hermitian structure $(TM, G, J)$ is almost Kählerian if and only if

$$\mu = \lambda'.$$

Thus the family of general almost Kählerian structures on $TM$ depends on five essential coefficients $a_1, a_3, b_1, b_3, \lambda$. Combining the results from the theorems (2.4), (2.3) and (2.6), we obtain that a general natural Kählerian structure $(G, J)$ on $TM$ is defined by three essential coefficients $a_1, a_3, \lambda$. However, these coefficients must satisfy the supplementary conditions $a_1 > 0, a_1 + 2t b_1 > 0, \lambda > 0, \lambda + 2t \mu > 0$. Examples of such structures can be found in [18] (see also [12]).

3. General natural Kähler structures of constant holomorphic sectional curvature on tangent bundles. The Levi-Civita connection $\nabla$ of the Riemannian manifold $(TM, G)$ is obtained from the formula

$$2G(\nabla_X Y, Z) = X(G(X, Z)) + Y(G(X, Z)) - Z(G(X, Y)) + G([X, Y], Z) - G([X, Z], Y) - G([Y, Z], X); \ \forall X, Y, Z \in \chi(M)$$

and is characterized by the conditions

$$\nabla G = 0, \ T = 0,$$

where $T$ is the torsion tensor of $\nabla$. 
In the case of the tangent bundle $TM$ we can obtain the explicit expression of $\nabla$. The symmetric $2n \times 2n$ matrix 

\[
\begin{pmatrix}
G^{(1)}_{ij} & G^{(3)}_{ij} \\
G^{(3)}_{ij} & G^{(2)}_{ij}
\end{pmatrix}
\]

associated to the metric $G$ in the base $(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n})$ has the inverse 

\[
\begin{pmatrix}
H^{(1)}_{ij} & H^{(3)}_{ij} \\
H^{(3)}_{ij} & H^{(2)}_{ij}
\end{pmatrix}
\]

where the entries are the blocks 

\[
H^{(k)}_{ij} = p^k g_{ij}^{(k)} + q^k y_i y_j
\]

Here $g^{kl}$ are the components of the inverse of the matrix $(g_{ij})$ and $p_1, q_1, p_2, q_2, p_3, q_3 : [0, \infty) \to \mathbb{R}$, some real smooth functions. Their expressions are obtained by solving the system:

\[
\begin{align*}
G^{(1)}_{ih} H^{(k)}_{hk} &+ G^{(3)}_{ih} H^{(k)}_{h(3)} = \delta^k_i \\
G^{(1)}_{ih} H^{(k)}_{h(3)} &+ G^{(3)}_{ih} H^{(k)}_{h(2)} = 0 \\
G^{(3)}_{ih} H^{(k)}_{h(1)} &+ G^{(2)}_{ih} H^{(k)}_{h(3)} = 0 \\
G^{(3)}_{ih} H^{(k)}_{h(3)} &+ G^{(2)}_{ih} H^{(k)}_{h(2)} = \delta^k_i ,
\end{align*}
\]

in which we substitute the relations (7) and (9). By using Lemma 1, we get $p_1, p_2, p_3$ as functions of $c_1, c_2, c_3$

\[
p_1 = \frac{c_2}{c_1 c_2 - c_1^2}, \quad p_2 = \frac{c_1}{c_1 c_2 - c_3^2}, \quad p_3 = -\frac{c_3}{c_1 c_2 - c_3^2}
\]

and $q_1, q_2, q_3$ as functions of $c_1, c_2, c_3, d_1, d_2, d_3, p_1, p_2, p_3$

\[
q_1 = -\frac{c_2 d_1 p_1 - c_3 d_3 p_1 - c_3 d_2 p_3 + c_2 d_3 p_3 + 2d_1 d_2 p_1 t - 2d_2^2 p_1 t}{c_1 c_2 - c_3^2 + 2c_2 d_1 t + 2c_1 d_2 t - 4c_3 d_3 t + 4d_1 d_2 t^2 - 4d_3^2 t^2}
\]

\[
q_2 = -\frac{d_2 p_2 + d_3 p_3}{c_2 + 2d_2 t}
\]
\[ q_3 = -\frac{(d_3 p_1 + d_3 p_2)(c_1 + 2d_1 t) - (d_1 p_1 + d_3 p_3)(c_3 + 2d_3 t)}{(c_1 + 2d_1 t)(c_2 + 2d_2 t) - (c_3 + 2d_3 t)^2}, \]

Next we can obtain the expression of the Levi Civita connection of the Riemannian metric \( G \) on \( TM \).

**Theorem 3.1.** The Levi-Civita connection \( \nabla \) of \( G \) has the following expression in the local adapted frame \( \left( \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}, \frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n} \right) \)

\[
\begin{align*}
\nabla_{\delta \partial} \frac{\partial}{\partial y^i} &= Q^h_{ij} \frac{\partial}{\partial y^h} + \tilde{Q}^h_{ij} \frac{\delta}{\delta x^h}, \\
\nabla_{\frac{\delta}{\delta x^j}} \frac{\partial}{\partial y^i} &= (\Gamma^h_{ij} + \tilde{P}^h_{ij}) \frac{\partial}{\partial y^h} + P^h_{ij} \frac{\delta}{\delta x^h}, \\
\nabla_{\frac{\delta}{\delta y^i}} \frac{\delta}{\delta x^j} &= (\Gamma^h_{ij} + \tilde{S}^h_{ij}) \frac{\delta}{\delta y^h} + S^h_{ij} \frac{\partial}{\partial y^h},
\end{align*}
\]

where \( \Gamma^h_{ij} \) are the Christoffel symbols of the connection \( \nabla \) and \( M \)-tensor fields appearing as coefficients in the above expressions are given as

\[
\begin{align*}
Q^h_{ij} &= \frac{1}{2}(\partial_h G^{(2)}_{jk} + \partial_j G^{(2)}_{ik} - \partial_k G^{(2)}_{ij}) H^{kh}_{(2)} + \frac{1}{2}(\partial_l G^{(3)}_{jk} + \partial_j G^{(3)}_{lk}) H^{kh}_{(3)}, \\
\tilde{Q}^h_{ij} &= \frac{1}{2}(\partial_h G^{(2)}_{jk} - \partial_j G^{(2)}_{ik} + \partial_k G^{(2)}_{ij}) H^{kh}_{(2)} + \frac{1}{2}(\partial_l G^{(3)}_{jk} - \partial_j G^{(3)}_{lk}) H^{kh}_{(3)}, \\
P^h_{ij} &= \frac{1}{2}(\partial_h G^{(3)}_{jk} - \partial_k G^{(3)}_{ij}) H^{kh}_{(3)} + \frac{1}{2}(\partial_l G^{(1)}_{jk} + R_{ijk} G^{(2)}_{lk}) H^{kh}_{(1)}, \\
\tilde{P}^h_{ij} &= \frac{1}{2}(\partial_h G^{(3)}_{jk} + \partial_k G^{(3)}_{ij}) H^{kh}_{(3)} + \frac{1}{2}(\partial_l G^{(1)}_{jk} + R_{ijk} G^{(2)}_{lk}) H^{kh}_{(1)}, \\
S^h_{ij} &= -\frac{1}{2}(\partial_h G^{(2)}_{ij} + R_{lij} G^{(2)}_{jk}) H^{kh}_{(2)} + c_3 R_{0lj} H^{kh}_{(3)}, \\
\tilde{S}^h_{ij} &= -\frac{1}{2}(\partial_h G^{(1)}_{ij} + R_{lij} G^{(2)}_{jk}) H^{kh}_{(3)} + c_3 R_{0lj} H^{kh}_{(1)},
\end{align*}
\]

where \( R^{h}_{kl} \) are the components of the curvature tensor field of the Levi Civita connection \( \nabla \) of the base manifold \( (M, g) \).

Taking into account the expressions (7), (9) and by using the formulas (11), (12) we can obtain the detailed expressions of \( P^h_{ij}, \tilde{Q}^h_{ij}, \tilde{P}^h_{ij}, \tilde{S}^h_{ij} \).

The curvature tensor field \( K \) of the connection \( \nabla \) is defined by the well known formula

\[
K(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad X,Y,Z \in \Gamma(TM).
\]
By using the local adapted frame \( \left( \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j} \right) \), \( i, j = 1, \ldots, n \) we obtain, after a standard straightforward computation

\[
K \left( \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right) \frac{\delta}{\delta x^k} = XX XX X_{ki j}^h \frac{\delta}{\delta x^h} + XXX Y_{ki j}^h \frac{\partial}{\partial y^h}, \\
K \left( \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right) \frac{\partial}{\partial y^k} = XX Y X_{ki j}^h \frac{\delta}{\delta x^h} + XXX Y_{ki j}^h \frac{\partial}{\partial y^h}, \\
K \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) \frac{\partial}{\partial y^k} = YY X X_{ki j}^h \frac{\delta}{\delta x^h} + YYY Y_{ki j}^h \frac{\partial}{\partial y^h}, \\
K \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) \frac{\partial}{\partial y^k} = YY Y X_{ki j}^h \frac{\delta}{\delta x^h} + YYY Y_{ki j}^h \frac{\partial}{\partial y^h}, \\
K \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial x^j} \right) \frac{\partial}{\partial y^k} = YY Y X_{ki j}^h \frac{\delta}{\delta x^h} + YYY Y_{ki j}^h \frac{\partial}{\partial y^h},
\]

where the \( M \)-tensor fields appearing as coefficients are given by

\[
XX XX X_{ki j}^h = \tilde{S}_{i k}^h \tilde{S}_{j k}^h + P_{i k}^h S_{j k}^h - \tilde{S}_{i k}^h \tilde{S}_{j k}^h - P_{i k}^h S_{i k}^h + R_{i k j}^h + R_{i k j}^h P_{i k}, \\
XX Y X_{ki j}^h = \tilde{P}_{k i}^h P_{j i}^h + P_{k i}^h \tilde{S}_{j i}^h - \tilde{P}_{k i}^h P_{j i}^h - P_{k i}^h \tilde{S}_{j i}^h + R_{i k j}^h \tilde{Q}_{i k}^h, \\
XX Y Y_{ki j}^h = \tilde{P}_{k i}^h \tilde{S}_{j i}^h + P_{k i}^h S_{j i}^h - \tilde{P}_{k i}^h \tilde{S}_{j i}^h - P_{k i}^h S_{k i}^h + \tilde{P}_{i k}^h R_{i k j}^h, \\
Y Y X X_{ki j}^h = \partial_{i} P_{j k}^h - \partial_{j} P_{i k}^h + \tilde{P}_{j k}^h \tilde{P}_{i k}^h + P_{j k}^h P_{i k}^h - \tilde{P}_{j k}^h \tilde{Q}_{i k}^h - P_{i k}^h \tilde{Q}_{i k}^h, \\
Y Y X Y_{ki j}^h = \partial_{i} \tilde{P}_{j k}^h - \partial_{j} \tilde{P}_{i k}^h + \tilde{P}_{j k}^h \tilde{P}_{i k}^h + P_{j k}^h \tilde{Q}_{i k}^h - \tilde{P}_{i k}^h \tilde{Q}_{i k}^h - P_{i k}^h \tilde{Q}_{i k}^h, \\
Y Y Y X_{ki j}^h = \partial_{i} Q_{j k}^h - \partial_{j} Q_{i k}^h + Q_{j k}^h \tilde{Q}_{i k}^h + \tilde{Q}_{j k}^h Q_{i k}^h - Q_{i k}^h \tilde{Q}_{i k}^h - Q_{i k}^h \tilde{Q}_{i k}^h, \\
Y Y Y Y_{ki j}^h = \partial_{i} Q_{j k}^h - \partial_{j} Q_{i k}^h + Q_{j k}^h \tilde{Q}_{i k}^h + \tilde{Q}_{j k}^h Q_{i k}^h - Q_{i k}^h \tilde{Q}_{i k}^h - Q_{i k}^h \tilde{Q}_{i k}^h, \\
X X X X_{ki j}^h = \partial_{i} S_{j k}^h + S_{j k}^h \tilde{S}_{i k}^h + \tilde{S}_{j k}^h P_{i k}^h - \tilde{P}_{i k}^h P_{j i}^h - P_{i k}^h S_{j k}^h - \tilde{P}_{i k}^h \tilde{Q}_{i k}^h, \\
X X Y X_{ki j}^h = \partial_{i} S_{j k}^h + S_{j k}^h \tilde{S}_{i k}^h + \tilde{S}_{j k}^h P_{i k}^h - \tilde{P}_{i k}^h P_{j i}^h - P_{i k}^h S_{j k}^h - \tilde{P}_{i k}^h \tilde{Q}_{i k}^h, \\
\]

\(^{(1)}\)
\[ YXYX^h_{ij} = \partial_i P^h_{kj} + \tilde{P}^h_{kj} \tilde{Q}^h_{ij} + \tilde{P}^h_{kj} P^h_{il} - Q^h_{ik} P^h_{lj} - \tilde{Q}^h_{ik} \tilde{S}^h_{lj}, \]
\[ YXYY^h_{ij} = \partial_i \tilde{P}^h_{kj} + \tilde{P}^h_{kj} Q^h_{il} + \tilde{P}^h_{kj} \tilde{P}^h_{il} - Q^h_{ik} \tilde{P}^h_{lj} - \tilde{Q}^h_{ik} \tilde{S}^h_{lj}. \]

Remark that, due to the condition for \((M, g)\) to have constant sectional curvature, we have \(\tilde{\nabla} R^h_{ij} = 0\), and the above formulas become simpler. In order to obtain the final expressions of the above \(M\)-tensor fields, we need the first and second order partial derivatives with respect to the tangential coordinates \(y^i\) of the usual tensor fields involved in the definition of the Riemannian metric \(G\).

\[ \partial_i G^{(\alpha)}_{jk} = c'\alpha g_{0i}g_{jk} + d'\alpha g_{0i}g_{0j}g_{0k} + d_\alpha g_{0i}g_{0j} + d_\alpha g_{0i}g_{jk} \]
\[ \partial_i H^{(\alpha)}_{jk} = p'\alpha g^{jk} g_{0i} + q'\alpha y^j y^k + q_\alpha y^j \delta_i^k \]
\[ \partial_i \partial_j G^{(\alpha)}_{kl} = c''\alpha g_{0i}g_{0j}g_{kl} + c'\alpha g_{0i}g_{0j}g_{0k} + d'\alpha g_{0i}g_{0j}g_{0k}g_{0l} + d_\alpha g_{0i}g_{0j}g_{0k} \]
\[ + d_\alpha g_{0i}g_{0j}g_{0k}g_{0l} + d_\alpha g_{0i}g_{0j}g_{0k}g_{0l} + d_\alpha g_{0i}g_{0j}g_{0k}g_{0l} + d_\alpha g_{0i}g_{0j}g_{0k}g_{0l}, \quad \alpha = 1, 2, 3 \]

Next we get the first order partial derivatives with respect to the tangential coordinates \(y_i\) of the \(M\)-tensor fields \(P^h_{ij}, Q^h_{ij}, S^h_{ij}, \tilde{P}^h_{ij}, \tilde{Q}^h_{ij}, \tilde{S}^h_{ij}\)

\[ \partial_i Q^h_{jk} = \frac{1}{2} \partial_i H^{kl}_{(2)} (\partial_j G^{(2)}_{kl} + \partial_k G^{(2)}_{jl} - \partial_l G^{(2)}_{jk}) \]
\[ + \frac{1}{2} H^{kl}_{(2)} (\partial_i \partial_j G^{(2)}_{kl} + \partial_i \partial_k G^{(2)}_{jl} - \partial_i \partial_l G^{(2)}_{jk}) \]
\[ + \frac{1}{2} \partial_i H^{kl}_{(3)} (\partial_j G^{(3)}_{kl} + \partial_k G^{(3)}_{jl}) + \frac{1}{2} H^{kl}_{(3)} (\partial_i \partial_j G^{(3)}_{kl} + \partial_i \partial_k G^{(3)}_{jl}), \]

\[ \partial_i \tilde{Q}^h_{jk} = \frac{1}{2} \partial_i H^{kl}_{(3)} (\partial_j G^{(2)}_{kl} + \partial_k G^{(2)}_{jl} - \partial_l G^{(2)}_{jk}) \]
\[ + \frac{1}{2} H^{kl}_{(3)} (\partial_i \partial_j G^{(2)}_{kl} + \partial_i \partial_k G^{(2)}_{jl} - \partial_i \partial_l G^{(2)}_{jk}) \]
\[ + \frac{1}{2} \partial_i H^{kl}_{(1)} (\partial_j G^{(3)}_{kl} + \partial_k G^{(3)}_{jl}) + \frac{1}{2} H^{kl}_{(1)} (\partial_i \partial_j G^{(3)}_{kl} + \partial_i \partial_k G^{(3)}_{jl}), \]

\[ \partial_i \tilde{P}^h_{jk} = \frac{1}{2} \partial_i H^{kl}_{(2)} (\partial_j G^{(2)}_{kl} - \partial_k G^{(2)}_{jl}) + \frac{1}{2} H^{kl}_{(2)} (\partial_i \partial_j G^{(3)}_{kl} - \partial_i \partial_k G^{(3)}_{jl}) \]
\[ + \frac{1}{2} \partial_i H^{kl}_{(3)} (\partial_j G^{(1)}_{kl} + R_{0kl} G^{(2)}_{rj}) \]
\[ + \frac{1}{2} H^{kl}_{(3)} (\partial_i \partial_j G^{(3)}_{kl} + R_{0kl} G^{(2)}_{rj} + R_{0kl} \partial_i G^{(2)}_{rj}), \]

\[ XXYX^h_{ij} = \partial_i P^h_{kj} + \tilde{P}^h_{kj} \tilde{Q}^h_{ij} + \tilde{P}^h_{kj} P^h_{il} - Q^h_{ik} P^h_{lj} - \tilde{Q}^h_{ik} \tilde{S}^h_{lj}, \]
\[ XXYY^h_{ij} = \partial_i \tilde{P}^h_{kj} + \tilde{P}^h_{kj} Q^h_{il} + \tilde{P}^h_{kj} \tilde{P}^h_{il} - Q^h_{ik} \tilde{P}^h_{lj} - \tilde{Q}^h_{ik} \tilde{S}^h_{lj}. \]
\[
\partial_t P_{jk}^h = \frac{1}{2} \partial_t H_{(3)}^{hl}(\partial_j G_{kl}^{(3)} - \partial_j G_{kl}^{(3)}) + \frac{1}{2} H_{(3)}^{hl}(\partial_i \partial_j G_{kl}^{(3)} - \partial_i \partial_j G_{kl}^{(3)}) + \frac{1}{2} \partial_i H_{(1)}^{hl}(\partial_j G_{kl}^{(1)} + R^{rl}_{0kl} G_{rl}^{(2)}) + \frac{1}{2} H_{(1)}^{hi}(\partial_i \partial_j G_{kl}^{(1)} + R^{rl}_{0kl} G_{rl}^{(2)}),
\]

\[
\partial_t S_{jk}^h = -\frac{1}{2} \{(\partial_i \partial_j G_{jk}^{(1)} + R_{ijk} G_{tr}^{(2)}) H_{(2)}^{rh} + (\partial_i G_{jk}^{(1)} + R_{0jk} G_{tr}^{(2)}) \partial_i H_{(2)}^{rh}\} + c_3 q_0 R_{j0kr} H_{(3)}^{rh} + c_3 (R_{jikr} H_{(3)}^{rh} + R_{j0kr} \partial_i H_{(3)}^{rh}),
\]

\[
\partial_t S_{jk}^h = -\frac{1}{2} \{(\partial_i \partial_j G_{jk}^{(1)} + R_{ijk} G_{tr}^{(2)}) H_{(3)}^{rh} + (\partial_i G_{jk}^{(1)} + R_{0jk} G_{tr}^{(2)}) \partial_i H_{(3)}^{rh}\} + c_3 q_0 R_{j0kr} H_{(1)}^{rh} + c_3 (R_{jikr} H_{(1)}^{rh} + R_{j0kr} \partial_i H_{(1)}^{rh}).
\]

Now we have to replace these derivatives, next the explicit expressions of the M-tensor fields \( P_{ij}^h, Q_{ij}^h, S_{ij}^h, \tilde{P}_{ij}^h, \tilde{Q}_{ij}^h, \tilde{S}_{ij}^h \), the expressions (11), (12) of the functions \( p_1, p_2, p_3, q_1, q_2, q_3 \) and of their derivatives in order to obtain the components of the curvature tensor as functions of \( a_1, a_2, a_3 \) and their derivatives of first, second and third order only. The expressions are obtained by using the Mathematica package RICCI. It was not convenient to think \( a_1, a_2, a_3, b_1, b_2, b_3 \) as well as \( c_1, c_2, c_3, d_1, d_2, d_3 \) and \( p_1, p_2, p_3, q_1, q_2, q_3 \) as functions of \( t \) since RICCI did not make some useful factorizations after the command TensorSimplify. We decided to consider these functions as well as their derivatives of first, second and third order, as constants, the tangent vector \( y \) as a first order tensor, the components \( G_{ij}^{(1)}, G_{ij}^{(2)}, G_{ij}^{(3)}, H_{(1)}^{ij}, H_{(2)}^{ij}, H_{(3)}^{ij} \) as second order tensors and so on, on the Riemannian manifold \( M \), the associated indices being \( h, i, j, k, l, r, s \).

The holomorphic sectional curvature of a Kählerian manifold \((TM, G, J)\) is given by the formula:

\[
K_0(X, Y)Z = \frac{k}{4} \left[ G(Y, Z)X - G(X, Z)Y + G(JY, Z)JX - G(JX, Z)JY + 2G(X, JY)JZ \right]
\]

In the case of the Kählerian manifold \((TM, G, J)\), we obtain, after a standard straightforward computation

\[
K_0 \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \frac{\partial}{\partial x^k} = XXXX0_{kij} \frac{\partial}{\partial x^h} + XXXY0_{kij} \frac{\partial}{\partial y^h},
\]
\[
K_0 \left( \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right) \frac{\partial}{\partial y^k} = XXYX_0^{h_{kij}} \frac{\delta}{\delta x^k} + XXYY_0^{h_{kij}} \frac{\partial}{\partial y^k},
\]

where

\[
XXX \delta_{x^i} \delta_{x^j} = \frac{k}{4} \left[ G_{i k}^{(1)} \delta_{i}^{(1)} - G_{i k}^{(1)} \delta_{i}^{(3)} - J_{j}^{(3)} h \left( J_{j}^{(1)} G_{i k}^{(3)} - J_{j}^{(3)} G_{i k}^{(1)} \right) \right],
\]

\[
XXY^{h_{kij}} = \frac{k}{4} \left[ J_{j}^{(1)} h \left( J_{j}^{(1)} G_{i k}^{(3)} - J_{j}^{(3)} G_{i k}^{(1)} \right) - J_{j}^{(3)} h \left( J_{j}^{(1)} G_{i k}^{(3)} - J_{j}^{(3)} G_{i k}^{(1)} \right) + 2 J_{k}^{(3)} h \left( J_{j}^{(1)} G_{i l}^{(1)} - J_{j}^{(3)} G_{i l}^{(1)} \right) \right],
\]

\[
XXYX^{h_{kij}} = \frac{k}{4} \left[ G_{i k}^{(3)} \delta_{i}^{(1)} - G_{i k}^{(3)} \delta_{i}^{(3)} - J_{j}^{(3)} h \left( J_{j}^{(1)} G_{i k}^{(2)} - J_{j}^{(3)} G_{i k}^{(1)} \right) + J_{j}^{(3)} h \left( J_{j}^{(1)} G_{i l}^{(1)} - J_{j}^{(3)} G_{i l}^{(1)} \right) - 2 J_{k}^{(3)} h \left( J_{j}^{(1)} G_{i l}^{(1)} - J_{j}^{(3)} G_{i l}^{(1)} \right) \right],
\]

\[
XXYY^{h_{kij}} = \frac{k}{4} \left[ J_{j}^{(1)} h \left( J_{j}^{(1)} G_{i k}^{(2)} - J_{j}^{(3)} G_{i k}^{(1)} \right) - J_{j}^{(3)} h \left( J_{j}^{(1)} G_{i k}^{(2)} - J_{j}^{(3)} G_{i k}^{(1)} \right) + 2 J_{k}^{(3)} h \left( J_{j}^{(1)} G_{i l}^{(1)} - J_{j}^{(3)} G_{i l}^{(1)} \right) \right],
\]

\[
YYX^{h_{kij}} = \frac{k}{4} \left[ J_{j}^{(2)} h \left( J_{j}^{(2)} G_{i l}^{(1)} - J_{j}^{(3)} G_{i l}^{(1)} \right) - J_{j}^{(3)} h \left( J_{j}^{(2)} G_{i k}^{(3)} - J_{j}^{(3)} G_{i k}^{(3)} \right) - 2 J_{k}^{(3)} h \left( J_{j}^{(2)} G_{i l}^{(3)} - J_{j}^{(3)} G_{i l}^{(3)} \right) \right],
\]

\[
YYXY^{h_{kij}} = \frac{k}{4} \left[ J_{j}^{(2)} h \left( J_{j}^{(2)} G_{i l}^{(1)} - J_{j}^{(3)} G_{i l}^{(1)} \right) - J_{j}^{(3)} h \left( J_{j}^{(2)} G_{i k}^{(3)} - J_{j}^{(3)} G_{i k}^{(3)} \right) + 2 J_{k}^{(3)} h \left( J_{j}^{(2)} G_{i l}^{(3)} - J_{j}^{(3)} G_{i l}^{(3)} \right) \right],
\]

\[
YYYX^{h_{kij}} = \frac{k}{4} \left[ J_{j}^{(2)} h \left( J_{j}^{(2)} G_{i l}^{(1)} - J_{j}^{(3)} G_{i l}^{(1)} \right) - J_{j}^{(3)} h \left( J_{j}^{(2)} G_{i k}^{(3)} - J_{j}^{(3)} G_{i k}^{(3)} \right) \right]
\]
of the components of the difference we have chosen the difference
\[ -J_j^{(2)} (J_i^{(2)} G_{ik}^{(1)} - J_i^{(3)} G_{ik}^{(3)}) - 2 J_j^{(3)} (J_j^{(3)} G_{il}^{(2)}) - J_j^{(3)} G_{il}^{(3)}) \]
\[
YXY\, Y^0_{kij} = \frac{k}{4} \left[ \left( G_{jk}^{(2)} \delta_i^h - G_{ik}^{(2)} \delta_j^h - J_i^{(3)} (J_j^{(2)} G_{ik}^{(3)}) - J_j^{(3)} G_{il}^{(3)}) \right) \right.
\]
\[
YXX\, Y^0_{kij} = \frac{k}{4} \left[ -G_{jk}^{(3)} \delta_i^h - J_i^{(3)} (J_j^{(2)} G_{ik}^{(1)}) - J_j^{(3)} G_{il}^{(3)}) \right.
\]
\[
YXY\, Y^0_{kij} = \frac{k}{4} \left[ J_i^{(1)} G_{ik}^{(1)} - J_i^{(2)} G_{ik}^{(1)} - J_i^{(3)} G_{ik}^{(3)}) \right.
\]
\[
YXX\, Y^0_{kij} = \frac{k}{4} \left[ J_i^{(1)} (J_j^{(2)} G_{ik}^{(2)}) - J_i^{(3)} G_{ik}^{(3)}) \right.
\]
\[
YXY\, Y^0_{kij} = \frac{k}{4} \left[ J_i^{(1)} (J_j^{(2)} G_{ik}^{(2)}) - J_i^{(3)} G_{ik}^{(3)}) \right.
\]
In order to get the conditions under which \((TM, G, J)\) is a Kählerian manifold of constant holomorphic sectional curvature, we study the vanishing of the components of the difference \(K - K_0\). In this study it is useful the following generic result similar to the lemma 2.1

**Lemma 3.2.** If \(\alpha_1, \ldots, \alpha_{10}\) are smooth functions on \(TM\) such that
\[
\alpha_1 \delta^h_{ij} g_{jk} + \alpha_2 \delta^h_{ij} g_{ik} + \alpha_3 \delta^h_{ij} g_{ij} + \alpha_4 \delta^h_{ik} g_{0j} + \alpha_5 \delta^h_{ij} g_{0k} + \alpha_6 \delta^h_{ij} g_{0k} g_{0k}
\]
\[
+ \alpha_7 g_{ij} g_{0j} + \alpha_8 g_{ij} g_{0j} y_{ij} + \alpha_9 g_{ij} g_{0k} y_{ij} + \alpha_{10} g_{ij} g_{0k} g_{0k} y_{ij} = 0,
\]
then \(\alpha_1 = \cdots = \alpha_{10} = 0\).

After a detailed analysis of several terms in the vanishing problem of the components of the above difference we have chosen the difference \(YXY\, Y^0_{kij} - YXY\, Y^0_{kij}^h\) since in this expression there appear two terms with shorter expressions. Namely, from the second term (which contain \(g_{ik} \delta^h_{ij}\)), by imposing the condition to be zero, we get

\[
\lambda' = \frac{4a_1 c - \lambda^k (2a_1 a_1' + 2c + 4a_3 a_3' c + 2a_2 c)}{k(a_1^2 + 2c + 2a_2 c)}.
\]
Replacing this value in the third one (which contains $g_{ij}\delta^h_k$) we obtain the value of $\lambda$:

\begin{equation}
\lambda = \frac{4a_1c}{k(a_1^2 + 2ct + 2a_3^2ct)}
\end{equation}

Next, by using this expression of $\lambda$ as well as the values of $\lambda', \lambda''$, and $\lambda'''$, obtained by straightforward computation, we get with RICCI that all the components of the difference $K - K_0$ are zero. We should mention the serious technical difficulties in the computation of the differences $Y_{XY}Y^h_{kij} - Y_{XY}Y^0_{kij}$, $Y_{XY}X^h_{kij} - Y_{XY}X^0_{kij}$, $Y_{XX}Y^h_{kij} - Y_{XX}Y^0_{kij}$, and $Y_{XXX}X^h_{kij} - Y_{XXX}X^0_{kij}$.

Hence we may state

**Theorem 3.3.** The Kählerian manifold $(TM, G, J)$ with $G$ and $J$ obtained as natural lifts of general type of the Riemannian metric $g$ on the Riemannian manifold $(M, g)$, has constant holomorphic sectional curvature $k$ if and only if the parameter $\lambda$ is expressed by the expression (15).

**Remark.** If $a_3 = 0$ we obtain the well known condition for $(TM, G, J)$ to have constant holomorphic sectional curvature in the case where $G, J$ are natural lifts of diagonal type (see [13], [3], and see also [16]).

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