SOME TYPES OF GENERALIZED QUASI EINSTEIN, PSEUDO RICCI-SYMMETRIC AND WEAKLY SYMMETRIC MANIFOLD

BY

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Abstract. In this paper we show that if in a generalized quasi Einstein manifold $G(QE)_n$ the associated scalars are constant and generator $U$ of the manifold is a recurrent vector field with the associated 1-form not being the 1-form of recurrence then the manifold is quasi-conformally conservative.

We have also obtained sufficient condition for a pseudo Ricci symmetric manifold to be a quasi Einstein manifold and also sufficient condition for a weakly symmetric manifold to be a Einstein manifold.

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1. Introduction. A non-flat Riemannian manifold $(M^n,g)$ of dimension $n$, $(n > 2)$ is defined to be a quasi Einstein manifold in [3] if its Ricci tensor $S$ of type $(0,2)$ satisfies

\[(1.1) \quad S(X,Y) = ag(X,Y) + bA(X)A(Y)\]

\[(1.2) \quad g(X,U) = A(X), \quad \forall X \in TM.\]

$U$ being a unit vector field. This type of manifold is denoted by $(QE)_n$. In [5], De and Ghosh defined generalized quasi Einstein manifold and studied many properties. A non-flat Riemannian manifold is called a generalized
quasi Einstein manifold if its Ricci-tensor $S$ of type $(0, 2)$ is non-zero and satisfies

\[ S(X, Y) = ag(X, Y) + b A(X) A(Y) + c B(X) B(Y) \]

where $a, b$ and $c$ are non-zero scalars and $A, B$ are two 1-forms such that

\[ g(X, U) = A(X), \quad g(X, V) = B(X) \]

$U, V$ being unit vectors which are orthogonal, i.e.

\[ g(U, V) = 0. \]

The vector fields $U$ and $V$ are called the generators of the manifold. This type of manifold is denoted by $G(QE)_n$.

In [1] author have studied various curvature properties on $G(QE)_n$.

A quasi conformal curvature tensor is defined in [6] as

\[ C^*(X, Y) Z = a_1 R(X, Y) Z + b_1 [S(Y, Z) X - S(X, Z) Y + g(Y, Z) Q X - g(X, Z) Q Y] \]

\[ - \frac{r}{n} \left[ \frac{a_1}{n} - 1 + 2 b_1 \right] [g(Y, Z) X - g(X, Z) Y], \]

where $a_1$ and $b_1$ are scalars of which $b_1 \neq 0$, $Q$ is a symmetric endomorphism [2] of the tangent space at each point corresponding to the Ricci tensor $S$ and

\[ g(Q X, Y) = S(X, Y) \]

\[ g(R(X, Y) Z, W) = R(X, Y, Z, W). \]

If $a_1 = 1$ and $b_1 = -\frac{1}{n-2}$, then (1.6) reduces to conformal curvature tensor.

A non-flat Riemannian manifold $(M^n, g)$ $(n \geq 2)$ is said to be a pseudo Ricci symmetric manifold if its curvature tensor $R$ satisfies the condition

\[ (\nabla_X R)(Y, Z) W = 2 B(X) R(Y, Z) W \]

\[ + B(Y) R(X, Z) W + B(Z) R(Y, X) W \]

where $B$ is a non-zero 1-form.
These relations will be required in next sections.

We know from [6] that \( \text{div} C^* = 0 \).

In this section we obtain a sufficient condition for a \( G(QE)_n \) to be quasi-conformally conservative. In a \( G(QE)_n \) if \( a, b, c \) are constant, then contracting (1.3) we get

\[
(2.1) \quad r = an + b + c, \quad \text{i.e.} \quad dr = 0,
\]

where \( r \) is the scalar curvature. Using (2.1) in (1.6) we get

\[
(\nabla_W C^*)(X, Y, Z) = a_1(\nabla_W R)(X, Y)Z + b_1[(\nabla_W S)(Y, Z)X - (\nabla_W S)(X, Z)Y + g(Y, Z)(\nabla_W Q)X - g(X, Z)(\nabla_W Q)Y].
\]

We know from [6] that

\[
(\text{div} R)(X, Y, Z) = (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)
\]

and from (1.3) we get

\[
\]

since \( a, b, c \) are constant.

Hence contracting (2.2) and using (2.3) we obtain

\[
(\text{div} C^*)(X, Y, Z) = 2b(a_1 + b_1)[(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(Y)] - (\nabla_Y A)(X)A(Z) - (\nabla_Y A)(Z)A(X)] + 2c(a_1 + b_1)[(\nabla_X B)(Y)B(Z) + (\nabla_X B)(Z)B(Y)] - (\nabla_Y B)(X)B(Z) - (\nabla_Y B)(Z)B(X)] + bb_1[(\nabla_U A)(X) + A(X)\text{div } U]g(Y, Z) + eb_1[(\nabla_U B)(X) + B(X)\text{div } U]g(Y, Z)
\]

A non-flat Riemannian manifold \((M^n, g) (n \geq 2)\) is said to be a weakly symmetric manifold [4] if its curvature tensor \( R \) satisfies the condition

\[
(1.10) \quad (\nabla_X R)(Y, Z)W = A(X)R(Y, Z)W + B(Y)R(X, Z)W + D(Z)R(Y, X)W + E(W)R(Y, Z)X + g[R(Y, Z)W, X]\mu,
\]

where \( A, B, D \) and \( E \) are non-zero 1-forms and \( \mu \) is a non-zero vector field. These relations will be required in next sections.

2. \( G(QE)_n (n > 3) \) with divergent free quasi conformal curvature tensor. We know quasi conformal curvature tensor is said to be conservative if divergence of \( C^* \) vanishes, i.e. \( \text{div} C^* = 0 \).

In this section we obtain a sufficient condition for a \( G(QE)_n \) to be quasi-conformally conservative. In a \( G(QE)_n \) if \( a, b, c \) are constant, then contracting (1.3) we get

\[
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\]

where \( r \) is the scalar curvature. Using (2.1) in (1.6) we get

\[
(\nabla_W C^*)(X, Y, Z) = a_1(\nabla_W R)(X, Y)Z + b_1[(\nabla_W S)(Y, Z)X - (\nabla_W S)(X, Z)Y + g(Y, Z)(\nabla_W Q)X - g(X, Z)(\nabla_W Q)Y].
\]
Now if we consider that the generator $U$ of the manifold is a recurrent vector field [7], [8] with associated 1-form $A$, not being the 1-form of recurrence, gives $\nabla_X U = D(X)U$, where $D$ is the 1-form of recurrence, we get

$$g(\nabla_X U, Y) = g(D(X)U, Y) \quad \text{i.e.} \quad (\nabla_X A)(Y) = D(X)A(Y).$$

In view of (2.5), (2.4) is expressed as follows

$$\text{(2.6)} \quad (\text{div} C^*)(X, Y, Z) = 2b(a_1 + b_1)[D(X)A(Y)A(Z) + D(X)A(Z)A(Y) - D(Y)A(X)A(Z) - D(Y)A(Z)A(X)] + 2c(a_1 + b_1)[D(X)B(Y)A(Z) + D(X)B(Z)A(Y) - D(Y)B(X)A(Z) - D(Y)B(Z)A(X)] + 2bb_1D(U)A(X)g(Y, Z) - 2bb_1D(U)A(Y)g(X, Z) + 2cb_1D(U)B(X)g(Y, Z) - 2cb_1D(U)B(Y)g(X, Z).$$

Since $(\nabla_X A)(U) = 0$, it follows from (2.5) that $D(X) = 0$. Hence from (2.6) it follows that $(\text{div} C^*)(X, Y, Z) = 0$.

Thus we can state the following:

**Theorem 2.1.** If in a $G(QE)_n(n > 3)$ the associated scalars are constants and generator $U$ of the manifold is a recurrent vector field with the associated 1-form $A$ not being the 1-form of recurrence, then the manifold is quasi-conformally conservative.

3. **Sufficient condition for a pseudo Ricci symmetric manifold to be a quasi Einstein manifold.** Now contracting (1.9) we have

$$\text{(3.1)} \quad (\nabla_X S)(Y, Z) = 2B(X)S(Y, Z) + B(Y)S(X, Z) + B(Z)S(Y, X).$$

Putting $Z = \rho$ in (3.1) we get

$$\text{(3.2)} \quad (\nabla_X S)(Y, \rho) = 2B(X)S(Y, \rho) + B(Y)S(X, \rho) + B(\rho)S(Y, X),$$

where $\rho$ is a vector field.

In a Riemannian geometry, a vector field $\rho$ defined by $g(X, \rho) = A(X)$, for any vector field $X$ is said to be a concircular vector field [8] if

$$\text{(3.3)} \quad (\nabla_X A)(Y) = \alpha g(X, Y) + \omega(X)A(Y),$$

where $\alpha$ is a non-zero scalar and $\omega$ is a closed 1-form. If $\rho$ is a unit vector, then the equation (3.3) can be written as

$$\text{(3.4)} \quad (\nabla_X A)(Y) = \alpha[g(X, Y) - A(X)A(Y)].$$
We suppose that a pseudo Ricci symmetric manifold admits a unit concircular vector field defined by (3.4), where $\alpha$ is a non-zero constant. Applying the Ricci identity to (3.4) we get

\[ A(R(X, Y)Z) = -\alpha^2 [g(X, Z)A(Y) - g(Y, Z)A(X)]. \]

Putting $Y = Z = e_i$ in (3.5), and taking summation over $i$, $1 \leq i \leq n$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, we get $A(QX) = (n - 1)\alpha^2 A(X)$ where $Q$ is the Ricci operator defined in (1.7), and which implies

\[ S(X, \rho) = (n - 1)\alpha^2 A(X). \]

From (3.6) we get

\[ (\nabla_Y S)(X, \rho) = (n - 1)\alpha^2 g(X, Y) - \alpha S(X, Y). \]

In (3.2) using (3.6) and (3.7) we have

\[ (n - 1)\alpha^3 g(Y, X) - \alpha S(Y, X) = 2\alpha^2 B(X)A(Y)(n - 1) + \alpha^2 A(X)B(Y)(n - 1) + S(Y, X)B(\rho). \]

From (3.8) we have

\[ \{\alpha + B(\rho)\}S(Y, X) = \alpha^3(n - 1)g(Y, X) - (n - 1)\alpha^2 A(X)B(Y) - 2(n - 1)\alpha^2 A(Y)B(X). \]

Putting $Y = \rho$ in (3.8) and using (3.6) we get

\[ B(X) = -A(X)B(\rho) \quad \forall \quad X \in TM. \]

Let us impose the condition

\[ \alpha + B(\rho) \neq 0 \]

i.e. 1-form $B(\rho)$ is not a constant, otherwise $B$ will be parallel to $A$. Putting (3.10) in (3.9) we obtain

\[ S(Y, X) = \frac{\alpha^3(n - 1)}{\alpha + B(\rho)} g(Y, X) + \frac{3(n - 1)\alpha^2}{\alpha + B(\rho)} A(X)A(Y)B(\rho) \]

i.e. $S(X, Y) = a g(X, Y) + bA(X)A(Y)$,
where \( a = \frac{\alpha^3(n-1)}{\alpha + B(\rho)} \) and \( b = \frac{3(n-1)\alpha^2}{\alpha + B(\rho)} B(\rho) \). Thus we can state the following theorem

**Theorem 3.1.** If a pseudo Ricci-symmetric manifold admits a unit concircular vector field whose associated scalar is a non-zero constant and satisfying the condition (3.11), then the manifold reduces to a quasi Einstein manifold and also the 1-forms \( A \) and \( B \) are opposite in sign.

4. Sufficient condition for a weakly symmetric manifold to be a Einstein manifold. Now contracting (1.10) we have

\[
\]

Putting \( Z = \rho \) in (4.1) we get

\[
(\nabla_X S)(Y, \rho) = A(X)S(Y, \rho) + B(Y)S(X, \rho) + D(\rho)S(Y, X) + E(R(X, Y)\rho) + E(R(X, \rho)Y).
\]

Using (3.5), (3.6) and (3.7) in (4.2) we have

\[
(n - 1)\alpha^2 g(Y, X) - \alpha S(Y, X) = A(X)(n - 1)\alpha^2 A(Y) + B(Y)(n - 1)\alpha^2 A(X) + D(\rho)S(Y, X)
\]
\[
-\alpha^2[A(Y)E(X) - A(X)E(Y)] - \alpha^2[g(X, Y)E(\rho) - A(Y)E(X)]
\]
\[
i.e. - \{\alpha + D(\rho)\}S(X, Y) = -\{\alpha^3(n - 1) + \alpha^2 E(\rho)\}g(X, Y)
\]
\[
+\alpha^2(n - 1)A(X)A(Y) + \alpha^2(n - 1)A(X)B(Y)
\]
\[
+\alpha^2 A(X)E(Y).
\]

Putting \( Y = \rho \) in (4.3) and using (3.6) we get

\[
(n - 1)\alpha^2 A(X)[1 + B(\rho) + D(\rho)] = 0.
\]

Considering \( 1 + B(\rho) + D(\rho) \neq 0 \), we have

\[
A(X) = 0.
\]

Let us impose the condition

\[
\alpha + D(\rho) \neq 0
\]
otherwise, 1-forms $D(\rho)$ and $B(\rho)$ will be constants. Putting (4.5) in (4.3) we obtain

\[(4.7) \quad S(X, Y) = \frac{\alpha^3(n-1) + \alpha^2E(\rho)}{\alpha + D(\rho)} g(X, Y) .\]

Thus we can state

**Theorem 4.1.** If a weakly symmetric manifold admits a unit concurcular vector field whose associated scalar is a non-zero constant and satisfying the condition (4.6), then the manifold reduces to a Einstein manifold.

**REFERENCES**


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