A LIMIT HARVESTING PROBLEM OF POPULATION DYNAMICS WITH LOGISTIC TERM

BY

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Abstract. In this paper, a nonlinear age-dependent population model with logistic term is considered. In the first part is reminded a large time behavior result, and next an optimal harvesting problem associated to a limit problem is investigated. Existence of an optimal control and necessary optimality conditions are established.

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1. Formulation of the problem. The starting point is the following nonlinear age-structured population model

\[
\begin{aligned}
& y_t + y_a + \mu(a)y + \Phi(Y(t))y = -u(a)y, \quad (a, t) \in Q \\
& Y(t) = \int_0^A y(a, t) \, da, \quad t \in (0, +\infty) \\
& y(0, t) = \int_0^A \beta(a)y(a, t) \, da, \quad t \in (0, +\infty) \\
& y(0, a) = y_0(a), \quad a \in (0, A),
\end{aligned}
\]

where \( Q = (0, A) \times (0, +\infty) \). Here \( A \in (0, +\infty) \) represents the maximal age for the population species.

\( y(a, t) \) is the population density of age \( a \in [0, A] \) at time \( t \in [0, +\infty) \) and by \( y_0 \) we have denoted the initial distribution of densities.

The third equation in (1) describes the birth process and it is known as the renewal law; \( y(0, t) \) gives the density of the newborn population at
time $t$, and $\beta$ is the fertility rate which depends in this case only on the age $a$. Therefore $\beta(a)y(a,t)$ stands for the density of newborns at time $t$, with parents of age $a$.

$\mu$ is the mortality rate and depends also only on age. System (1) describes the evolution of an age-structured population which includes a logistic term depending on the total population density at moment $t$, $Y(t)$. $\Phi(Y(t))$ represents an additional mortality rate caused by overpopulation. $u$ is the harvesting effort (and depends on age) and $u(a)y(a,t)$ gives the harvested population of age $a$ at the moment $t$, while $\int_0^A u(a)y(a,t)da$ gives the total harvest at the moment $t$.

The goal of the paper is to find the harvesting effort (control)

$$u \in L^\infty(0, A), \quad 0 \leq u(a) \leq L \text{ a.e. in } (0, A),$$

($L \in (0, +\infty)$) which maximizes

$$\lim_{t \to \infty} \int_0^A u(a)y^u(a, t) \, da,$$

where $y^u$ is the solution to (1) corresponding to $u$. In fact, we will show that this problem is equivalent to the following optimal harvesting problem:

$$\max_{u \in \mathcal{U}} \int_0^A u(a)\tilde{y}^u(a) \, da,$$

where $\mathcal{U} = \{v \in L^\infty(0, A): 0 \leq v(a) \leq L \text{ a.e. in } (0, A)\}$ denotes the set of admissible controls, and $\tilde{y}^u$ is the nontrivial nonnegative solution to

$$\begin{cases}
y'(a) + \mu(a)y(a) + \Phi(Y_0)y(a) = -u(a)y(a), & a \in (0, A) \\
y(0) = \int_0^A y(a) \, da \\
y(0) = \int_0^A \beta(a)y(a) \, da.
\end{cases}$$

The paper is structured as follows: in the next section we present an asymptotic behavior result; it will be shown that, under proper assumptions, the solution to (1) is stabilized toward the nontrivial nonnegative solution to (3).

Then we consider the optimal harvesting problem (2)-(3); we prove the existence of an optimal control and provide the necessary optimality conditions.
We mention that in the case without control, it has been proved that the system (1) has a unique nonnegative nontrivial solution (see e.g. [4]).

A large amount of papers has been devoted to optimal control problems for age-structured population dynamics, see e.g. [1], [5], [8], [7], [9], [12].

For the asymptotic behavior of the solutions to age-structured systems we refer to [2], [3], [6] and [11].

Here are the assumptions of this paper:

(A1) \( \beta \in L^\infty(0, A), \beta(a) \geq 0 \text{ a.e. } a \in (0, A); \)

(A2) \( \mu \in L^1_{loc}([0, A]), \mu(a) \geq 0 \text{ a.e. } a \in (0, A), \int_0^A \mu(a) \, da = +\infty; \)

(A3) \( y_0 \in L^\infty(0, A), y_0(a) \geq 0 \text{ a.e. } a \in (0, A); \)

(A4) \( \Phi : [0, +\infty) \rightarrow [0, +\infty) \) is an increasing continuously differentiable function which satisfies \( \Phi(0) = 0 \) and \( \lim_{r \to +\infty} \Phi(r) = +\infty; \)

(A5) \( \int_0^A \beta(a) \exp\left\{ -\int_0^a (\mu(s) + L) \, ds \right\} da > 1. \)

2. Large time behavior of the solution. Let us notice that the solution to (1) is a separable one, i.e., \( y(a, t) = x(t)p(a, t), (a, t) \in Q, \) where \( p \) is the solution to

\[
\begin{cases}
  p_t + p_a + \mu(a)p = -u(a)p, & (a, t) \in Q \\
  p(0, t) = \int_0^A \beta(a)p(a, t) \, da, & t \in (0, +\infty) \\
  p(a, 0) = y_0(a), & a \in (0, A)
\end{cases}
\]

and \( x \) is the solution to

\[
\begin{cases}
  x'(t) + \Phi(x(t)P(t))x(t) = 0, & t \in (0, +\infty) \\
  P(t) = \int_0^A p(a, t) \, da \\
  x(0) = 1.
\end{cases}
\]

It is obvious that \( y \) is a solution to (1). Note that also the solution \( x \) depends on the control \( u \) through the dependence of the function \( \Phi(x(t)P(t)) \). For every \( u \in \mathcal{U} \) fixed, (4) has a unique nonnegative solution and (5) has a
unique Carathéodory solution (which is also nonnegative). Actually the solution \( p \) to (4) is given by
\[
p(a, t) = \begin{cases} 
  b(t - a) \exp\{- \int_0^a (\mu(s) + u(s)) \, ds\}, & a < t \\
  y_0(a - t) \exp\{- \int_0^t (\mu(a - t + s) + u(a - t + s)) \, ds\}, & a > t.
\end{cases}
\]
Therefore, for \( a < t \), the solution is written as
\[
p(a, t) = b(t - a) \exp\{- \int_0^a (\mu(s) + u(s)) \, ds\},
\]
where \( b \) satisfies the following Volterra equation
\[
(6) \quad b(t) = F(t) + \int_0^t K(t - s) b(s) \, ds.
\]
Here
\[
F(t) = \begin{cases} 
  \int_0^A \beta(a) e^{- \int_0^a (\mu(a-t+s) + u(a-t+s)) \, ds} \, y_0(a-t) \, da, & t < A \\
  0, & \text{otherwise}
\end{cases}
\]
and
\[
K(t) = \begin{cases} 
  \beta(t) \exp\{- \int_0^t (\mu(s) + u(s)) \, ds\}, & t < A \\
  0, & \text{otherwise.}
\end{cases}
\]
We have that \( F \in C(R^+) \), \( K \in L^\infty(R^+) \) and, consequently, \( b \in C(R^+) \).

The following result holds (see e.g. [10]):

**Theorem 1.** The solution \( b \) to (6) satisfies
\[
b(t) = e^{\alpha^* t} b_0(t), \quad \forall t \in R^+,
\]
where \( \lim_{t \to \infty} b_0(t) = \bar{b}_0 \geq 0 \) and \( \alpha^* \) is the solution to the following equation:
\[
\int_0^A \beta(a) \exp\{- \int_0^a (\mu(s) + u(s) + \alpha) \, ds\} \, da = 1.
\]
Taking into account the previous result, we obtain that, for \( a < t \), the solution \( p \) can be written as
\[
p(a, t) = e^{\alpha^*(t-a)} b_0(t-a) e^{- \int_0^a (\mu(s) + u(s)) \, ds},
\]
and we can state the next result:
Theorem 2. In the case of non-trivial datum, i.e., \( \bar{b}_0 > 0 \), we infer that
\[
\begin{align*}
\lim_{t \to \infty} \|p(t)\|_{L^\infty(0,A)} &= 0 \quad \text{if} \ \alpha^* < 0, \\
\lim_{t \to \infty} \|p(t)\|_{L^1(0,A)} &= +\infty \quad \text{if} \ \alpha^* > 0, \\
\lim_{t \to \infty} \|p(t) - \tilde{p}\|_{L^2(0,A)} &= 0 \quad \text{if} \ \alpha^* = 0,
\end{align*}
\]
where \( \tilde{p} = \bar{b}_0 e^{-\int_0^a (\mu(s)+u(s)) \, ds} \) is a nontrivial steady state solution to (4).

For more details and complete proofs of the previous theorems see [4].

Denoting by
\[
R = \int_0^A \beta(a) e^{-\int_0^a (\mu(s)+u(s)) \, ds} \, da,
\]
the statements of Theorem 2 are equivalent to:

- If \( R < 1 \), then \( \|p(t)\|_{L^\infty(0,A)} \to 0 \), as \( t \to +\infty \);
- If \( R > 1 \), then \( \|p(t)\|_{L^1(0,A)} \to +\infty \), as \( t \to +\infty \);
- If \( R = 1 \), then \( \|p(t) - \tilde{p}\|_{L^2(0,A)} \to 0 \), as \( t \to +\infty \), where \( \tilde{p} \) is the same as in Theorem 2.

Let now \( x \) be the unique Carathéodory solution to (5); obviously \( x \) is nonincreasing and nonnegative, therefore there exists \( \lim_{t \to \infty} x(t) \in [0,1] \).

Analyzing the previous results, we can see that in the case \( R < 1 \), the solution converges to 0 as \( t \to \infty \), even without the additional mortality rate expressed by \( \Phi(Y(t)) \). Therefore, in what follows we shall consider the assumption (A5) (which implies \( R > 1 \)).

Theorem 3. If \( p_0 \) is a non-trivial datum and assumption (A5) is satisfied, then the solution \( y \) to (1) satisfies
\[
\lim_{t \to \infty} \|y(t) - \tilde{y}\|_{L^\infty(0,A)} = 0,
\]
where
\[
\tilde{y}(a) = \exp\{-\alpha^*a\} \bar{b}_0 \exp\{-\int_0^a (\mu(s) + u(s)) \, ds\} \frac{1}{h_0} \Phi^{-1}(\alpha^*),
\]
\( a \in [0,A] \), is a stationary solution to (1) and \( h_0 = \bar{b}_0 \int_0^A e^{-\int_0^a (\mu(s)+u(s)) \, ds} \, da \).
Proof. By (A5) we have that $P(t) = e^{\alpha^* t} h(t)$, $t \geq 0$, where $\lim_{t \to \infty} h(t) = h_0 > 0$. Then, the corresponding equation in $x$ takes the form

$$x'(t) + \Phi(x(t)h(t)e^{\alpha^* t})x(t) = 0, \quad t > 0,$$

$$x(0) = 1,$$

or, equivalently, taking $z(t) = e^{\alpha^* t} x(t)$ ($t \geq 0$):

$$z'(t) = (\alpha^* - \Phi(h(t)z(t)))z(t), \quad t > 0,$$

$$z(0) = 1.$$ 

Obviously, $z(t) > 0$, $\forall t \in \mathbb{R}^+$; in fact

$$\lim_{t \to \infty} z(t) = z_0, \quad z_0 > 0,$$

where $z_0$ is the unique solution to $\alpha^* - \Phi(h_0z_0) = 0$. Indeed, if we denote by $z_h$ the unique solution to $\alpha^* - \Phi(hz) = 0$, and let $0 < h_1 < h_0 < h_2$; it follows that $z_{h_1} > z_0 > z_{h_2}$; as $\lim_{t \to \infty} h(t) = h_0$, it can be shown that

$$\lim_{t \to \infty} \text{dist}(z(t), [z_{h_2}, z_{h_1}]) = 0,$$

which implies that

$$\lim_{t \to \infty} z(t) = z_0,$$

where

$$z_0 = \frac{\Phi^{-1}(\alpha^*)}{h_0}.$$

Therefore we have obtained that

$$\lim_{t \to \infty} y(t) = \tilde{y} \quad \text{in} \quad L^\infty(0, A),$$

where

$$\tilde{y}(a) = \exp\{-\alpha^* a\bar{h}_0\exp\{-\int_0^a (\mu(s) + u(s)) \, ds\} \frac{1}{h_0} \Phi^{-1}(\alpha^*)},$$

$a \in [0, A]$, is a stationary solution to (2)$_{1-3}$.

In fact (3) has only two nonnegative solutions, one of them being the trivial one (see [4] and [6]).

We mention that this asymptotic behavior result for the linear age structured population dynamics has been proved first in [6]. \qed
3. Optimal harvesting. Assume in addition that \( y_0(a) > 0 \) a.e. in \((0, A)\). We shall consider next the optimal harvesting problem (2)-(3). First note that for every fixed \( u \in \mathcal{U} \), the nontrivial and nonnegative solution to (3) is given by \( \tilde{y}(a) = \tilde{y}(0)e^{-\int_0^a (u(s)+\mu(s)+\Phi(Y_0)) \, ds} \), \( a \in [0, A] \) and substituting this in the third equation in (3) we get

\[
1 = \int_0^A \beta(a)e^{-\int_0^a (u(s)+\mu(s)+\Phi(Y_0)) \, ds} \, da.
\]

We can solve equation (7) for the single variable \( Y_0 \), and from

\[
Y_0 = \int_0^A \tilde{y}(a) \, da = \tilde{y}(0) \int_0^A e^{-\int_0^a (u(s)+\mu(s)+\Phi(Y_0)) \, ds} \, da,
\]

we get the initial value \( \tilde{y}(0) \).

Existence of an optimal solution

**Theorem 4.** Problem (2)-(3) admits at least one optimal control.

**Proof.** Denote by \( d = \sup_{u \in \mathcal{U}} J(u) \), where \( J(u) = \int_0^A u(a)\tilde{y}^u(a) \, da \) and \( \tilde{y}^u \) denotes the nontrivial and nonnegative solution to (3) corresponding to the control \( u \in \mathcal{U} \). By a comparison result we get that \( 0 \leq J(u) \leq L \int_0^A \overline{y}(a) \, da \), where \( \overline{y} \) is the solution to (3) corresponding to \( u \equiv 0 \). Let now a sequence \( \{u_n\}_{n \in \mathbb{N}^*} \subset \mathcal{U} \) be such that \( d - \frac{1}{n} < J(u_n) \leq d \), \( \forall n \in \mathbb{N}^* \). Since \( \{u_n\} \) is bounded in \( L^\infty(0, A) \) it follows that is also bounded in \( L^2(0, A) \), so we can take a subsequence (also denoted by \( \{u_n\} \)) such that \( u_n \to u^* \) weakly in \( L^2(0, A) \). Since \( \mathcal{U} \) is a convex and closed subset of \( L^2(0, A) \), it is also weakly closed, therefore \( u^* \in \mathcal{U} \). We have that

\[
0 \leq \tilde{y}^{u_n} \leq \overline{y} \quad \text{a.e} \quad a \in (0, A),
\]

which implies that, on a subsequence, \( \tilde{y}^{u_n} \to \tilde{y}^* \) weakly in \( L^2(0, A) \) and \( \tilde{y}^{u_n}(0) \to \gamma \) in \( \mathbb{R} \). Using a corollary to Mazur’s theorem, we get that there exists a sequence \( \tilde{y}_n \) such that

\[
\tilde{y}_n = \sum_{i=n+1}^{k_n} \lambda^n_i \tilde{y}^{u_n}, \quad \lambda^n_i \geq 0, \quad \sum_{i=n+1}^{k_n} \lambda^n_i = 1,
\]
and \( \tilde{y}_n \to \tilde{y}^* \) in \( L^2(0, A) \). Let \( \tilde{u}_n \) be defined as

\[
\tilde{u}_n(a) = \begin{cases} \\
\sum_{i=n+1}^{k_n} \lambda^i y^n u_i(a) / \sum_{i=n+1}^{k_n} \lambda^i y^n u_i(a), & \text{if } \sum_{i=n+1}^{k_n} \lambda^i y^n u_i(a) \neq 0 \\
0, & \text{if } \sum_{i=n+1}^{k_n} \lambda^i y^n u_i(a) = 0.
\end{cases}
\]

For these controls we have \( \tilde{u}_n \in U \). In the following, we shall denote by

\[
Y^u = \int_0^A \tilde{y}^u(a) \, da,
\]

and

\[
Y^* = \int_0^A \tilde{y}^*(a) \, da.
\]

Since \( \tilde{y}^u_n \to \tilde{y}^* \) weakly in \( L^2(0, A) \) it follows that \( Y^u_n \to Y^* \). Let us consider now the system (3) corresponding to the controls \( u_i \):

\[
\begin{cases} \\
(\tilde{y}^u)' + \mu(a)\tilde{y}^u + \Phi(Y^u_0)\tilde{y}^u = -u_i \tilde{y}^u, & a \in (0, A) \\
\tilde{y}^u(0) = \int_0^A \beta(a)\tilde{y}^u(a) \, da.
\end{cases}
\]

Multiplying the system (9) by \( \lambda^i \) and summarizing from \( n + 1 \) to \( k_n \), we obtain:

\[
\begin{cases} \\
(\tilde{y}_n)' + \mu(a)\tilde{y}_n + \sum_{n+1}^{k_n} \lambda^i \Phi(Y^0_0)\tilde{y}_n = -u_n \tilde{y}_n, & a \in (0, A) \\
\tilde{y}_n(0) = \int_0^A \beta(a)\tilde{y}_n(a) \, da.
\end{cases}
\]

The solution to the above system, \( \tilde{y}_n \), is given by

\[
\tilde{y}_n(a) = \tilde{y}_n(0) e^{- \int_0^a (\mu(s) + \tilde{u}_n(s)) \, ds} - \int_0^a e^{- \int_0^s (\mu(\tau) + \tilde{u}_n(\tau)) \, d\tau} \sum_{n+1}^{k_n} \lambda^i \Phi(Y^0_0)\tilde{y}_n(s) \, ds,
\]

for \( a \in [0, A] \). Passing to the limit, one obtain that

\[
\tilde{y}^*(a) = \gamma e^{- \int_0^a (\mu(s) + u^*(s)) \, ds} - \int_0^a e^{- \int_0^s (\mu(\tau) + u^*(\tau)) \, d\tau} \Phi(Y^0_0)\tilde{y}^*(s) \, ds,
\]
a.e. \( a \in (0, A) \). It follows immediately that there exists a representant of \( \tilde{y}^* \) in \( L^2(0, A) \) continuous on \([0, A] \) (also denoted by \( \tilde{y}^* \)) which verifies

\[
\tilde{y}^*(a) = \tilde{y}^*(0) e^{-\int_0^a \Phi(Y_0^*(\tau)) \, d\tau} - \int_0^a e^{-\int_s^a \Phi(Y_0^*(\tau)) \, d\tau} \Phi(Y_0^*(s)) \tilde{y}^*(s) \, ds,
\]

\( a \in [0, A] \), i.e. \( \tilde{y}^* \) is a nonnegative solution to (3), corresponding to \( u^* \). We will prove immediately that this solution is nontrivial (i.e. \( \tilde{y}^* = \tilde{y}^u^* \)). We will show in addition that \( \tilde{y}_n \to \tilde{y}^* \) in \( C([0, A]) \). This yields

\[
|\tilde{y}_n(a) - \tilde{y}^*(a)| \leq |\tilde{y}_n(0) e^{-\int_0^a \Phi(Y_0^*(\tau)) \, d\tau} - \tilde{y}^*(0) e^{-\int_0^a \Phi(Y_0^*(\tau)) \, d\tau}|
\]

\[
+ \left| \int_0^a \left[ e^{-\int_s^a \Phi(Y_0^*(\tau)) \, d\tau} - e^{-\int_s^a \Phi(Y_0^*(\tau)) \, d\tau} \right] \Phi(Y_0^*(s)) \tilde{y}^*(s) \, ds \right| \]

\[
\leq e^{-\int_0^a \Phi(Y_0^*(\tau)) \, d\tau} |\tilde{y}_n(0) - \tilde{y}^*(0)|
\]

\[
+ |\tilde{y}^*(0)| \left| e^{-\int_0^a \Phi(Y_0^*(\tau)) \, d\tau} - e^{-\int_0^a \Phi(Y_0^*(\tau)) \, d\tau} \right| ds
\]

\[
+ \int_0^a e^{-\int_s^a \Phi(Y_0^*(\tau)) \, d\tau} \sum_{k=0}^{n-1} \lambda_k^n \Phi(Y_0^*(s)) \tilde{y}^*(s) \, ds - e^{-\int_s^a \Phi(Y_0^*(\tau)) \, d\tau} \Phi(Y_0^*(s)) \tilde{y}^*(s) \, ds,
\]

for \( a \in [0, A] \). Then taking \( \sup_{a \in [0, A]} \) and having in mind the boundedness of the functions involved, we get that

\[
\sup_{a \in [0, A]} |\tilde{y}_n(a) - \tilde{y}^*(a)|
\]

\[
\leq C_1 |\tilde{y}_n(0) - \tilde{y}^*(0)| + C_2 \left[ e^{-\int_0^a \Phi(Y_0^*(\tau)) \, d\tau} - e^{-\int_0^a \Phi(Y_0^*(\tau)) \, d\tau} \right] ds
\]

\[
+ C_3 \int_0^A \left( \left| \Phi(Y_0^*(\tau)) \right| \left| \sum_{k=0}^{n-1} \lambda_k^n \tilde{y}^*(s) - \tilde{y}^*(s) \right| \right) ds,
\]

where \( C_1, C_2, C_3 \) are positive constants. Now, since \( \tilde{u}_n \to u^* \), \( \tilde{y}_n \to \tilde{y}^* \) in \( L^2(0, A) \) and \( Y_0^{\tilde{u}_n} \to Y_0^* \) in \( R \) (which implies \( \Phi(Y_0^{\tilde{u}_n}) \to \Phi(Y_0^*) \) due to the
continuity of \( \Phi \), it follows that \( \tilde{y}_n \to \tilde{y}^* \) in \( C([0, A]) \). Actually, (3) has two nonnegative solutions: a trivial and a nontrivial one.

Since
\[
J(\tilde{u}_n) = \sum_{i=n+1}^{k_n} \lambda_i^n J(u_i) \to d, \quad \text{as} \quad n \to +\infty,
\]
we conclude that \( J(u^*) = d > 0 \) and consequently that \( \tilde{y}^* = \tilde{y}^{u^*} \).

**Necessary optimality conditions.** Let us denote by \( q \) the adjoint state, i.e., \( q \) satisfies:

\[
\begin{cases}
q' - \mu(a)q - \Phi(Y^*_0)q + \Psi = u^*(a)(1 + q), & a \in (0, A) \\
\Psi = \beta(a)q(0) - \Phi'(Y^*_0) \int_0^A q(a)\tilde{y}^*(a) \, da \\
q(A) = 0,
\end{cases}
\]

where \((u^*, \tilde{y}^*)\) is an optimal pair for (2)-(3) and \( Y^*_0 = \int_0^A \tilde{y}^*(a) \, da \).

Then the necessary optimality conditions are given by:

**Theorem 5.** Let \( q \) the corresponding adjoint state (which exists and is unique). Then

\[
u^*(a) = \begin{cases} 0, & \text{if} \ 1 + q(a) < 0 \\ L, & \text{if} \ 1 + q(a) > 0. \end{cases}
\]

**Proof.** The existence and uniqueness of the adjoint variable \( q \) can be proved via Banach’s fixed point theorem.

Let \( v \in L^\infty(0, A) \) such that \( u^* + \varepsilon v \in \mathcal{U} \) for any \( \varepsilon > 0 \) small enough. From the optimality of \( u^* \) we get:

\[
\int_0^A u^*(a)\tilde{y}^*(a) \geq \int_0^A (u^*(a) + \varepsilon v(a))\tilde{y}^{u^*+\varepsilon v}(a) \, da,
\]

which implies that

\[
\int_0^A u^*(a) \frac{\tilde{y}^{u^*+\varepsilon v}(a) - \tilde{y}^*(a)}{\varepsilon} \, da + \int_0^A v(a)\tilde{y}^{u^*+\varepsilon v}(a) \, da \leq 0.
\]

In order to prove the maximum principle, we need to prove first the following convergencies:

\[
\lim_{\varepsilon \to 0} \tilde{y}^{u^*+\varepsilon v} = \tilde{y}^* \quad \text{in} \quad C([0, A]),
\]
and

\[
\lim_{\varepsilon \to 0} \frac{\tilde{y}^{u^*+\varepsilon v} - \tilde{y}^*}{\varepsilon} = z \quad \text{in} \quad L^2(0, A),
\]

where \(z\) is the solution to

\[
\begin{cases}
z' + [\mu(a) + \Phi(Y^*_0)]z + \Phi'(Y^*_0)\tilde{y}^* Z = -u^* z - v\tilde{y}^*, \quad a \in (0, A) \\
Z = \int_0^A z(a) \, da \\
z(0) = \int_0^A \beta(a) z(a) \, da.
\end{cases}
\]

Note that \(\tilde{y}^{u^*+\varepsilon v}\) is the solution to

\[
\begin{cases}
(\tilde{y}^{u^*+\varepsilon v})' + r(a)\tilde{y}^{u^*+\varepsilon v} = -(\varepsilon v + \Phi(Y^*_0)\tilde{y}^{u^*+\varepsilon v}), \quad a \in (0, A) \\
Y^*_0 = \int_0^A \tilde{y}^{u^*+\varepsilon v}(a) \, da \\
\tilde{y}^{u^*+\varepsilon v}(0) = \int_0^A \beta(a)\tilde{y}^{u^*+\varepsilon v}(a) \, da,
\end{cases}
\]

where \(r(a) = \mu(a) + u^*(a)\). This implies \(0 < \tilde{y}^{u^*+L} \leq \tilde{y}^{u^*+\varepsilon v}(a) \leq \tilde{y}(a), \quad a \in [0, A]\), (for \(\varepsilon \in (0, 1)\) small enough) and, since the sequence is uniformly equicontinuous, we get that there exists a sequence \(\varepsilon_n\) such that \(\tilde{y}^{u^*+\varepsilon_n v}\) converges uniformly to a limit \(\bar{y}^*\). The uniform convergence of \(\tilde{y}^{u^*+\varepsilon v}\) also implies

\[
Y^*_0 = \int_0^A \tilde{y}^{u^*+\varepsilon v}(a) \, da \to Y^*_0 = \int_0^A \bar{y}^*(a) \, da, \quad \text{in} \quad \mathbb{R}.
\]

Writing the solution to (16), we get for every \(a \in [0, A]\):

\[
\tilde{y}^{u^*+\varepsilon v}(a) = \tilde{y}^{u^*+\varepsilon v}(0) \exp\{-\int_0^a r(s) \, ds\} - \int_0^a \tilde{y}^{u^*+\varepsilon v}(s) [\Phi(Y^*_0)\tilde{y}^{u^*+\varepsilon v} + \varepsilon v(s)] \exp\{-\int_s^a r(\tau) \, d\tau\} \, ds.
\]

Then, passing to the limit we obtain that:

\[
\bar{y}^*(a) = \bar{y}^*(0) \exp\{-\int_0^a r(s) \, ds\} - \int_0^a \bar{y}^*(s) \Phi(Y^*_0) \exp\{-\int_s^a r(\tau) \, d\tau\} \, ds, \quad a \in [0, A],
\]
(here \( \bar{y}^*(a) \geq \bar{y}^{u^* + L}(a) \), \( \forall a \in (0, A) \)), which is obviously the nontrivial non-negative solution to (3) corresponding to \( u^* \), i.e. \( \bar{y}^* = \bar{y}^u \). In order to prove the second assertion, let us denote by \( z^\varepsilon = \frac{1}{\varepsilon} [\bar{y}^{u^* + \varepsilon v} - \bar{y}^u] \). Subtracting (3) from (16) and dividing by \( \varepsilon \), we obtain that \( z^\varepsilon \) is solution to:

\[
\begin{cases}
(z^\varepsilon)' + f(a)z^\varepsilon + \frac{1}{\varepsilon} \left[ \Phi(Y_0^{u^*+\varepsilon v}) - \Phi(Y_0^*) \right] \bar{y}^{u^*+\varepsilon v} = -v \bar{y}^{u^*+\varepsilon v}, \\
\Phi(Y_0^{u^*+\varepsilon v}) = \int_0^A \bar{y}^{u^*+\varepsilon v}(a) \, da \\
z^\varepsilon(0) = \int_0^A \beta(a)z^\varepsilon(a) \, da,
\end{cases}
\]

where \( f(a) = \mu(a) + u^*(a) + \Phi(Y_0^*), \ a \in (0, A) \).

Let us take next \( w_\varepsilon = z^\varepsilon - z \) and prove that \( w_\varepsilon \to 0 \) as \( \varepsilon \to 0 \) in \( L^2(0, A) \). Note that, due to the continuous differentiability of \( \Phi \), the following relation hold:

\[
\frac{1}{\varepsilon} \left[ \Phi(Y_0^{u^*+\varepsilon v}) - \Phi(Y_0^*) \right] = \Phi'(Y_0^*) \int_0^A z^\varepsilon(a) \, da + \varphi(\varepsilon),
\]

where \( \varphi(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \).

Subtracting now (15) from (17), we obtain that \( w_\varepsilon \) is the solution to

\[
\begin{cases}
w_\varepsilon' + f(a)w_\varepsilon + \Phi'(Y_0^*)\bar{y}^u \int_0^A w_\varepsilon(a) \, da = F_\varepsilon(a), \ a \in (0, A) \\
w_\varepsilon(0) = \int_0^A \beta(a)w_\varepsilon(a) \, da,
\end{cases}
\]

where we have denoted by

\[
F_\varepsilon(a) = -v(a) + \Phi'(Y_0^*) \int_0^A z(a) \, da [\bar{y}^{u^*+\varepsilon v}(a) - \bar{y}^u(a)] - \varphi(\varepsilon)\bar{y}^{u^*+\varepsilon v}(a).
\]

Since \( \bar{y}^{u^*+\varepsilon v} \to \bar{y}^u \) in \( C([0, A]) \), it is obvious that \( F_\varepsilon \to 0 \) in \( L^\infty(0, A) \), as \( \varepsilon \to 0 \). Then we get that \( w_\varepsilon \to w \) as \( \varepsilon \to 0 \), where \( w \) is the solution to:

\[
\begin{cases}
w' + f(a)w + \Phi'(Y_0^*)\bar{y}^u \int_0^A w(a) \, da = 0, \ a \in (0, A) \\
w(0) = \int_0^A \beta(a)w(a) \, da.
\end{cases}
\]
Actually, via Banach’s fixed point theorem we infer that the above system has a unique solution. Therefore we have that \( w_\varepsilon \to 0 \) in \( L^\infty(0, A) \), which concludes the proof of the second assertion.

Making now \( \varepsilon \to 0 \) in (12), and using that \( \tilde{y}^{u^*+\varepsilon v} \to \tilde{y}^* \) in \( C([0, A]) \), and that \( z^\varepsilon \to z \) in \( L^2(0, A) \), we obtain that

\[
\int_0^A u^*(a)z(a) \, da + \int_0^A v(a)\tilde{y}^*(a) \, da \leq 0,
\]

for all \( v \in L^\infty(0, A) \) such that \( u^* + \varepsilon v \in \mathcal{U} \) for any \( \varepsilon > 0 \) small enough. Multiplying the first equation in (10) by \( z \) and integrating over \( (0, A) \) we get:

\[
\int_0^A z(a) \left[ q'(a) - \mu(a)q(a) - \Phi(Y_0^*)q(a) + \Psi \right] da = \int_0^A z(a)u^*(a)(1+q(a)) \, da.
\]

Integrating by parts, replacing \( z' \) and eliminating the identical terms, it follows

\[
\int_0^A z(a)u^*(a) \, da = \int_0^A v(a)\tilde{y}^*(a)q(a) \, da.
\]

Therefore, we obtain that

\[
\int_0^A \tilde{y}^*(a)(1+q(a))v(a) \, da \leq 0,
\]

for any \( v \in L^\infty(0, A) \) such that \( u^* + \varepsilon v \in \mathcal{U} \) for any \( \varepsilon > 0 \) small enough, which is equivalently to (11).

\[ \square \]

**Corollary 6.** By (10) and (11) it follows that the adjoint state satisfies

\[
\begin{cases}
q' - \mu(a)q - \Phi(Y_0^*)q + \beta(a)q(0) \\
-\Phi'(Y_0^*) \int_0^A q(a)\tilde{y}^*(a) \, da = L(1+q)^+, \quad a \in (0, A) \\
q(A) = 0.
\end{cases}
\]

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REFERENCES


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