ON THE ALGEBRAICALLY COMPACT ABELIAN Q-GROUPS II

BY

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Abstract. We prove that any algebraically compact Abelian Q-group with at most countable maximal torsion subgroup is bounded. This supplies a recent result of ours in (Compt. rend. Acad. bulg. Sci., 2007).


Key words: algebraically compact groups, divisible groups, bounded groups, Q-groups, maximal torsion subgroups.

Throughout the present short note, let all groups into consideration be arbitrary Abelian groups (possibly mixed). All our notion and notation are standard as we refer to [3] for a more detailed information too. For instance, $A^1$ will always denote the subgroup of a group $A$ consisting of all elements of infinite height as computed in $A$, and $|M|$ will always denote the cardinality of any set $M$.

In ([1], Problem 2) we pose the following question.

Problem. Find the group class which is the intersection between the classes of algebraically compact and Q-groups. Whether or not this is, in fact, the class of bounded groups?

Recently, we obtained in [2] a positive answer to that query in the case when the set $\operatorname{supp}(G) = \{p|G_p \neq 0\}$ is finite, where $G$ is a group with $p$-primary component of torsion $G_p$. Specifically, the following statement was established.

Theorem ([2]). Let $G$ be a group such that $\operatorname{supp}(G)$ is finite. Then $G$ is an algebraically compact Q-group if and only if $G$ is bounded.
We also conjectured there that the Problem in general holds in the affirmative. The goal of this brief paper is to settle it provided that the torsion subgroup \( G_t = \oplus_p G_p \) is with no more than of countable cardinality. Before doing that, we need a few preliminaries to make the article more nearly self-contained and friendly to the reader.

**Definition ([3]).** A group \( G \) is called algebraically compact if \( G \) separates as a direct summand of each group which contains it as a pure subgroup.

**Definition ([4]).** A group \( G \) is said to be a Q-group if \( G^1 = 0 \) and for each \( H \leq G \) with \( |H| \geq \aleph_0 \) the inequality \(|(G/H)^1| \leq |H|\) holds.

Clearly, direct sums of cyclic groups, and even direct sums of countable groups without elements of infinite height, are themselves Q-groups.

We shall say that the group \( G \) satisfies Q-property if for every infinite subgroup \( H \) of \( G \) the following inequality holds: \(|(G/H)^1| \leq |H|\). So, a group is a Q-group if it is without elements of infinite height and satisfies the Q-property. It is straightforward that subgroups of groups which satisfy the Q-property retain the same property. Indeed, assume that \( C \leq A \leq G \) and \( C \) is with infinite cardinality. Therefore, \(|(A/C)^1| \leq |(G/C)^1| = |C|\), that is precisely the desired inequality. Likewise, it is worthwhile noticing and simple to check that the Q-property is closed with respect to isomorphism, i.e., if two groups are isomorphic and one of them satisfies the Q-property, then the other one satisfies the Q-property as well. As we said, this fact has an immediate verification, so we omit its proof leaving it to the reader. Moreover, if \( G \) satisfies the Q-property and its reduced part \( G_r \) is with infinite cardinality, then \(|(G/G_r)^1| = |G/G_r| \leq |G_r| \) which is equivalent to \(|G| = |G_r|\).

We are now prepared to show the truthfulness of the following statement.

**Theorem.** Suppose that \( G \) is a group whose torsion part \( G_t \) is at most countable. Then \( G \) is an algebraically compact Q-group if and only if it is bounded.

**Proof.** The sufficiency is immediate since bounded groups are both algebraically compact groups (see [3], Theorem 27.5) and Q-groups as early noted above.
As for the necessity, suppose now that \( G \) is an algebraically compact Q-group. First of all, we claim that \( G/G_t \) satisfies the Q-property. Indeed, letting \( H/G_t \leq G/G_t \) for some \( H \leq G \) with \( |H/G_t| \geq 8_0 \), hence \( |H| \geq |H/G_t| \geq 8_0 \) and \( |(G/Z_1)/H/G_t| = |(G/H)| \leq |H| = |H/G_t| \) since \( G/G_t/H/G_t \cong G/H \). \( G \) is a Q-group and \( |G_t| \leq 8_0 \). So, the claim is sustained. At the same time, the assumptions on \( G/G_t \) ensure that it is a co-torsion group (see [3]) and this implies by ([3], v. I., p. 119, Exercise 4.) in view of ([3], Corollary 54.5) it is algebraically compact. Since \( G/G_t \) is torsion-free, this enables one to deduce from ([3], Corollary 40.4) that if its reduced part is nontrivial, then it contains a direct summand isomorphic to the group \( J_p \) of all \( p \)-adic integers for some prime number \( p \). But we claim that \( J_p/Z \) is divisible, where \( Z \) is the additive group of the ring of all integers. (See also [3], v. I., p. 119, Exercise 4.) In order to argue this, let \( x \in J_p \), hence \( x = s_0 + s_1 p + \cdots + s_n p^n + \cdots \), where \( s_n = 0, 1, \ldots, p - 1 \) for each natural number \( n \). But this is tantamount to \( x = s_0 + p(s_1 + \cdots + s_n p^{n-1} + \cdots) \in Z + pJ_p \) since, for all naturals \( n \), the coefficients \( s_n \) run on a finite set of integers and so \( s_n \in Z \). Thereby, it follows that \( J_p = Z + pJ_p \) and thus \( J_p/Z = p(J_p/Z) \), i.e., \( J_p/Z \) is \( p \)-divisible. But, on the other hand, \( J_p \) is \( q \)-divisible for each prime \( q \neq p \), that is \( J_p = qJ_p \). This allows us to infer that \( q(J_p/Z) = (qJ_p + Z)/Z = J_p/Z \), i.e., it is \( q \)-divisible for every prime number \( q \neq p \). As a result, via [3], \( J_p/Z \) is divisible, which substantiates our claim. Furthermore, by what we have demonstrated in our comments alluded to above, \( J_p \) must satisfy the Q-property. That is why, \( |J_p/Z| \leq |Z| \) which gives that \( |J_p| = |Z| = 8_0 \). But this is wrong because it is well known that \( |J_p| = 2^{8_0} \) (see, for example, [3], v. II, p. 147). (Note also that the wanted contradiction for \( J_p \) about its cardinality may also be obtained by imitation of our idea in proving the main theorem from [2].) Therefore, this contradiction shows that the reduced part of \( G/G_t \) has to be zero and so it is readily seen that \( G/G_t \) must be divisible. Consequently, since \( G \) is a Q-group, we deduce that \( |G/G_t| \leq |G_t| \) which yields that \( |G| = |G_t| \leq 8_0 \), that is, \( G \) is of necessity at most countable. Finally, employing ([3], v. I, p. 200, Exercise 3(a) or p. 274, Exercise 4), we conclude that \( G \) is really bounded, as asserted. □

In closing, we notice as the proof already demonstrated that the results remain valid for the more general class of co-torsion groups, which properly contains the class of algebraically compact groups. (Compare with [3].)

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REFERENCES


