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SOBRIETY VIA θ-OPEN SETS

BY

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Abstract. SÜNDERHAUF studied the important notion of sobriety in terms of nets. In this paper, by the same token, we present and study the notion of θ-sobriety by utilizing the notion of θ-open sets.

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1. Introduction. In 1943, FOMIN [4] (see, also [5]) introduced the notion of θ-continuity. The notions of θ-open subsets, θ-closed subsets and θ-closure were introduced by VELIČKO [11] for the purpose of studying the important class of H-closed spaces in terms of arbitrary filterbases. DICKMAN and PORTER [2], [3], JOSEPH [7] continued the work of Veličko. Recently NOIRI and JAFARI [9] have also obtained several new and interesting results related to these sets.

In what follows (X, τ) and (Y, σ) (or X and Y) denote topological spaces. Let A be a subset of X. We denote the interior and the closure of a set A by $Int(A)$ and $Cl(A)$, respectively. A point $x \in X$ is called a θ-cluster point of A if $A \cap Cl(U) \neq \emptyset$ for every open set U of X containing x. The set of all θ-cluster points of A is called the θ-closure of A. A subset A is called θ-closed if A and its θ-closure coincide. The complement of a θ-closed set is called θ-open. We denote the collection of all θ-open sets by $\theta(X, \tau)$. It is shown in [8] that the collection of θ-open sets in a space X form a topology denoted by τ_θ. A topological space (X, τ) is called θ-compact [6] if every cover of the space by θ-open sets has a finite subcover. We denote the filter of θ-open neighbourhoods [1] of some point x in X by $\Omega_\theta(x)$.

Definition 1. *A function $f : (X, \tau) \to (Y, \sigma)$ is called θ-continuous if for each $x \in X$ and each V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(Cl(U)) \subset Cl(V)$.*

Definition 2. *Two topological spaces (X, τ) and (Y, σ) are θ-homeomorphic* [11] *if there exists a one-to-one and onto function $f : (X, \tau) \to (Y, \sigma)$ such that f and f^{-1} are both θ-continuous.*

2. θ-sobriety

Definition 3. *A space (X, τ) is said to be θ-sober if it is θ-homeomorphic with the space of points of its frame of θ-open sets.*

Recall that the θ-saturated set and the θ-kernel of a set A [1] are the same, i.e. $\cap\{O \in \theta(X, \tau) \mid A \subset O\}$.

θ-sobriety implies the existence of LUB(= least upper bound) of subsets which are directed with respect to the order of θ-specialization, i.e. if (X, τ) is a topological space, then the order of θ-specialization of X is defined by $x \leq_r y \Leftrightarrow x \in Cl_\theta(\{y\})$.

Theorem 1. *Let X be a θ-sober space for which finite intersection of θ-compact θ-saturated subsets are θ-compact.*

(1) *Every cover of a θ-compact set by θ-open sets contains a finite subcover.*

(2) *If the intersection of θ-compact θ-saturated sets is contained in a θ-open set, then the same is true for an intersection of finitely many of them.*

Now we offer a new notion called θ-observative net by which we characterize θ-sobriety.

Definition 4. *A net $(x_i)_i \in I$ on a space X is θ-observative if for all $i \in I$ and for all $U \in \theta(X, \tau)$ we have that $x_i \in U$ implies that the net is eventually in the set U.*

Recall that a filter base $\mathcal{F}$ is called θ-convergent [12] to a point x in X if for any open set U containing x there exists $B \in \mathcal{F}$ such that $B \in Cl(U)$.

Definition 5. *A θ-observative $(x_i)_i \in I$ strongly θ-converges to a point x in X if it θ-converges to x with respect to $\theta(X, \tau)$, and also if it satisfies that x is an element of every θ-open set which eventually contains the net. We denote it by $x_i \xrightarrow{\theta} x$.*

Lemma 1. *If $(x_i)_i \in I$ is a θ-observative net on a space (X, τ), then $x_i \xrightarrow{\theta} x$ if and only if $x_i \longrightarrow x$ with respect to the τ_θ.*

Proof. Let $x_i \xrightarrow{\theta} x$ and $x \in A$ for some θ-closed set A. If the net is not eventually contained in A, then it is frequently in the θ-open set $X - A$. By hypothesis, the net is θ-observative and therefore $[x]_{\geq i} \subseteq X - A$ for some tail. Hence $x \in X - A$ as a consequence of strong θ-convergence. But this is a contradiction and hence the claim.

Now suppose that $x_i \longrightarrow x$ with respect to the τ_θ. Then a θ-open set which eventually contains the net but does not contain x establishes a θ-neighbourhood $X - U$ of x which has been forgotten by the net. Therefore strong θ-convergence follows readily. □

Here we establish the θ-derived filter $\mathcal{F}(x)\mathcal{I}$ for a net $(x_i)_i \in I$ as follows:

$$\mathcal{F}(x)\mathcal{I} = \{U \in \theta(X, \tau) \mid \exists i \in I \cdot [x]_{\geq i} \subseteq U\}.$$

Definition 6. *A filter $\mathcal{F} \subseteq \tau$ is called θ-completely prime if for every $O \in \mathcal{F}$ and for any family of θ-open sets $(O_i)_{i \in I}$ such that $O \subseteq \bigcup_I O_i$, then $O_k \in \mathcal{F}$ for some $k \in I$.*

Theorem 2. *A filter derived from a θ-observative is θ-completely prime.*

Proof. Let the net $(x_i)_i \in I$ be a θ-observative and $[x]_{\geq i} \subseteq \bigcup_{j \in J} U_j$ for some collectection of θ-open sets and some index $i \in I$. Hence $x_i \in \bigcup_{j \in J} U_j$. Therefore there is some $j_0 \in J$ with $x_j \in U_{j_0}$. Since the net is θ-observative, then it follows that some tail is contained in U_{j_0}. Thus the set is a filter.□

Proposition 1. *If $(x_i)_i \in I$ is a θ-observative net, then $x_i \xrightarrow{\theta} x$ if and only if $\mathcal{F}(x)\mathcal{I} = \Omega_\theta(x)$.*

Proof. Since $(x_i)_i \in I$ strongly θ-converges to x if and only if it is the case that $x \in U$ is equivalent to the existence of some $i \in I$ with $[x]_{\geq i} \subseteq U$.

But how can we deal with the situation where a space is θ-sober if all its θ-observative nets strongly θ-converge?

In such situation, we need the following construction:

Assign to each θ-completely prime filter $\mathcal{F}$ a θ-observative net such that $\mathcal{F}(x)\mathcal{I} = \mathcal{F}$. □

Theorem 3. *Let $\mathcal{F}$ be a filter of θ-open subsets of the space (X, τ). Then $\mathcal{F}$ is θ-completely prime if and only if for all $U \in \mathcal{F}$, there exists $x \in U$ with the property that $x \in G$ implies $G \in \mathcal{F}$ for every $G \in \theta(X, \tau)$.*

Proof. Suppose that $\mathcal{F}$ has this property and $\bigcup_{j\in J} U_j \in \mathcal{F}$. Take $x \in \bigcup_{j\in J} U_j$ with $x \in G \Rightarrow G \in \mathcal{F}$. Clearly, $x \in U_{j_0}$ for some $j_0 \in J$. Therefore $U_{j_0} \in \mathcal{F}$. This means that the filter is θ-completely prime.
Conversely. assume that $U \in \mathcal{F}$ has not this property. It follows that for each $x \in U$, this is $G_x \in \theta(X, \tau)$ with $G_x \notin \mathcal{F}$. Put $U_x := G_x \cap U$. Now we have $U_x \notin \mathcal{F}$ for all $x \in U$ and $U = \bigcup_{x\in U} U_x \in \mathcal{F}$. But this is against our hypothesis that $\mathcal{F}$ is θ-completely prime and hence the claim. □

Now we give a new appropriate construction. Let $\mathcal{F}$ be a θ-completely prime filter of θ-open sets on (X, τ). Take $\mathcal{F}$ with reserved set inclusion as order to be the index set of our net. If $U \in \mathcal{F}$, pick $x_U \in U$ with the property that $x_U \in G$ implies $G \in \mathcal{F}$. This is possible by the above Theorem. A net established in this way is called a θ-derived net from the filter.

Lemma 2. *A θ-derived net from a θ-completely prime filter is θ-observative.*

Proof. Let $U \in \mathcal{F}$ and $x_u \in G$. Then $G \in \mathcal{F}$ by choice of x_U. If $V \subseteq G$, then $x_v \in V \subseteq G$. Hence $[x]_{\geq i} \subseteq G$. Therefore the net is θ-observative. □

Theorem 4. *Every θ-completely prime filter equals the θ-derived filter of any of its θ-derived nets.*

Proof. Clearly, $[x]_{\geq i} \subseteq U$ for $U \in \mathcal{F}$. Thus $U \in \mathcal{F} \Rightarrow \mathcal{F}(x)\mathcal{U}$. Conversely, $U \in \mathcal{F}(x)\mathcal{U} \Rightarrow [x]_{\geq i} \subseteq U$ for some $G \in \mathcal{F}$. Therefore, $x_U \in U$ which implies that $U \in \mathcal{F}$ by choice of x_G. □

Theorem 5. *A topological space is θ-sober if and only if every θ-observative net strongly θ-converges to a unique point.*

Proof. Obvious. □

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