ON A PROBLEM OF TRUESDELL FOR ANISOTROPIC ELASTIC SHELLS

BY

MIRCEA BÎRSAN

Abstract. In this paper we study the equilibrium of cylindrical elastic shells under the action of resultant forces and moments on the end edges. We employ the linear theory of Cosserat surfaces to describe the deformation of anisotropic and inhomogeneous cylindrical shells with arbitrary (open or closed) cross-section. In this context, we prove a minimum energy characterization for the solution of the relaxed Saint–Venant’s problem determined previously. Then, we treat the problem of Truesdell associated to the deformation of cylindrical shells.

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1. Introduction

In this paper, we employ the theory of Cosserat surfaces to investigate the deformation of thin cylindrical shells made from an elastic, anisotropic and inhomogeneous material.

The model of Cosserat shells has been presented in details in the monograph of NAGHDI [12]. The Cosserat theories for shells, rods and points are presented also in the book of RUBIN [15] from a modern perspective, together with several applications. According to [12,15], the Cosserat shell is a two–dimensional continuum endowed with a single deformable vector (called director) assigned to each of its points. Thus, the two–dimensional continuum can describe the deformation of the middle surface of a three–dimensional shell, while the variations of the director field give information about the three–dimensional effects in the mechanics of thin shells.
In the last decades, the problem of Saint–Venant in the three-dimensional elasticity for anisotropic materials has been intensively studied (see e.g., [5,10]). Due to its importance in industrial applications, the Saint–Venant’s problem for shells has also been investigated in many articles, within the classical shell theories (see e.g., [1,11,14]). ERICKSEN [7] has discussed this problem in the context of the Cosserat theory for shells. For isotropic and homogeneous Cosserat shells, the torsion problem has been treated previously by WENNER [18], and the relaxed Saint-Venant’s problem has been solved in [2].

We begin our paper by presenting a summary of the basic equations for the linear theory of Cosserat shells. Then, we confine our attention to cylindrical shells in Section 2. We consider anisotropic and inhomogeneous shells, which constitutive coefficients are assumed to be independent of the axial coordinate. We study the static deformation of cylindrical surfaces with arbitrary (open or closed) cross-section, under the action of resultant forces and resultant moments prescribed on the end edges. Our analysis is based on some results concerning the relaxed Saint–Venant problem established in [4], which are summarized in Section 3 and in the Appendix. In Section 4 we prove that the solutions previously determined for the relaxed Saint–Venant’s problem are the minimizers of the strain energy on certain classes of solutions. Section 5 is devoted to investigate, in the context of Cosserat shell theory, the counterpart of a problem formulated by Truesdell for the torsion of cylinders.

2. Deformation of cylindrical Cosserat shells

In this section we summarize the basic equations of equilibrium in the linear theory of Cosserat shells. We present first the equations for the general case of anisotropic and inhomogeneous materials, and then we confine to the case of cylindrical shells.

Let us denote by $S$ the reference configuration of a Cosserat surface and by $\theta^\alpha$ ($\alpha = 1, 2$) a curvilinear material coordinate system on $S$. The static deformation of the Cosserat shell is defined by the position vector $r \left(\theta^1, \theta^2\right)$ and the deformable director $d \left(\theta^1, \theta^2\right)$ assigned to every point of the surface. The reference configuration $S$, which is assumed to coincide with the initial configuration, is characterized by the position vector $R \left(\theta^1, \theta^2\right)$ and the
director field \( D(\theta^1, \theta^2) \). We introduce the following fields

\[
(1) \quad A_\alpha = \frac{\partial R}{\partial \theta^\alpha}, \quad A_3 = \frac{A_1 \times A_2}{|A_1 \times A_2|}, \quad A_{\alpha\beta} = A_\alpha \cdot A_\beta, \quad B_{\alpha\beta} = A_3 \cdot A_{\alpha\beta},
\]

which represent the covariant base vectors along the \( \theta^\alpha \)-curves, the unit normal to \( S \) and the first and second fundamental forms of the surface \( S \), respectively. Throughout this paper, a subscript comma stands for partial differentiation with respect to the coordinates \((\theta^\alpha)\), while a subscript vertical bar denotes the covariant differentiation with respect to the metric tensor \( A_{\alpha\beta} \). Also, we make use of the summation convention over repeated indices and assume that the Latin indices take the values \( \{1, 2, 3\} \), while the Greek indices are confined to the range \( \{1, 2\} \).

In the linear theory, we introduce the infinitesimal displacement \( u = r - R \) and director displacement \( \delta = d - D \). The strain measures \( e_{\alpha\beta}, \gamma_i \) and \( \rho_i \) are defined by

\[
(2) \quad e_{\alpha\beta} = \frac{1}{2} (\hat{u}_{\alpha|\beta} + \hat{u}_{\beta|\alpha}) - B_{\alpha\beta} \hat{u}_3, \quad \gamma_\alpha = \hat{\delta}_\alpha + \hat{u}_{3,\alpha} + B_\alpha^3 \hat{u}_\beta, \quad \gamma_3 = \hat{\delta}_3, \quad \rho_{3\alpha} = \hat{\delta}_{3,\alpha},
\]

where \( \hat{u}_i = u \cdot A_i \) and \( \hat{\delta}_i = \delta \cdot A_i \). We consider the case of shells with constant thickness in the reference configuration, which means that we have \( D = A_3 \) (cf. [12], page 447).

For an arbitrary curve \( c \) on \( S \) (which may also be the boundary \( \partial S \)) we denote by \( \nu \) the (outward) unit normal to \( c \) tangent to the surface \( S \). Let \( N \) and \( M \) denote the contact force and the contact director couple acting per unit length of \( c \), and \( \nu_\alpha = \nu \cdot A_\alpha \). Then, we have the decompositions of Cauchy type

\[
(3) \quad N = \left( N^{\alpha\beta} A_\beta + V^\alpha A_3 \right) \nu_\alpha, \quad M = \left( M^\alpha A_\alpha \right) \nu_\alpha.
\]

Let us introduce the surface tensor defined by \( N^{\alpha\beta} = N^{\alpha\beta} + B_\gamma^i M^{\gamma\alpha} \). For anisotropic and inhomogeneous shells, the constitutive equations have the form

\[
W = \frac{1}{2} C_1^{\alpha\beta\gamma\delta} e_{\alpha\beta} e_{\gamma\delta} + \frac{1}{2} C_2^{\alpha\beta j\gamma} \rho_{\alpha j} \rho_{\beta \gamma} + C_3^{\alpha\beta\gamma} \rho_{\alpha\beta} \rho_{\gamma i} + C_4^{\alpha i \gamma j} \gamma_i \gamma_j + \frac{1}{2} C_5^{\alpha i \gamma j} \hat{\gamma}_i \hat{\gamma}_j,
\]

\[
N^{\alpha\beta} = \frac{1}{2} \left( \frac{\partial W}{\partial e_{\alpha\beta}} + \frac{\partial W}{\partial e_{\beta\alpha}} \right), \quad V^i = \frac{\partial W}{\partial \gamma_i}, \quad M^{\alpha i} = \frac{\partial W}{\partial \rho_{\alpha i}},
\]

\[
(4) \quad \rho_{3\alpha} = \hat{\delta}_{3,\alpha}.
\]
where $W$ represents the strain energy density per unit area of $S$. The function $W = W(e_{\alpha\beta}, \gamma_i, \rho_{i\alpha})$ is a quadratic form of its arguments, which characterize each specific anisotropic material. The coefficients $C_{(n)}^{\alpha\beta\gamma\delta}$ satisfy the following symmetry conditions

$$
C_{(1)}^{\alpha\beta\gamma\delta} = C_{(1)}^{\beta\alpha\gamma\delta} = C_{(1)}^{\gamma\delta\alpha\beta},
C_{(2)}^{\alpha\beta\gamma\delta} = C_{(2)}^{\beta\alpha\gamma\delta},
C_{(3)}^{\alpha\beta\gamma\delta} = C_{(3)}^{\beta\alpha\gamma\delta},
C_{(1)}^{\alpha\beta\gamma\delta} = C_{(1)}^{\beta\alpha\gamma\delta},
C_{(1)}^{\alpha\beta\gamma\delta} = C_{(1)}^{\beta\alpha\gamma\delta},
C_{(1)}^{\alpha\beta\gamma\delta} = C_{(1)}^{\beta\alpha\gamma\delta}.
$$

The equations of equilibrium for Cosserat shells in the absence of assigned body forces can be written as

$$
N_{\alpha} = 0, \quad V^\alpha = 0, \quad M^\alpha = 0.
$$

For any displacement field $v = (u, \delta)$, the strain energy of the Cosserat shell is

$$
U(v) = \int_S W(e_{\alpha\beta}(v), \gamma_i(v), \rho_{i\alpha}(v)) \, da.
$$

Also, we consider the energy norm $\| \cdot \|$ and the scalar product $\langle \cdot, \cdot \rangle$ given by

$$
\|v\|^2 = 2U(v) = \langle v, v \rangle,
\langle v, \tilde{v} \rangle = \int_S [N_{\alpha\beta}(v)e_{\alpha\beta}(\tilde{v}) + V^i(v)\gamma_i(\tilde{v}) + M^\alpha(v)\rho_{i\alpha}(\tilde{v})] \, da.
$$

In the second part of this section we confine our attention to cylindrical Cosserat shells and write the relevant field equations for this particular geometry of the surface.

We assume that the reference configuration $S$ of the Cosserat shell is a cylindrical (open or closed) surface with arbitrary cross section, which generators are parallel to the axis $Ox_3$ of the rectangular Cartesian coordinate frame $Ox_1x_2x_3$ (see Figure 1).

The cylindrical surface $S$ is situated between the planes $x_3 = 0$ and $x_3 = \bar{z}$, and we denote by $C_z$ the cross-section curve which belongs to the plane $x_3 = z$, $z \in [0, \bar{z}]$. Let us choose the surface curvilinear coordinates $\theta^1 = s$, $\theta^2 = z$ on $S$, where $s \in [0, \bar{s}]$ represents the arc parameter along the curves $C_z$ and $z = x_3$, $z \in [0, \bar{z}]$. We denote by $e_i$ the unit vectors along the $Ox_i$ axes. The parametric equation of $S$ is given by

$$
R = R(s, z) = x_\alpha(s) e_\alpha + z e_3, \quad D = D(s) = \epsilon_{\alpha\beta}x_\beta'(s) e_\alpha,
$$

where $\epsilon_{\alpha\beta}$ is the permutation symbol.
where the functions $x_\alpha(s)$ are assumed to be of class $C^3[0, \bar{s}]$. We denote by 
\[(\cdot)' = \frac{d}{ds}(\cdot)\] and $\epsilon_{\alpha\beta}$ is the two–dimensional alternator ($\epsilon_{12} = -\epsilon_{21} = 1$, 
$\epsilon_{11} = \epsilon_{22} = 0$). The end edge curves are $C_0$ and $C_z$. For closed shells, the 
functions $x_\alpha(s)$ (and their derivatives) satisfy the continuity conditions at 
s = 0 and s = \bar{s}. For open shells, we denote by $L_1$ and $L_2$ the lateral edges 
characterized by s = 0 and s = \bar{s}, respectively. We remark that, for the 
surface $S$ given by (8), we have 

\[
A_1 = \mathbf{\tau}(s) = x_\alpha'(s)\mathbf{e}_\alpha, \quad A_2 = \mathbf{e}_3, \quad A_3 = \mathbf{n}(s) = \epsilon_{\alpha\beta}x'_\beta(s)\mathbf{e}_\alpha,
\]

(9) 
\[
A_{\alpha\beta} = \delta_{\alpha\beta}, B_{11} = -r^{-1}, \quad B_{12} = B_{21} = B_{22} = 0,
\]

where $\mathbf{\tau}$ and $\mathbf{n}$ represent the unit tangent and normal vectors to $C_z$, $r$ 
is the curvature radius of $C_z$ and $\delta_{\alpha\beta}$ is the Kronecker symbol. Thus, the 
physical components of any tensor on the cylindrical surface coincide with 
the covariant and with the contravariant components of the same tensor. 
Taking into account that $\theta^1 = s$, $\theta^2 = z$ and $A_3 = \mathbf{n}$, in what follows we 
shall employ the subscripts $s$, $z$ and $n$ instead of the indices 1, 2 and 3,
respectively, for any tensor components. In particular, we can decompose any vector \( \mathbf{v} \) as \( \mathbf{v} = v_\theta \mathbf{e}_3 + v_n \mathbf{n} \). Using this notation convention together with (9), the geometrical relations (2) become

\[
e_{ss} = \frac{\partial u_s}{\partial s} + \frac{u_n}{r}, \quad e_{sz} = e_{zs} = \frac{1}{2} \left( \frac{\partial u_s}{\partial z} + \frac{\partial u_z}{\partial s} \right), \quad e_{zz} = \frac{\partial u_z}{\partial z}, \quad \gamma_n = \delta_n,
\]

(10)

\[
\gamma_s = \delta_s - \frac{u_s}{r} + \frac{\partial u_n}{\partial s}, \quad \gamma_z = \delta_z + \frac{\partial u_n}{\partial z}, \quad \rho_{ss} = \frac{\partial \delta_s}{\partial s} + \frac{1}{r} \frac{\partial u_s}{\partial r} + \frac{u_n}{r^2},
\]

\[
\rho_{zz} = \frac{\partial \delta_z}{\partial z}, \quad \rho_{sz} = \frac{\partial \delta_s}{\partial z}, \quad \rho_{zs} = \frac{\partial \delta_z}{\partial s} + \frac{1}{r} \frac{\partial u_s}{\partial z}, \quad \rho_{ns} = \frac{\partial \delta_n}{\partial s}, \quad \rho_{nz} = \frac{\partial \delta_n}{\partial z}.
\]

The equations of equilibrium (5) can be written for cylindrical shells in the form

\[
\frac{\partial}{\partial s} N_{ss} + \frac{\partial}{\partial z} N_{zs} + \frac{1}{r} V_s = 0, \quad \frac{\partial}{\partial s} N_{zs} + \frac{\partial}{\partial z} N_{zz} = 0,
\]

(11)

\[
\frac{\partial}{\partial s} V_s + \frac{\partial}{\partial z} V_z - \frac{1}{r} N_{ss} = 0, \quad \frac{\partial}{\partial s} M_{ss} + \frac{\partial}{\partial z} M_{zs} - V_s = 0,
\]

\[
\frac{\partial}{\partial s} M_{zz} + \frac{\partial}{\partial z} M_{z_z} - V_z = 0, \quad \frac{\partial}{\partial s} M_{sn} + \frac{\partial}{\partial z} M_{zn} - V_n = 0.
\]

The boundary conditions on lateral edges (in the case of open cylindrical shells) are

(12) \( N = 0, \quad M = 0 \) on \( L_\gamma \) \( (\gamma = 1, 2) \).

For simplicity, we use the notation \( s_1 = 0, s_2 = \bar{s} \). In the case of closed cylindrical shells, we consider the following continuity conditions for the displacement field \( \mathbf{v} = (u, \delta) \) at the endpoints of the interval \([s_1, s_2]\)

\[
v(s_1, z) = v(s_2, z), \quad \frac{\partial v}{\partial s}(s_1, z) = \frac{\partial v}{\partial s}(s_2, z),
\]

(13)

\[
\frac{\partial^2 v}{\partial s^2}(s_1, z) = \frac{\partial^2 v}{\partial s^2}(s_2, z), \quad z \in [0, \bar{z}].
\]

Adopting the usual approach of the relaxed Saint–Venant’s problem we consider that, on the end edges, the resultant forces and moments are prescribed. For any displacement field \( \mathbf{v} = (u, \delta) \), we define the vectors

(14) \( \mathbf{R}(v) = \int_{C_0} \mathbf{N}(v) \, dl, \quad \mathbf{M}(v) = \int_{C_0} \left[ \mathbf{R} \times \mathbf{N}(v) + \mathbf{D} \times \mathbf{M}(v) \right] \, dl, \)
which represent the resultant force and the resultant moment about $O$ of the contact forces and contact director couples acting on $C_0$, corresponding to the displacement field $v$. Relations (14) can be expressed using the tensor components as

$$
\mathcal{R}_i(v)e_i = -\int_{C_0} \left[ x'_\alpha N_{zs}(v) + \epsilon_{\alpha\beta} x'_\beta V_z(v) \right] \mathrm{d}l e_\alpha - \int_{C_0} N_{zz}(v) \mathrm{d}l e_3, 
$$

(15)

$$
\mathcal{M}_i(v)e_i = \int_{C_0} \left[ \epsilon_{\beta\gamma} x_\beta N_{zz}(v) + x'_\alpha M_{zz}(v) \right] \mathrm{d}l e_\alpha + \int_{C_0} \left[ \epsilon_{\alpha\beta} x'_\alpha x'_\beta N_{zs}(v) + x_\alpha x'_\alpha V_z(v) - M_{zs}(v) \right] \mathrm{d}l e_3.
$$

The boundary conditions on the end edge $C_0$ are

$$
\mathcal{R}(v) = \mathcal{R}^0, \quad \mathcal{M}(v) = \mathcal{M}^0.
$$

From the conditions of equilibrium for the shell and the relations (12) and (16), we can readily deduce the boundary conditions on the other end edge $C_z$.

### 3. Saint–Venant’s problem for anisotropic shells

The problem under investigation herein consists in determining the equilibrium of anisotropic and inhomogeneous cylindrical shells under the action of resultant forces and resultant moments acting on the end edges. The equations of equilibrium are (11), the boundary conditions on the lateral edges are (12) for open shells (alternatively, for closed cylindrical shells we have the continuity conditions (13)), and the boundary conditions on the end edge $C_0$ are (16). By analogy with the corresponding situation in the three-dimensional elasticity, this problem is called the relaxed Saint–Venant’s problem for cylindrical shells. Let us designate the problem formulated above by $P(\mathcal{R}^0, \mathcal{M}^0)$.

In the remainder of this paper, we consider anisotropic and inhomogeneous Cosserat shells which constitutive coefficients are independent of the axial coordinate $z$. Thus, we assume that the coefficients $C^{\alpha}_{(n)}$ of the strain energy density $W$ in (4) are functions of the circumferential coordinate $s$ only. We denote by $\mathcal{D}$ the set of all displacement fields $v = (u, \delta) \in C^1(\mathcal{S}) \cap C^2(\mathcal{S})$ which satisfy the equations of equilibrium (11), together with the conditions on the lateral edges (12) for open shells,
and the continuity conditions (13) for closed shells. Then, the problem $P(\mathcal{R}^0, \mathcal{M}^0)$ can be formulated in the following terms: find a displacement field $v \in \mathcal{D}$ which satisfies the end edge conditions (16).

In this section, we summarize some results concerning the relaxed Saint–Venant’s problem for anisotropic cylindrical shells which have been obtained previously in [4]. The results and the notations presented in this section will be useful in the subsequent developments.

We shall denote by $K(\mathcal{R}^0, \mathcal{M}^0)$ the set of all solutions to the problem $P(\mathcal{R}^0, \mathcal{M}^0)$. We observe that $P(\mathcal{R}^0, \mathcal{M}^0)$ can be decomposed into two problems:

$(P_1)$: the extension–bending–torsion problem, characterized by $\mathcal{R}^0_\alpha = 0$, $\mathcal{M}^0_i = 0$.

$(P_2)$: the flexure problem, characterized by $\mathcal{R}^0_3 = 0$.

We designate by $K_I(\mathcal{R}^0_3, \mathcal{M}^0_1, \mathcal{M}^0_2, \mathcal{M}^0_3)$ and $K_{II}(\mathcal{R}^0_1, \mathcal{R}^0_2)$ the sets of all solutions $v = (u, \delta)$ to the problems $(P_1)$ and $(P_2)$, respectively.

In [4], we have determined a solution of the extension–bending–torsion problem $(P_1)$ expressed as a linear combination of four displacement fields $v^{(1)}, v^{(2)}, v^{(3)}$ and $v^{(4)}$. These displacement fields $v^{(k)}$ are known and their expressions are recorded in the Appendix.

For any constants $a_k$, $k = 1, 2, 3, 4$, we denote by $\hat{a}$ the four–dimensional vector $\hat{a} = (a_1, a_2, a_3, a_4) \in \mathbb{R}^4$ and let $v\{\hat{a}\}$ be the displacement field given by the linear combination $v\{\hat{a}\} = a_1 v^{(1)} + a_2 v^{(2)} + a_3 v^{(3)} + a_4 v^{(4)}$. According to [4], the displacement field $v\{\hat{a}\}$ has the following properties: (i) $\partial v\{\hat{a}\}/\partial x_3$ is a rigid displacement field; (ii) $v\{\hat{a}\}$ satisfies the equations of equilibrium and the boundary conditions on the lateral edges for vanishing body loads and contact loads, i.e. $v\{\hat{a}\} \in \mathcal{D}$; and (iii) the resultant force and resultant moment corresponding to the field $v\{\hat{a}\}$ are given by

$$R(v\{\hat{a}\}) = -\left(\sum_{k=1}^{4} D_{3k} a_k\right) e_3,$$

$$M(v\{\hat{a}\}) = \epsilon_{\beta \alpha} \left(\sum_{k=1}^{4} D_{3k} a_k\right) e_\alpha - \left(\sum_{k=1}^{4} D_{4k} a_k\right) e_3,$$

where the coefficients $D_{kr}$ are defined by

$$D_{kr} = \frac{1}{\bar{z}} \langle v^{(k)}, v^{(r)} \rangle, \quad k, r \in \{1, 2, 3, 4\}.$$
We have \( \det(D_{kr})_{4 \times 4} \neq 0 \).

The solution of the relaxed Saint–Venant’s problem is specified by the following result.

**Theorem 1.** (i) The extension–bending–torsion problem \((P_1)\) for cylindrical Cosserat shells admits a solution \(v^0 \in K_I(\mathcal{R}^0_3, \mathcal{M}^0_1, \mathcal{M}^0_2, \mathcal{M}^0_3)\) such that \(\frac{\partial v^0}{\partial x_3}\) is a rigid displacement field. This solution is given by

\[
(19) \quad v^0 = v\{\hat{a}\},
\]

where the constants \(a_k\) are determined by the system of linear equations

\[
(20) \quad \left( \sum_{r=1}^{4} D_{kr} a_r \right)_{k=1,\ldots,4} = (M^0_2, -M^0_1, -R^0_3, -M^0_3). \]

(ii) The flexure problem \((P_2)\) admits a solution \(v^F \in K_{II}(\mathcal{R}^0_1, \mathcal{R}^0_2)\) of the form

\[
(21) \quad v^F = \int_0^{x_3} v\{\hat{b}\} \, dx_3 + v\{\hat{c}\} + w(s),
\]

where \(\hat{b} = (b_1, b_2, b_3, b_4)\), \(\hat{c} = (c_1, c_2, c_3, c_4)\) are constants and \(w(s)\) is a displacement field which depends only on \(s\). For this solution, the constants \(b_k\) are given by

\[
(22) \quad \left( \sum_{r=1}^{4} D_{kr} b_r \right)_{k=1,\ldots,4} = (-R^0_1, -R^0_2, 0, 0). \]

Theorem 1 has been proved in Section 5 (see Theorems 4 and 5) of [4]. We mention that the constants \(\hat{c}\) and the field \(w(s)\) appearing in (21) have also been determined in [4] and their expressions are recorded in the Appendix.

We notice that \(v^0\) and \(v^F\) possess properties which are analogous to those of the classical Saint–Venant’s solutions in linear elasticity (see e.g., [10]). The analogy will be extended in the next section, where we prove that these solutions for Cosserat shells can be characterized as minimizers of energy.
4. Properties of the solution

The aim of this section is to establish minimum energy characterizations for the solution of Saint–Venant’s problem presented in the preceding section.

We shall use the notation \( f(s, z)\big|_{z=\bar{z}} = f(s, \bar{z}) - f(s, 0) \), for any function \( f \). Let us consider the linear subspace \( \hat{D} \) of all displacement fields \( v \in D \) which satisfy the following symmetry conditions on the end edges \( C_0 \) and \( C_\varepsilon \)

\[
\left[ N_{\bar{z}}(v), N_{zz}(v), V_z(v), M_{\bar{z}}(v), M_{zz}(v), M_{zn}(v) \right]_{z=\bar{z}} = 0.
\]

The next result will be useful.

**Lemma 2.** For every displacement field \( v \in \hat{D} \), we have \( \mathcal{R}_\alpha(v) = 0 \).

**Proof.** The conditions of equilibrium for the shell imply that

\[
\int_{C_0 \cup C_\varepsilon} (R \times N + D \times M) dl = 0.
\]

Hence, using the tensor components, we find that

\[
\left[ \int_{C_\varepsilon} \left( \epsilon_{\beta\alpha} x_\beta N_{zz}(v) + x'_\alpha M_{zz}(v) \right) dl \right]_{z=\bar{z}} = \bar{\varepsilon} \epsilon_{\alpha\beta} \mathcal{R}_\beta(v), \quad \forall v \in D.
\]

Then, taking \( v \in \hat{D} \) and using (23), we obtain the desired result. \( \square \)

Let us introduce the subspaces \( \mathcal{R} \) and \( \mathcal{V} \) of \( \hat{D} \) given by

\[
\mathcal{R} = \left\{ v \in \hat{D} \mid \mathcal{R}_\alpha(v) = M_\alpha(v) = 0 \right\}, \quad \mathcal{V} = \left\{ v\{\hat{a}\} \mid \hat{a} \in \mathbb{R}^4 \right\}.
\]

Indeed, we have \( v^{(k)} \in \hat{D} \) by virtue of (56) and (58), so that \( \mathcal{V} \subset \hat{D} \) is the linear subspace spanned by the displacement fields \( v^{(k)} \), \( k = 1, \ldots, 4 \). Concerning the subspaces (25), the following result holds.

**Lemma 3.** The subspaces \( \mathcal{R} \) and \( \mathcal{V} \) realize an orthogonal decomposition of the space \( \hat{D} \), i.e. we have: (i) \( \mathcal{R} \perp \mathcal{V} \); (ii) For every \( v \in \hat{D} \) there exist (uniquely) the fields \( \bar{v} \in \mathcal{R} \) and \( v\{\hat{a}\} \in \mathcal{V} \) such that \( v = \bar{v} + v\{\hat{a}\} \).
Proof. For any two displacement fields \( v, \tilde{v} \in D \), the scalar product (7)_2 can be written

\[
\langle \tilde{v}, v \rangle = \left[ \int_C \left( N_{zs}(\tilde{v})u_s + N_{zz}(\tilde{v})u_z + V_z(\tilde{v})u_n + M_{zs}(\tilde{v})\delta_s \\
+ M_{zz}(\tilde{v})\delta_z + M_{zn}(\tilde{v})\delta_n \right) dl \right]_{z=0}^{z=\pi},
\]

Then, using the relations (16), (56), (58) and (24), we find that

\[
\langle v, v\{\hat{a}\} \rangle = -\frac{1}{2} \bar{z}^2 a_\alpha R_{\alpha}(v) \\
- \bar{z} \left[ \epsilon_{a3} a_\beta M_{\alpha}(v) + a_3 R_3(v) + a_4 M_3(v) \right], \quad \forall v \in \hat{D}, \hat{a} \in \mathbb{R}^4.
\]

Next, if we choose \( v \in \mathcal{R} \) in (27) and take into account Lemma 2 and (25), we deduce that \( \langle v, v\{\hat{a}\} \rangle = 0 \), for all \( v \in \mathcal{R} \), \( v\{\hat{a}\} \in \mathcal{V} \), and hence, \( \mathcal{R} \perp \mathcal{V} \).

To prove (ii), let us consider an arbitrary field \( v \in \hat{D} \). In view of (17), we can determine (uniquely) the constants \( \hat{a} \) such that \( R_3(v\{\hat{a}\}) = R_3(v) \) and \( M_i(v\{\hat{a}\}) = M_i(v) \). Then, we define the field \( \tilde{v} = v - v\{\hat{a}\} \) and notice that \( R_3(\tilde{v}) = M_i(\tilde{v}) = 0 \). Consequently, \( \tilde{v} \in \mathcal{R} \) and the proof is complete.

On the basis of the above lemmas, we prove minimum energy characterizations for the solution \( v^0 \) of the extension–bending–torsion problem and the solution \( v^F \) of the flexure problem.

**Theorem 4.** Let \( v^0 \in K_I(R_0^0, M_i^0) \) be the solution of the extension–bending–torsion problem given by (19), (20), corresponding to the resultant force \( R_3^0 \) and the resultant moments \( M_i^0 \). If we denote by \( Y \) the subclass of solutions \( v = (u, \delta) \) which satisfy the symmetry conditions (23), i.e.

\[
Y = K_I(R_3^0, M_i^0) \cap \hat{D},
\]

then we have

\[
U(v^0) \leq U(v), \quad \forall v \in Y,
\]

and the equality holds if and only if \( v \) differs from \( v^0 \) by a rigid displacement field.

Proof. Consider an arbitrary field \( v \in Y \). By the definitions (25) and (28), we have \( v - v^0 \in \mathcal{R} \). On the other hand, we have \( v^0 \in \mathcal{V} \) and by virtue
of Lemma 3 we get \( \langle v - v^0, v^0 \rangle = 0 \). Hence, \( \|v^0\|^2 + \|v - v^0\|^2 = \|v\|^2 \) and we deduce that (29) holds true. \( \square \)

Let \( D_F \) be the subspace of all displacement fields \( v \in D \) which satisfy \( \partial v/\partial x_3 \in C^1(\bar{S}) \cap C^2(S) \) and the following symmetry conditions on the end edges

\[
\left[ \left( N_{zz}, N_{zz}, V_z, M_{zz}, M_{zn}, M_{zn} \right) \left( \frac{\partial v}{\partial x_3} \right) \right]_{z=0}^{z=\bar{z}} = 0.
\]

We notice that \( D_F \) can be characterized as

\[
D_F = \{ v \in D \mid \frac{\partial v}{\partial x_3} \in \hat{D} \}.
\]

The following result holds.

**Theorem 5.** Let \( v^F \in K_{II}(R^0_\alpha) \) be the solution of the flexure problem given by Theorem 1, corresponding to the resultant forces \( R^0_\alpha \), and let \( Y_F \) be the set

\[
Y_F = \{ v \in D_F \mid \mathcal{R}_\alpha(v) = R^0_\alpha \}.
\]

Then, we have

\[
U\left( \frac{\partial v^F}{\partial x_3} \right) \leq U\left( \frac{\partial v}{\partial x_3} \right), \quad \forall v \in Y_F,
\]

and the equality holds if and only if \( \frac{\partial v}{\partial x_3} \) and \( \frac{\partial v^F}{\partial x_3} \) differ only by a rigid displacement field.

**Proof.** First, we notice that \( v^F \in Y_F \). From Theorem 1 of [4] and (32) we deduce that

\[
\mathcal{R}_\alpha\left( \frac{\partial v}{\partial x_3} \right) = 0, \quad \mathcal{M}_3 \left( \frac{\partial v}{\partial x_3} \right) = 0, \quad \mathcal{M}_\alpha \left( \frac{\partial v}{\partial x_3} \right) = \epsilon_{\alpha\beta} R^0_\beta, \quad \forall v \in Y_F.
\]

Consequently, we have

\[
\mathcal{R} \left( \frac{\partial(v - v^F)}{\partial x_3} \right) = 0, \quad \mathcal{M} \left( \frac{\partial(v - v^F)}{\partial x_3} \right) = 0, \quad \forall v \in Y_F,
\]

and hence,

\[
\frac{\partial(v - v^F)}{\partial x_3} \in \mathcal{R}, \quad \forall v \in Y_F.
\]
On the other hand, the relation (21) yield

\[(36) \quad \frac{\partial v^F}{\partial x_3} = v\{b\} \in V,\]

where we have neglected a rigid displacement field of the Cosserat shell. From (35), (36) and the Lemma 3 we find

\[\langle \frac{\partial v}{\partial x_3} - \frac{\partial v^F}{\partial x_3}, \frac{\partial v^F}{\partial x_3} \rangle = 0, \quad \forall v \in Y_F,\]

and we obtain

\[U\left(\frac{\partial v^F}{\partial x_3}\right) + U\left(\frac{\partial v}{\partial x_3} - \frac{\partial v^F}{\partial x_3}\right) = U\left(\frac{\partial v}{\partial x_3}\right), \quad \forall v \in Y_F.\]

The proof is complete. \(\square\)

Theorems 4 and 5 establish properties of the solutions \(v^0\) and \(v^F\) for cylindrical Cosserat shells, which are analogous to the minimum energy characterizations of Saint–Venant’s classical solution in the three-dimensional theory of elasticity. In [8], ERICKSEN discusses the significance of symmetry conditions of the type (23).

5. A problem of Truesdell

In this section, we approach a problem concerning the deformation of cylindrical shells which, in the context of three–dimensional elasticity, was formulated by Truesdell. In [16, 17], TRUESDELL proposed the following problem for the torsion of elastic cylinders: to define the functional \(\tau(\cdot)\) on the set of all solutions \(u\) of the torsion problem, corresponding to a scalar torque \(M^0_3\), such that

\[\tau(u) = -\frac{M^0_3}{\mu D},\]

where \(\mu D\) is the torsional rigidity of the cylinder. The solution of this problem was given by DAY [6]. Later, PODIO-GUIDUGLI [13] rephrased the problem for extension and bending, while IESAN [9] considered the flexure of elastic cylinders.

Returning to the extension–bending–torsion problem \((P_1)\) for cylindrical Cosserat shells, we mention that in the isotropic case the constants \(a_3,\)
which satisfy the system (20) can be interpreted as the global measures of axial strain, axial curvature and twist, respectively (see [3]).

Suggested by (20), we consider the following problem: to define the functionals \( \tau_k(\cdot), \ k = 1, \ldots, 4 \), on the set of solutions \( K_I(\mathcal{R}_0^0, \mathcal{M}_0^0) \) such that

\[
\left( \sum_{r=1}^4 D_{kr} \tau_r(v) \right)_{k=1, \ldots, 4} = (\mathcal{M}_2^0, -\mathcal{M}_1^0, -\mathcal{R}_3^0, -\mathcal{M}_3^0),
\]

(37)

\( \forall v \in K_I(\mathcal{R}_0^0, \mathcal{M}_0^0) \).

We shall solve this problem on the subclass of solutions (28). Let \( v \in Y \) be an arbitrary fixed displacement field. Since \( v \in \hat{D} \), by Lemma 3 we see that \( v \) decomposes as

\[
v = \hat{v} + \check{v}, \quad \check{v} \in \mathcal{R}, \quad \hat{v} \in \mathcal{V}.
\]

We notice that the field \( \hat{v} \) is unique and it minimizes the functional \( g : \mathcal{V} \to \mathbb{R}, \ g(w) = \|v - w\| \). In view of (18), the fields \( v^{(1)}, \ldots, v^{(4)} \) form a basis in the linear space \( \mathcal{V} \). Consider \( \alpha_k = \alpha_k(v) \), \( k = 1, \ldots, 4 \), the components of the field \( \hat{v} \) relative to the basis \{\( v^{(1)}, \ldots, v^{(4)} \)\} and denote by \( \hat{\alpha} = (\alpha_1, \ldots, \alpha_4) \). Then, we have

\[
\hat{v} = \sum_{k=1}^4 \alpha_k v^{(k)} = v\{\hat{\alpha}\}.
\]

(39)

Since \( v \in K_I(\mathcal{R}_3^0, \mathcal{M}_1^0) \), we obtain from (17), (38) and (39) that

\[
\left( \sum_{r=1}^4 D_{kr} \alpha_r(v) \right)_{k=1, \ldots, 4} = (\mathcal{M}_2^0, -\mathcal{M}_1^0, -\mathcal{R}_3^0, -\mathcal{M}_3^0), \quad \forall v \in Y.
\]

(40)

The comparison of (37) and (40) leads to

\[
\tau_k(v) = \alpha_k(v), \quad \forall v \in Y, \ k = 1, \ldots, 4,
\]

i.e. \( \tau_k(v) \) coincide with the components of \( \hat{v} \in \mathcal{V} \) referred to the basis \{\( v^{(1)}, \ldots, v^{(4)} \)\}.

In order to obtain an analytical expression for the functionals \( \tau_k(v) \), we use the relations (27) to get

\[
\langle v, v^{(\alpha)} \rangle = \varepsilon_{\alpha\beta\gamma} \mathcal{M}_\beta^0, \langle v, v^{(3)} \rangle = -\varepsilon \mathcal{R}_3^0, \langle v, v^{(4)} \rangle = -\varepsilon \mathcal{M}_3^0, \forall v \in Y.
\]

(42)
From (40)–(42) we deduce

\[ \sum_{r=1}^{4} D_{kr} \tau_r(v) = \frac{1}{2} \langle v, v^{(k)} \rangle, \quad \forall v \in Y, \quad k = 1, \ldots, 4, \]

which can be written with the help of (26) in the form

\[ \sum_{r=1}^{4} D_{kr} \tau_r(v) = \frac{1}{2} \int_{C_0 \cup C_1} [N(v^{(k)}) \cdot u + M(v^{(k)}) \cdot \delta] \, dl, \forall v = (u, \delta) \in Y. \]

The relation (43) gives the answer to Truesdell’s problem for the extension–bending–torsion of cylindrical Cosserat shells.

For the isotropic case, the functional \( \tau_4(v), \tau_3(v) \) and \( \tau_2(v) \), respectively, are called the generalized twist, generalized axial strain and generalized axial curvature [3].

In the remainder of this section, we shall investigate Truesdell’s problem for the flexure of cylindrical shells. Taking into account the relations (22), we are led to consider the problem: to define the functionals \( \sigma_k(\cdot) \), \( k = 1, \ldots, 4 \), on the set of solutions to the flexure problem such that

\[ \left( \sum_{r=1}^{4} D_{kr} \sigma_r(v) \right)_{k=1, \ldots, 4} = (-R^0_1, -R^0_2, 0, 0), \quad \forall v \in K_{II}(R^0_0). \]

We shall determine a solution to this problem on the subclass of solutions \( K_{II}(R^0_0) \cap D_F \). Let us choose an arbitrary element \( v \in K_{II}(R^0_0) \cap D_F \). It follows that \( \partial v / \partial x_3 \in \mathcal{D} \), and by virtue of Lemma 3, we have the decomposition

\[ \frac{\partial v}{\partial x_3} = \hat{v} + \tilde{v}, \quad \hat{v} \in \mathcal{R}, \quad \tilde{v} \in \mathcal{V}. \]

The field \( \hat{v} \) is uniquely determined and it minimizes the functional \( h : \mathcal{V} \to \mathbb{R}, \ h(w) = \| \frac{\partial w}{\partial x_3} - w \| \) . We denote by \( \beta_k = \beta_k(v) \) the components of the field \( \hat{v} \) relative to the basis \( \{ v^{(1)}, \ldots, v^{(4)} \} \) and \( \hat{\beta} = (\beta_1, \ldots, \beta_4) \). Then,

\[ \hat{v} = \sum_{k=1}^{4} \beta_k v^{(k)} = v\{ \hat{\beta} \}. \]
If we apply the Theorem 1 of [4], for the field \( v \in K_{II}(R_0^0) \) we obtain that \( v\{\hat{\beta}\} \in K_I(0, R_0^0, -R_1^0, 0) \). The last relation together with (20) yield

\[
(47) \left( \sum_{r=1}^{4} D_{kr} \beta_r(v) \right)_{k=1, \ldots, 4} = (-R_1^0, -R_2^0, 0, 0), \quad \forall v \in K_{II}(R_0^0) \cap D_F.
\]

From (44) and (47) we get

\[
(48) \sigma_k(v) = \beta_k(v), \quad \forall v \in K_{II}(R_0^0) \cap D_F, \quad k = 1, \ldots, 4.
\]

Further, let us put the functionals (48) in another form. If we write the relation (27) for \( \partial v / \partial x^3 \) and use the equations (34), we obtain

\[
\left\langle \frac{\partial v}{\partial x^3}, v\{\hat{a}\} \right\rangle = -\bar{\varepsilon} a_\alpha R_\alpha(v), \quad \forall v \in D_F, \quad \hat{a} \in \mathbb{R}^4.
\]

Consequently, we have

\[
(49) \left\langle \frac{\partial v}{\partial x^3}, v^{(\alpha)} \right\rangle = -\bar{\varepsilon} R_\alpha^0, \quad \left\langle \frac{\partial v}{\partial x^3}, v^{(\alpha+2)} \right\rangle = 0, \quad \forall v \in K_{II}(R_\alpha^0) \cap D_F, \alpha = 1, 2.
\]

Then, from (47)–(49) we deduce

\[
(50) \left( \sum_{r=1}^{4} D_{kr} \sigma_r(v) \right)_{k=1, \ldots, 4} = \left( \frac{1}{\bar{\varepsilon}} \left\langle \frac{\partial v}{\partial x^3}, v^{(\alpha)} \right\rangle, 0, 0 \right), \quad \forall v \in K_{II}(R_\alpha^0) \cap D_F.
\]

On the other hand, in virtue of (26), we can write

\[
(51) \left\langle \frac{\partial v}{\partial x^3}, v^{(\alpha)} \right\rangle = \int_{\partial C \setminus \bar{C}_\varepsilon} \left[ N(v^{(\alpha)}) \cdot \frac{\partial u}{\partial x^3} + M(v^{(\alpha)}) \cdot \frac{\partial \delta}{\partial x^3} \right] dl, \quad v = (u, \delta).
\]

From (50) and (51) we can obtain the analytical expressions of \( \sigma_k(v) \). These functionals correspond to the global measures of strain appropriate to flexure, associated to the displacement field \( v \in K_{II}(R_\alpha^0) \cap D_F \), in the isotropic case [3].

**Appendix**

**The cross-section plane problem.** We recall that the solution of the relaxed Saint-Venant’s problem for anisotropic three-dimensional cylinders
reduces to solving some generalized plane strain problems associated to the cross-section of the cylinder (see e.g., [10]). As a counterpart of these generalized plane strain problems we consider, in the case of cylindrical shells, the following problem (called the *cross-section plane problem*): find the displacement field \( v(s) = (u(s), \delta(s)) \) which depends only on the circumferential coordinate \( s \), and which satisfies the equilibrium equations

\[
A(v(s)) = -F(s),
\]

where \( F(s) = (f_s, f_z, f_n, l_s, l_z, l_n) \), together with the boundary conditions

(i) for open cylindrical shells (here \( N^{(\gamma)} \) and \( M^{(\gamma)} \) are given constant vectors)

\[
N(v(s_\gamma)) = N^{(\gamma)}, \quad M(v(s_\gamma)) = M^{(\gamma)} \quad \text{on } L_\gamma \quad (\gamma = 1, 2),
\]

(ii) for closed cylindrical shells

\[
v(s_1) = v(s_2), \quad v'(s_1) = v'(s_2).
\]

In the relation (52), \( f \) and \( l \) represent the assigned force and director force which depend only on \( s \), while the operator \( A(v) = (A_1(v), ..., A_6(v)) \) is given by

\[
A_1(v) = \frac{\partial}{\partial s} N_{ss}(v) + \frac{\partial}{\partial z} N_{zs}(v) + \frac{1}{r} V_s(v),
\]
\[
A_2(v) = \frac{\partial}{\partial s} N_{sz}(v) + \frac{\partial}{\partial z} N_{zz}(v),
\]
\[
A_3(v) = \frac{\partial}{\partial s} V_s(v) + \frac{\partial}{\partial z} V_z(v) - \frac{1}{r} N_{ss}(v),
\]
\[
A_4(v) = \frac{\partial}{\partial s} M_{ss}(v) + \frac{\partial}{\partial z} M_{zs}(v) - V_s(v),
\]
\[
A_5(v) = \frac{\partial}{\partial s} M_{sz}(v) + \frac{\partial}{\partial z} M_{zz}(v) - V_z(v),
\]
\[
A_6(v) = \frac{\partial}{\partial s} M_{sn}(v) + \frac{\partial}{\partial z} M_{zn}(v) - V_n(v).
\]

According to Theorem 2 from [4], the necessary and sufficient conditions for the existence of the solution \( v(s) \) to the cross-section plane problem
(52)–(54) are the following
\[
\int_{C_0} f dl + (1 - \varepsilon) \left( N^{(1)} + N^{(2)} \right) = 0,
\]
\[
\left[ \int_{C_0} (R \times f + D \times l) dl + (1 - \varepsilon) \sum_{\gamma = 1}^{2} (R^{(\gamma)}(0) \times N^{(\gamma)} + D^{(\gamma)} \times M^{(\gamma)}) \right] \cdot e_3 = 0,
\]
where \( \varepsilon \) takes the values \( \varepsilon = 0 \) for open cylindrical shells and \( \varepsilon = 1 \) for closed shells, and we denote by \( R^{(\gamma)}(z) = [R(s, z)]_{s = s_3}, D^{(\gamma)} = [D(s)]_{s = s_3} (\gamma = 1, 2) \).

The solution of the problem (52)–(54) can be determined as in Sect. 4 of [4], provided the conditions (55) are satisfied.

The displacement fields \( v^{(k)} \). Let us define the fields \( v^{(k)} \), \( k = 1, 2, 3, 4 \), which are employed to construct the solution of the relaxed Saint-Venant’s problem for cylindrical Cosserat shells.

First, we introduce four displacement fields denoted by \( v_c^{(k)} \), \( k = 1, \ldots, 4 \), and given by
\[
v_c^{(1)} = (-z^2 e_\alpha + xz e_3, z \alpha x e_3), \ n_c^{(3)} = (ze_3, 0),
\]
\[
v_c^{(4)} = (-ze_\alpha x e_3, x \alpha e_3).
\]
We observe that the strain measures corresponding to the fields (56) are independent of the axial coordinate \( x_3 \) and that \( \frac{\partial v_c^{(k)}}{\partial x_3} \) is a rigid displacement field.

For each \( k = 1, \ldots, 4 \), let us consider the cross–section plane problem (52)–(54) for the given data
\[
F(s) = A(v_c^{(k)}), \ N^{(\gamma)} = -N(v_c^{(k)}(s_\gamma)),
M^{(\gamma)} = -M(v_c^{(k)}(s_\gamma)), \ \gamma = 1, 2.
\]
We can verify that the fields (57) satisfy the conditions (55), so that the problem (52)–(54) with the system of external loads (57) admits a solution, denoted by \( w^{(k)}(s) \), \( k = 1, \ldots, 4 \).

Then, we define the displacement fields
\[
v^{(k)} = v_c^{(k)} + w^{(k)}(s), \ k = 1, 2, 3, 4.
\]
We remark that the fields \( v^{(k)} \) have the following properties (see [4], Sect. 5)

\[
v^{(k)} \in D \quad \text{and} \quad R_\alpha(v^{(k)}) = 0, \quad k = 1, \ldots, 4.
\]

The field \( w(s) \) and the constants \( \hat{c} \). For the sake of completeness, we present here the expressions for the field \( w(s) \) and the constants \( \hat{c} \) which appear in the solution (21) of the flexure problem \( (P_2) \). The displacement field \( w(s) \) is determined by solving the cross–section plane problem (52)-(54) for the external loads given by (see Theorem 5 of [4])

\[
\begin{align*}
\mathcal{F}(s) &= A\left(\int_0^{x_3} v\{\hat{b}\} \, dx_3\right), \\
\mathcal{N}(\gamma) &= -N\left(\int_0^{x_3} v\{\hat{b}\} \, dx_3\right)(s_\gamma), \\
\mathcal{M}(\gamma) &= -M\left(\int_0^{x_3} v\{\hat{b}\} \, dx_3\right)(s_\gamma), \quad \gamma = 1, 2.
\end{align*}
\]

Then, the constants \( \hat{c} = (c_1, c_2, c_3, c_4) \) are given by the system of equations

\[
\begin{pmatrix}
\sum_{r=1}^4 D_{kr} c_r
\end{pmatrix}_{k=1,\ldots,4} = \left( -M_2(\hat{w}), M_1(\hat{w}), R_3(\hat{w}), M_3(\hat{w}) \right),
\]

where we denote by \( \hat{w}(s) = w(s) + \int_0^{x_3} v\{\hat{b}\} \, dx_3 \). In this manner, we have specified precisely the solution \( v^F \) of the flexure problem, introduced by Theorem 1.

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Department of Mathematics,
“Al.I. Cuza” University of Iași,
Iași,
ROMANIA
bmircea@uaic.ro